The domatic numbers of factors of graphs

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Abstract

The maximum cardinality of a partition of the vertex set of a graph G into dominating sets is the *domatic number* of G, denoted $d(G)$. We consider Nordhaus-Gaddum type results involving the domatic number of a graph, where a Nordhaus-Gaddum type result is a (tight) lower or upper bound on the sum or product of a parameter of a graph and its complement. Thereafter we investigate the upper bounds on the sum and product of the domatic numbers $d(G_1)$, $d(G_2)$ and $d(G_3)$ where $G_1 \oplus G_2 \oplus G_3 = K_n$. We show that the upper bound on the sum is $n+2$, while the maximum value of the product is $\lfloor n/3 \rfloor^3$ for $n \geq 57$.

1 Introduction

In a graph $G = (V, E)$ the open neighborhood of a vertex $v \in V$ is $N(v) = \{x \in V \mid vx \in E\}$, the set of vertices adjacent to v. The closed neighborhood

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is $N[v] = N(v) \cup \{v\}$. A set $S \subseteq V$ is a dominating set if every vertex in V is either in S or is adjacent to a vertex in S, that is, $V = \cup_{s \in S} N[s]$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set. A *domatic partition* is a partition of V into dominating sets and the *domatic number* $d(G)$ is the largest number of sets in a domatic partition [3]. The domatic number of a graph has been extensively studied, see for example [1, 3, 12, 13]. It follows from the definition that $\gamma(G) \cdot d(G) \leq n$, and hence $d(G) \leq n/\gamma(G)$. We shall use this fact and the following results, the first of which is based on a result of Ore [9] while the second result is due to Cockayne and Hedetniemi [3].

Theorem 1 [9] *For every graph G, $d(G) = 1$ if and only if G has an isolated vertex.*

Theorem 2 [3] *For every graph G, $d(G) \leq \delta(G) + 1$.*

If $G_1, G_2, \ldots, G_t$ are graphs on the same vertex set V with disjoint edge sets, then $G = G_1 \oplus G_2 \oplus \ldots \oplus G_t$ denotes the graph with vertex set V and edge set $E(G) = E(G_1) \cup E(G_2) \ldots \cup E(G_t)$ and the graphs $G_1, G_2, \ldots, G_t$ are called a t-factoring of G.

The special case of a 2-factoring of the complete graph K_n is simply a factoring of K_n into a graph G and its complement $\overline{G}$. A Nordhaus-Gaddum type result is a (tight) lower or upper bound on the sum or product of a parameter of a graph and its complement. In 1956 the original paper [8] by Nordhaus and Gaddum appeared. In it they gave sharp bounds on the sum and product of the chromatic numbers of a graph and its complement. Since then such results have been given for several parameters (see [4]). They include results on the domination number (see [7, 11], for example) and the domatic number (see [3, 5]). In particular, it is shown in [5] that $d(G) \cdot d(\overline{G}) \leq n^2/4$.

In this paper we consider the domatic number and two variations of Nordhaus-Gaddum type inequalities. First, we extend the concept of a Nordhaus-Gaddum type result by considering $G_1 \oplus G_2 = K_{s,s}$ rather than $G_1 \oplus G_2 = K_n$. We establish sharp lower and upper bounds on the sums and products of $d(G_1)$ and $d(G_2)$. We show that the upper bound on the sum is $s + 2$, while the maximum product is $\lfloor s/2 \rfloor^2$ for $s \geq 10$.

Second, we look at the complete graph factored into three edge-disjoint graphs. We investigate upper bounds on the sum and product of the domatic numbers $d(G_1)$, $d(G_2)$ and $d(G_3)$ where $G_1 \oplus G_2 \oplus G_3 = K_n$ and $n \geq 3$. We show that the upper bound on the sum is $n + 2$, while the maximum value of the product is $\lfloor n/3 \rfloor^3$ for $n \geq 57$.

2 Domatic number and relative complement

In [2] the idea of a relative complement of a graph was suggested. If G is a subgraph of H, then the graph $H - E(G)$ is the *complement of G relative to H*. In [6] it is shown that the complete bipartite graph $K_{s,s}$ is a suitable graph to consider for results on relative complements. In particular, $K_{s,s}$ is shown to be an obvious replacement for the complete graph K_n in Nordhaus-Gaddum results.

The independence number of a graph is the cardinality of a maximum independent set and the independent domination number is the minimum cardinality of a maximal independent set. In [6] the sums and products of $\psi(G_1)$ and $\psi(G_2)$ are examined where $G_1 \oplus G_2 = K_{s,s}$, and ψ is the independence, domination, or independent domination number, inter alia.

Theorem 3 [6] *Let $s \geq 3$ be an integer and let $G_1 \oplus G_2 = K_{s,s}$. Then the following table represents some sharp bounds on the sum and product of $\psi(G_1)$ and $\psi(G_2)$ for certain parameters ψ.*

	SUM		*PRODUCT*	
ψ	*Lower*	*Upper*	*Lower*	*Upper*
γ	5	$2s+2$	6	$\lfloor (s/2+2)^2 \rfloor$
i	5	$3s$	6	$2s^2$
β	$2s$	$3s$	s^2	$\lfloor 9s^2/4 \rfloor$

Here we consider $G_1 \oplus G_2 = K_{s,s}$, where $s \geq 2$, and lower and upper bounds on the sums and products of $d(G_1)$ and $d(G_2)$. In what follows in this section, we simplify the notation by letting $d_i = d(G_i)$, $\gamma_i = \gamma(G_i)$, $\delta_i = \delta(G_i)$, and $\Delta_i = \Delta(G_i)$ for $i = 1, 2$. Observe that $\gamma_i \geq 2$ for $i = 1, 2$.

Since $d(G) \geq 1$ for all graphs G, $d_1 + d_2 \geq 2$ and $d_1 d_2 \geq 1$. That these lower bounds are sharp, may be seen by taking $G_1 \cong K_1 \cup K_{s-1,s}$. Then $d_1 = 1$ and $d_2 = d(K_{1,s} \cup \overline{K}_{s-1}) = 1$ since each of G_1 and G_2 contains an isolated vertex.

The upper bounds are more interesting. We show first that the upper bound on the sum is $s+2$. We shall assume that $K_{s,s}$ has partite sets $\mathcal{L}$ and $\mathcal{R}$ (representing "left" and "right"). For the remainder of this section, we may also assume that $\gamma_2 \geq \gamma_1$. Observe that $\gamma_1 \geq 2$ always.

Theorem 4 *Let $s \geq 2$ be an integer and let $G_1 \oplus G_2 = K_{s,s}$. Then*

$$d_1 + d_2 \leq s + 2.$$

Furthermore, $d_1 + d_2 = s + 2$ if and only if either G_1 or G_2 is isomorphic to sK_2.

Proof. Let v be any vertex of G_1. Then, by Theorem 2, $d_1 \leq \delta(G_1) + 1 \leq \deg_{G_1} v + 1$ and $d_2 \leq \deg_{G_2} v + 1$. Thus, $d_1 + d_2 \leq \deg_{G_1} v + \deg_{G_2} v + 2 = s + 2$. This establishes the upper bound.

Assume $G_1 \cong sK_2$. Then G_2, the complement of G_1 relative to $K_{s,s}$, may be obtained from $K_{s,s}$ by removing the edges of a 1-factor. Thus, $d_1 = 2$ and $d_2 = s$, so $d_1 + d_2 = s + 2$. Hence the sufficiency is clear.

To prove the necessity, assume $d_1 + d_2 = s + 2$. If G_1 or G_2, say G_2, has an isolated vertex, then $d_1 \leq s$ and $d_2 = 1$, whence $d_1 + d_2 \leq s + 1$, a contradiction. Hence $\delta_1 \geq 1$ and $\delta_2 \geq 1$. Thus $\Delta_1 \leq s - 1$ and $\Delta_2 \leq s - 1$. In particular, if $s = 2$, then $G_1 \cong G_2 \cong 2K_2$. So we may assume that $s \geq 3$, for otherwise the result follows. Recall that $\gamma_2 \geq \gamma_1 \geq 2$. If $\gamma_1 \geq 4$, then $d_j \leq |V(G_j)|/\gamma_j \leq s/2$ for $j = 1, 2$, and thus $d_1 + d_2 \leq s$, a contradiction. Hence $\gamma_1 = 2$ or $\gamma_1 = 3$.

If $\gamma_1 = 3$, then $d_1 \leq |V(G_1)|/\gamma_1 = 2s/3$. Furthermore, since $\gamma_1 = 3$, it is evident that there is a vertex in $\mathcal{L}$ or $\mathcal{R}$ of degree at least $s - 2$ in G_1, and therefore of degree at most 2 in G_2. Thus Theorem 2 gives that $d_2 \leq 3$. Hence $d_1 + d_2 \leq 2s/3 + 3$. If $s \geq 4$, or if $s = 3$ and $d_1 + d_2 < 2s/3 + 3$, then $d_1 + d_2 < s + 2$, a contradiction. Hence $s = 3$, $d_1 = 2s/3 = 2$ and $d_2 = 3$. Since $d_2 = 3 \leq \delta_2 + 1$, we have $\delta_2 \geq 2$. Furthermore, $\delta_1 \geq 1$, so G_2 must be 2-regular, i.e., $G_2 \cong C_6$, $G_1 \cong 3K_2$, and the theorem holds.

Assume, then, that $\gamma_1 = 2$. Then $d_1 \leq |V(G_1)|/\gamma_1 = s$. Furthermore, since $\gamma_1 = 2$, there are vertices in each partite set $\mathcal{L}$ and $\mathcal{R}$ of degree $s - 1$ in G_1, and therefore of degree 1 in G_2. Thus $d_2 \leq 2$, whence $d_1 + d_2 \leq s + 2$ with equality if and only if $d_1 = s$ and $d_2 = 2$. If $d_1 = s$, then we may partition the vertex set of G_1 into s dominating sets each of cardinality 2. Since $s \geq 3$, each such dominating set consists of a vertex u (say) from $\mathcal{L}$ and a vertex v (say) from $\mathcal{R}$. Thus, u dominates $\mathcal{R} - \{v\}$ and v dominates $\mathcal{L} - \{u\}$. Since $\Delta_1 \leq s - 1$, uv cannot be an edge. Consequently, G_1 may be obtained from $K_{s,s}$ by removing the edges of a 1-factor. Thus, $G_2 \cong sK_2$. $\square$

We consider next the maximum product $d_1 d_2$. Three lemmas are given to establish the upper bound.

Lemma 5 *Let $s \geq 2$ be an integer and let $G_1 \oplus G_2 = K_{s,s}$. If $\gamma_1 = 2$, then*

$$d_1 \cdot d_2 \leq 2s.$$

Furthermore, $d_1 d_2 = 2s$ if and only if either G_1 or G_2 is isomorphic to sK_2.

Proof. If G_2 has an isolated vertex, then $d_1 \leq s$ and $d_2 = 1$, whence $d_1 d_2 \leq s < 2s$. Hence we may assume that $\delta_1 \geq 1$ and $\delta_2 \geq 1$. Thus $\Delta_1 \leq s-1$ and $\Delta_2 \leq s-1$. In particular, if $s = 2$, then $G_1 \cong 2K_2$ and $G_2 \cong 2K_2$, and $d_1 = d_2 = 2$. So we may assume that $s \geq 3$. Since $\gamma_1 = 2$, it follows, as in the proof of Theorem 4, that $d_1 \leq s$ and $d_2 \leq 2$, whence $d_1 d_2 \leq 2s$ with equality if and only if $d_1 = s$ and $d_2 = 2$ if and only if $G_2 \cong sK_2$. $\square$

Lemma 6 *Let $s \geq 3$ be an integer and let $G_1 \oplus G_2 = K_{s,s}$. If $\gamma_1 = 3$, then*

$$d_1 \cdot d_2 \leq 2s,$$

and this bound is sharp.

Proof. Since $\gamma_1 = 3$, we have, as in the proof of Theorem 4, that $d_1 \leq 2s/3$ and $d_2 \leq 3$, whence $d_1 d_2 \leq 2s$. This establishes the upper bound. To see that this bound can be realised, let s be a multiple of 3. If $s = 3$, then we may take $G_1 \cong sK_2$. For $s \geq 6$, let $K_{s,s}$ have partite sets $\mathcal{L}$ and $\mathcal{R}$, say, and partition $\mathcal{L}$ ($\mathcal{R}$) into three sets $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ (respectively, $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$) each of cardinality $s/3$. We construct the edge set of G_2 as follows. Add all edges between $\mathcal{L}_1$ and $\mathcal{R}_2$, and add all edges between $\mathcal{L}_2$ and $\mathcal{R}_1$. We then join each vertex of $\mathcal{L}_3$ to one vertex of $\mathcal{R}_1$ and to one vertex of $\mathcal{R}_2$ in such a way that every vertex of $\mathcal{R}_1 \cup \mathcal{R}_2$ is adjacent to exactly one vertex of $\mathcal{L}_3$ (so in G_2 the subgraph induced by $\mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{L}_3$ is isomorphic to $s/3K_{1,2}$). Similarly, we join each vertex of $\mathcal{R}_3$ to one vertex of $\mathcal{L}_1$ and to one vertex of $\mathcal{L}_2$ in such a way that every vertex of $\mathcal{L}_1 \cup \mathcal{L}_2$ is adjacent to exactly one vertex of $\mathcal{R}_3$. Then $\mathcal{L}_i \cup \mathcal{R}_i$ is a dominating set of G_2 for $i = 1, 2, 3$. Thus $d_2 \geq 3$. However, since G_2 has minimum degree 2, $d_2 \leq 3$. Consequently, $d_2 = 3$. For each vertex $v \in \mathcal{L}_3 \cup \mathcal{R}_3$, let $N_v = N[v]$ in G_2. Then the sets $\{N_v \mid v \in \mathcal{L}_3 \cup \mathcal{R}_3\}$ form a domatic partition in G_1 of cardinality $2s/3$. Hence $d_1 \geq 2s/3$. Since no two vertices dominate G_1 and each N_v dominates G_1, it follows that $\gamma_1 = 3$. Therefore, $d_1 \leq 2s/3$, and consequently, $d_1 = 2s/3$. Thus, $d_1 d_2 = 2s$. □

Lemma 7 *Let $s \geq 4$ be an integer and let $G_1 \oplus G_2 = K_{s,s}$. If $\gamma_1 \geq 4$, then*

$$d_1 \cdot d_2 \leq \lfloor s/2 \rfloor^2,$$

and this bound is sharp.

Proof. Since $\gamma_j \geq 4$, we know that $d_j \leq 2s/\gamma_j = s/2$ for $j = 1, 2$. Hence $d_j \leq \lfloor s/2 \rfloor$ for $j = 1, 2$, and thus $d_1 \cdot d_2 \leq \lfloor s/2 \rfloor^2$.

That the bound of $s^2/4$ can be realized for $s \geq 4$ with s even, may be seen as follows. Let $K_{s,s}$ have partite sets $\mathcal{L}$ and $\mathcal{R}$, say. Partition $\mathcal{L}$ ($\mathcal{R}$) into two sets $\mathcal{L}_1$ and $\mathcal{L}_2$ (respectively, $\mathcal{R}_1$ and $\mathcal{R}_2$) each of cardinality $s/2$. Let the edges of G_1 (say) be given by all edges between $\mathcal{L}_1$ and $\mathcal{R}_1$ and between $\mathcal{L}_2$ and $\mathcal{R}_2$. If $s = 4$, then $G_1 \cong G_2 \cong 2K_{2,2}$, whence $d_1 = d_2 = 2 = s/2$. If $s \geq 6$, then any minimum dominating set of G_1 contains a vertex from each of $\mathcal{L}_1$, $\mathcal{L}_2$, $\mathcal{R}_1$, and $\mathcal{R}_2$. Thus, $\mathcal{L} \cup \mathcal{R}$ can be partitioned into $s/2$ dominating sets each of cardinality 4. Hence, $d_1 = s/2$. Similarly, $d_2 = s/2$. Thus, $d_1 \cdot d_2 = s^2/4$.

That the bound of $(s-1)^2/4$ can be realized for $s \geq 5$ with s odd, may be seen as follows. Let $K_{s,s}$ have partite sets $\mathcal{L}$ and $\mathcal{R}$, say. Partition $\mathcal{L}$ ($\mathcal{R}$) into two sets $\mathcal{L}_1$ and $\mathcal{L}_2$ (respectively, $\mathcal{R}_1$ and $\mathcal{R}_2$) of sizes $(s+1)/2$ and $(s-1)/2$. Then with G_1 and G_2 defined as in the preceeding paragraph, we have $d_1 = d_2 = (s-1)/2$. Thus, $d_1 \cdot d_2 = (s-1)^2/4$. □

An immediate consequence of Lemmas 5, 6 and 7 now follows.

Theorem 8 *Let $s \geq 2$ be an integer and let $G_1 \oplus G_2 = K_{s,s}$. Then*

$$d(G_1) \cdot d(G_2) \leq \begin{cases} 2s & \text{for } s \leq 9 \\ \lfloor s/2 \rfloor^2 & \text{for } s \geq 10 \end{cases}$$

and these bounds are sharp.

3 Triple factors of the complete graph

In this section, we consider another generalization of Nordhaus-Gaddum-type results, i.e., we look at the complete graph factored into three edge-disjoint graphs. This direction was pursued by Plesník [10] who extended

Nordhaus and Gaddum's results on the chromatic number to the case where the complete graph is factored into more than two factors.

Upper bounds on the sum and product of the domination numbers $\gamma(G_1)$, $\gamma(G_2)$ and $\gamma(G_3)$, where $G_1 \oplus G_2 \oplus G_3 = K_n$, are presented in [6]. It is proven in [6] that $\gamma(G_1) + \gamma(G_2) + \gamma(G_3) \leq 2n + 1$, while the maximum value of the triple product $\gamma(G_1) \cdot \gamma(G_2) \cdot \gamma(G_3)$ is $n^3/27 + \Theta(n^2)$. We investigate upper bounds on the sum and product of the domatic numbers $d(G_1)$, $d(G_2)$, and $d(G_3)$ where $G_1 \oplus G_2 \oplus G_3 = K_n$ and $n \geq 3$.

3.1 The Triple Sum

We consider first the sum of the domatic numbers $d(G_1)$, $d(G_2)$ and $d(G_3)$. We show that the upper bound on the sum is $n + 2$. To do this, we prove the following stronger result.

Theorem 9 *Let $n \geq 3$ be an integer and let $G_1 \oplus G_2 \oplus \cdots \oplus G_t = K_n$ be a t-factoring of K_n. Then*

$$t \leq \sum_{i=1}^{t} d(G_i) \leq n + t - 1.$$

Proof. The proof of the lower bound is trivial, since $d(G) \geq 1$ for all graphs G. That this lower bound is sharp for $t \geq 3$, may be seen by taking $G_1 \cong K_1 \cup K_{n-1}$, $G_2 \cong K_2 \cup \overline{K}_{n-2}$, $G_3 \cong K_1 \cup K_{1,n-2}$, and, if $t \geq 4$, $G_i \cong \overline{K}_n$ for $i = 4, \ldots, t$. Then G_i contains an isolated vertex for $i = 1, 2, \ldots, t$, so $d(G_i) = 1$. Hence, $\sum_{i=1}^{t} d(G_i) = t$. To prove the upper bound, we note that for each factor G_i, $d(G_i) \leq \delta(G_i) + 1$. Since $G_1 \oplus G_2 \oplus \cdots \oplus G_t = K_n$,

$$\sum_{i=1}^{t} d(G_i) \leq \sum_{i=1}^{t} (\delta(G_i) + 1) \leq n - 1 + t.$$

That the upper bound is sharp may be seen by taking $G_1 \cong K_n$ (so $G_i \cong \overline{K}_n$ for $i = 2, 3, \ldots, t$). □

In particular, if $t = 2$, then we have the following result of Cockayne and Hedetniemi [3].

Corollary 10 *For any graph G, $d(G) + d(\overline{G}) \leq n + 1$.*

Furthermore, Theorem 9 yields the following bounds on the triple sum.

Corollary 11 *Let $n \geq 3$ be an integer and let $G_1 \oplus G_2 \oplus G_3 = K_n$. Then*

$$3 \leq d(G_1) + d(G_2) + d(G_3) \leq n + 2,$$

and these bounds are sharp.

It is shown in [3] that equality for the bound of Corollary 10 holds if and only if $G \cong K_n$ or $G \cong \overline{K}_n$. As we saw in the proof to Theorem 9, the upper bound of $n - 1 + t$ for $t \geq 3$ factors is achieved by a generalization of K_n and $\overline{K}_n$, i.e., for $G_1 = K_n$ and $G_i = \overline{K}_n$ for $2 \leq i \leq t$. However, these graphs are not the only factors achieving the upper bound for the triple product. Although the characterization of these extremal graphs remains an open problem, we conclude this subsection with another factorization that achieves the upper bound. Let $n \geq 2$ be even. Let $G_1 \cong 2K_{n/2}$ and $G_2 \cong n/2K_2$. Thus G_3 is obtainable from a complete bipartite graph $K_{n/2,n/2}$ by removing the edges of a 1-factor. Then $d(G_1) = d(G_3) = n/2$, while $d(G_2) = 2$. Thus, $d(G_1) + d(G_2) + d(G_3) = n + 2$.

3.2 The Triple Product

As before, we simplify the notation by letting $d_i = d(G_i)$, $\gamma_i = \gamma(G_i)$, $\delta_i = \delta(G_i)$, and $\Delta_i = \Delta(G_i)$ for each $i = 1, 2, 3$.

We now turn our attention to the product of the domatic numbers d_1, d_2 and d_3. Since $d(G) \geq 1$ for all graphs G, $d_1 d_2 d_3 \geq 1$ and this lower bound is readily seen to be sharp. To establish a sharp upper bound on the product is, however, less trivial.

Lemma 12 *If $\gamma_i = 1$ for some i, then $d_1 d_2 d_3 \leq n$.*

Proof. We may assume that $\gamma_1 = 1$. Then each of G_2 and G_3 contains an isolated vertex, so $d_2 = d_3 = 1$. Hence $d_1 \cdot d_2 \cdot d_3 \leq n$. □

We now consider the maximum value for the triple product for large n. We shall prove:

Theorem 13 *Let $n \geq 27$ be an odd integer with $n \notin \{29, 35, 37, 53\}$ or let $n \geq 42$ be an even integer with $n \notin \{44, 50, 52, 56\}$. If $G_1 \oplus G_2 \oplus G_3 = K_n$, then*

$$d_1 d_2 d_3 \leq \lfloor n/3 \rfloor^3,$$

and this bound is sharp.

We may assume that $\gamma_i \geq 2$ for $i = 1, 2, 3$, for otherwise the result follows from Lemma 12. Further, we may assume that $\gamma_1 \leq \gamma_2 \leq \gamma_3$. To prove Theorem 13 we shall prove a series of lemmas.

Lemma 14 *If $\gamma_i \geq 3$ for all $i = 1, 2, 3$, then*

$$d_1 d_2 d_3 \leq \lfloor n/3 \rfloor^3.$$

Proof. For each i, we know that $d_i \leq n/\gamma_i \leq n/3$, whence $d_1 d_2 d_3 \leq \lfloor n/3 \rfloor^3$. □

In view of Lemma 14, we may assume in what follows that $\gamma_1 = 2$, for otherwise $d_1 d_2 d_3 \leq \lfloor n/3 \rfloor^3$.

Lemma 15 *If n is even, then*

$$d_1 d_2 d_3 \leq \frac{n}{2} \left\lfloor \frac{n+4}{4} \right\rfloor^2.$$

Proof. Since $\gamma_1 = 2$, $d_1 \leq n/\gamma_1 = n/2$. Furthermore, $\gamma_1 = 2$ implies that $\Delta_1 \geq n/2 - 1$. Let x be a vertex of degree Δ_1 in G_1. Then $\deg_{G_2} x + \deg_{G_3} x \leq n - 1 - \Delta_1 \leq n/2$. Letting $a = \deg_{G_2} x$, we observe therefore that $d_2 \leq a + 1$ and $d_3 \leq \deg_{G_3} x + 1 \leq n/2 - a + 1$. Thus,

$$d_2 d_3 \leq \frac{1}{2}(a+1)(n - 2a + 2).$$

The second derivative of the function $(a+1)(n-2a+2)/2$ is always negative, and the function is maximized when $a^* = n/4$. Hence

$$d_2 d_3 \leq \left\lfloor \frac{n+4}{4} \right\rfloor^2.$$

Thus,

$$d_1 d_2 d_3 \leq \frac{n}{2} \left\lfloor \frac{n+4}{4} \right\rfloor^2. \qquad \square$$

Lemma 16 *If n is odd, then*

$$d_1 d_2 d_3 \leq \frac{n-1}{2} \left\lfloor \frac{n+3}{4} \right\rfloor^2.$$

Proof. Since $\gamma_1 = 2$, $d_1 \leq \lfloor n/\gamma_1 \rfloor = (n-1)/2$. Furthermore, $\gamma_1 = 2$ implies that $\Delta_1 \geq \lceil n/2 - 1 \rceil = (n-1)/2$. Let x be a vertex of degree Δ_1 in G_1. Then $\deg_{G_2} x + \deg_{G_3} x \leq n - 1 - \Delta_1 \leq (n-1)/2$. Letting $a = \deg_{G_2} x$, we observe therefore that $d_2 \leq a + 1$ and

$d_3 \leq \deg_{G_3} x + 1 \leq (n-1)/2 - a + 1$. Thus,

$$d_2 d_3 \leq \frac{1}{2}(a+1)(n-2a+1).$$

The second derivative of the function $(a+1)(n-2a+1)/2$ is always negative, and the function is maximized when $a^* = (n-1)/4$. Hence

$$d_2 d_3 \leq \left\lfloor \frac{n+3}{4} \right\rfloor^2.$$

Thus,

$$d_1 d_2 d_3 \leq \frac{n-1}{2} \left\lfloor \frac{n+3}{4} \right\rfloor^2. \qquad \square$$

Theorem 13 now follows from Lemmas 14, and 15 and 16. That the bound of $\lfloor n/3 \rfloor^3$ can be realized may be seen as follows. Partition the vertex set of K_n into three sets A, B and C as follows. If n is a multiple of 3, then each of A, B and C has cardinality $n/3$. If $n \equiv 1 \,(mod\, 3)$, then A has cardinality $\lfloor n/3 \rfloor + 1$ and each of B and C has cardinality $\lfloor n/3 \rfloor$. If $n \equiv 2 \,(mod\, 3)$, then each of A and B has cardinality $\lfloor n/3 \rfloor + 1$ and C has cardinality $\lfloor n/3 \rfloor$. Let the edges of G_1 be given by all edges between A and B and all of the edges of the (complete) graph induced by C. Let the edges of G_2 be given by all edges between B and C and all of the edges of the (complete) graph induced by A. Thus the edges of G_3 are given by all edges between A and C and all of the edges of the (complete) graph induced by B. Then for $j = 1, 2, 3$, any minimum dominating set of G_j contains a vertex from each of A, B and C. Thus, $A \cup B \cup C$ can be partitioned into $\lfloor n/3 \rfloor$ dominating sets of G_j, so $d_j = \lfloor n/3 \rfloor$. Hence $d_1 \cdot d_2 \cdot d_3 = \lfloor n/3 \rfloor^3$.

We have also investigated the problem for small values of n. The following table summarizes our findings.

n	Maximum product	Realization G_1, G_2, G_3
1	1	K_1, K_1, K_1
2	2	K_2, $\overline{K}_2$, $\overline{K}_2$
3	3	K_3, $\overline{K}_3$, $\overline{K}_3$
4	8	$2K_2$, $2K_2$, $2K_2$
5	8	$K_3 \cup K_2$, $P_3 \cup K_2$, $P_3 \cup K_2$ or
		P_5, $P_3 \cup K_2$, $P_3 \cup K_2$
6	18	$2K_3$, C_6, $3K_2$
		See discussion (a)
7	18	See discussion (b)
8	32	See discussion (a)
9	36	See discussion (c)
10	50	See discussion (a)
11	50	See discussion (b)
12	75	See discussion (d)

Table 1. Optimal values of the triple product for small n.

In some cases these realizations may be obtained via general constructions which we now describe.

(a) For $n \geq 4$ with n even, partition the vertices of K_n into two sets A (B) each of cardinality $n/2$. Let the edges of G_1 be given by all the edges of the (complete) graph induced by A and all the edges of the (complete) graph induced by B. Let the edges of G_3 consist of those edges between A and B that induce a perfect matching. Hence, the edges of G_2 are given by those edges between A and B that do not belong to G_3. Then $d_1 = d_2 = n/2$ and $d_3 = 2$, whence $d_1 d_2 d_3 \leq n^2/2$.

(b) For $n \geq 5$ with n odd, partition the vertices of K_n into two sets A (B) with A having cardinality $(n+1)/2$ and B having cardinality $(n-1)/2$. Let the edges of G_1 be given by all the edges of the (complete) graph induced by A and all the edges of the (complete) graph induced by B. Let the edges of G_3 consist of those edges between A and B that induce the graph $(n-3)/2K_2 \cup P_3$. Hence, the edges of G_2 are given by those edges between A and B that do not belong to G_3. Then $d_1 = d_2 = (n-1)/2$ and $d_3 = 2$, whence $d_1 d_2 d_3 \leq (n-1)^2/2$.

(c) For $n = 9$, let G_1 be the circulant $C_9\langle 1, 2\rangle$ with jump sequence $\{1, 2\}$; that is, if we label the vertices of G_1 by $v_0, v_1, \ldots, v_8$, then $v_i v_j \in E(G_1)$ if and only if $i - j \equiv 1 \,(mod\,9)$ or $i - j \equiv 2 \,(mod\,9)$. Let G_2 be the circulant $C_9\langle 3\rangle$ (so $v_i v_j \in E(G_2)$ if and only if $i - j \equiv 3 \,(mod\,9)$). Thus $G_2 \cong 3K_3$. Finally, let G_3 be the circulant $C_9\langle 4\rangle$. Thus $G_3 \cong C_9$. Then each of G_2 and G_3 has domatic number 3, while G_1 has domatic number $(n-1)/2$ (for example, $\{v_0, v_4\}$, $\{v_1, v_5\}$, $\{v_2, v_6\}$, $\{v_3, v_7, v_8\}$ is a domatic partition of G_1 of cardinality 4). Hence $d_1 d_2 d_3 = 4 \cdot 3 \cdot 3 = 36 = (n-1)(n-3)^2/8$.

(d) For $n = 12$, partition the vertex set of K_{12} into four sets $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2, b_3\}$, $C = \{c_1, c_2, c_3\}$ and $D = \{d_1, d_2, d_3\}$. Let the edges of G_1 be given by the edges of the complete graph on A, the edges of the complete graph on C, the edges $\{b_i d_i \,|\, i = 1,2,3\}$, all edges between B and $\{c_1, c_2, a_3\}$, and all edges between D and $\{a_1, a_2, c_3\}$. Let the edges of G_2 be given by the edges of the complete graph on B, the edges of the complete graph on D, the edges $\{a_i c_i \,|\, i = 1,2,3\}$, all edges between D and $\{c_1, c_2, a_3\}$, and all edges between C and $\{a_1, a_2, c_3\}$. Finally, G_3 consists of the two 6-cycles $b_1, d_2, b_3, d_1, b_2, d_3, b_1$ and $a_1, c_2, a_3, c_1, a_2, c_3, b_1$. Then $d_3 = 3$. Furthermore, $d_1 = d_2 = n/2 - 1 = 5$ (for example, $\{a_1, c_1\}$, $\{a_2, c_2\}$, $\{a_3, c_3\}$, $\{b_1, b_2, d_3\}$, $\{d_1, d_2, b_3\}$ is a domatic partition of G_1 of cardinality 5). Hence $d_1 d_2 d_3 = 5 \cdot 5 \cdot 3 = 75 = 3 \cdot (n-2)^2/4$.

It is a simple exercise to characterize the extremal factors for $1 \leq n \leq 6$. We note that the upper bounds for $1 \leq n \leq 6$ are achieved if and only if we have the realisation given in Table 1.

We close with the following.

Conjecture 1 *Let $n \geq 15$ be an integer with $n \notin \{16, 17, 20\}$, and let $G_1 \oplus G_2 \oplus G_3 = K_n$. Then*

$$d_1 d_2 d_3 \leq \lfloor n/3 \rfloor^3 .$$

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