On Graphs with Strong α-valuations

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Abstract

The concept of a strong α-valuation was introduced by Maheo, who showed that if a graph G has a strong α-valuation, then so does $G \times K_2$. We show that for various graphs G, $G \times Q_n$ has a strong α-valuation and $G \times P_n$ has an α-valuation, where Q_n is the n-cube and P_n the path with n edges, including $G = K_{m,2}$ for any m. Yet we show that $K_{m,n} \times K_2$ does not have a strong α-valuation if m and n are distinct odd integers.

1 Introduction

Only graphs without loops and multiple edges will be considered in this paper. If m and n are integers we denote $\{m, m+1, \ldots, n\}$ by $[m,n]$. Let N denote the set of nonnegative integers. The *cartesian product* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \times G_2 = (V, E)$ where $V = V_1 \times V_2$ and $E = \{\{(u_1, u_2), (v_1, v_2)\} : u_1 = v_1$ and $\{u_2, v_2\} \in E_2$ or $u_2 = v_2$ and $\{u_1, u_2\} \in E_1\}$.

For any graph G we call an injective function $\gamma : V(G) \to N$ a *valuation* of G. Rosa [11] called such a function γ on a graph G with q edges a *β-valuation* if γ is an injection from $V(G)$ into $[0,q]$ such that

$$\{|\gamma(u) - \gamma(v)| : \{u,v\} \in E(G)\} = [1,q].$$

The number $|\gamma(u) - \gamma(v)|$ is called the *label* of the edge $\{u,v\}$. A β-valuation is now more commonly called a *graceful valuation*. An *α-valuation* is a graceful valuation having the additional property that there exists an integer λ such that if $\{u,v\} \in E(G)$, then $\{u,v\} = \{a,b\}$, where $\gamma(a) \leq \lambda < \gamma(b)$. The number λ, which is unique, is called the *critical*

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value of the α-valuation. Note that if G admits an α-valuation then G is bipartite with parts A and B, where $A = \{u \in V(G) : \gamma(u) \leq \lambda\}$, and $B = \{u \in V(G) : \gamma(u) > \lambda\}$. We denote the set A by $S(\gamma)$ (the "small" vertices). Valuations of graphs are particularly interesting because of their applications to graph decompositions (see Rosa [11]).

A graph G admitting an α-valuation was defined by Maheo [10] to be "strongly graceful" if in addition to all of the above there exists an automorphism π of G taking A onto B and the following conditions are satisfied.

(1) $|A| = |B| = s$ and $|E| = 2l + s$ for some integers $l \geq 0$ and $s \geq 0$.

(2) We have $\lambda \in [l - s, l + s - 1]$.

(3) If $a \in A$, then $\{a, \pi(a)\}$ is an edge with label in $[l + 1, l + s]$.

(4) The map π is its own inverse.

Note that since $|A| = s$, condition (3) implies that the labels on the edges $\{a, \pi(a)\}$ for $a \in A$ are exactly the integers of $[l + 1, l + s]$. The left side of Figure 1 shows a strong α-valuation of $K_{2,3} \times K_2$ (note that $K_{2,3}$ has $s = 5$ vertices and $l = 6$ edges).

In fact condition (2) is redundant. For any graceful graph has an edge with label 1, so there exist vertices $a \in A$ and $b \in B$ such that $\gamma(a) = \lambda$ and $\gamma(b) = \lambda + 1$. Then the label on the edge $\{a, \pi(a)\}$ is $\gamma(\pi(a)) - \lambda$, which is in $[\lambda + 1, 2l + s] - \lambda = [1, 2l + s - \lambda]$. By (3) this label is in $[l + 1, l + s]$. Thus $2l + s - \lambda \geq l + 1$, which implies that $\lambda \leq l + s - 1$. Likewise the label on the edge $\{b, \pi^{-1}(b)\}$ is in $\lambda + 1 - [0, \lambda] = [1, \lambda + 1]$. By (3) this label is in $[l + 1, l + s]$. Thus $\lambda + 1 \geq l + 1$ and so $\lambda \geq l \geq l - s$.

We prefer to call the γ in a strongly graceful graph a *strong α-valuation* since this draws attention to the fact that G admits an α-valuation rather than merely a graceful (or β-) valuation.

In [10], Maheo proved the following powerful and in our opinion under-appreciated result.

Theorem 1 *If G has a strong α-valuation, then so does $G \times K_2$.*

Moreover, Maheo showed that the n-cube Q_n (the Cartesian product of n copies of K_2), the ladder L_n (the graph $P_n \times K_2$, where P_n is the path with n edges), and the book B_{2n} (the graph $K_{1,2n} \times K_2$) all have strong α-valuations.

Numerous researchers have investigated valuations of graphs (see Gallian [5] and [6] for general surveys). Several have investigated valuations of products of graphs (see [1]–[10],[12],[13],[14] for some examples). Unfortunately, many authors did not distinguish between β and α-valuations and few appear to be aware of Maheo's result. Thus rather than showing

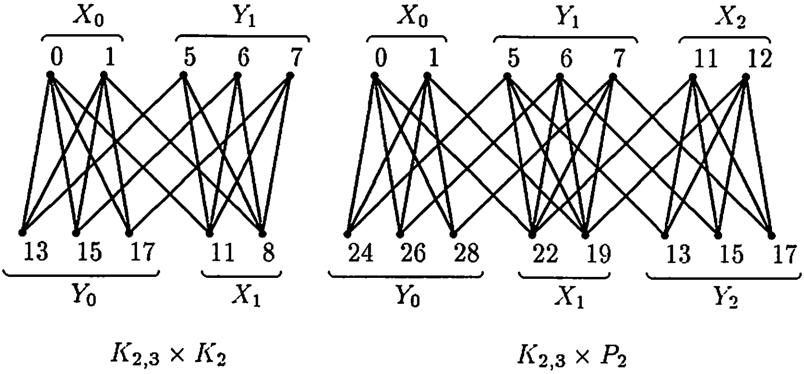


Figure 1: $K_{2,3}$ is prestrong

that a certain graph G has a strong α-valuation and applying Theorem 1, they would show directly that $G \times Q_n$ admits an α-valuation or is simply graceful.

There are some graphs G such that $G \times K_2$ has a strong α-valuation enjoying certain additional properties. Let G be a bipartite graph with vertex partition X, Y. Suppose that G has s vertices and l edges, and set $t = s + l$. Let $X_i = \{(x,i) : x \in X\}$ and $Y_i = \{(y,i) : y \in Y\}$, $i = 0, 1$. We can consider $G \times K_2$ to be the bipartite graph with vertex partition $A = X_0 \bigcup Y_1$, $B = Y_0 \bigcup X_1$ and edges $\{(x,i),(y,i)\}$, $i = 0, 1$, where $\{x, y\}$ runs through the edges of G, along with all edges $\{(v,0),(v,1)\}$ for $v \in V(G)$. We say the graph G is *prestrong* if $G \times K_2$ has a strong α-valuation γ with $S(\gamma) = A$ such that the map π in the definition of a strong α-valuation interchanges $(v,0)$ and $(v,1)$ for all $v \in V(G)$, and such that if we define γ^* on $X_0 \bigcup Y_0$ to be γ on X_0 and $\gamma - t$ on Y_0, then γ^* is an α-valuation on the subgraph of $G \times K_2$ induced by $X_0 \bigcup Y_0$ with $S(\gamma^*) = X_0$. (This implies that G has an α-valuation.)

Note that if $t = 13$ is subtracted from the values 19, 18, 15, and 17 of the vertices in the graph $G \times K_2$ in Figure 2, this does *not* give an α-valuation on the left-hand copy of G, since two vertices get the value 2. In fact the particular graph G shown is not prestrong, since it is known that G has no α-valuation. On the other hand the left side of Figure 1 shows that $K_{2,3}$ is prestrong. Here subtracting $t = 11$ from the values 13, 15, and 17 gives an α-valuation of the left-hand copy of $K_{2,3}$.

We will show that if G is any graph with one more vertex than edges, and if G has an α-valuation such that the cardinalities of the sets of the corresponding bipartition of the vertices differ by at most 1, then G is prestrong (and so by Maheo's result $G \times Q_d$ has a strong α-valuation for

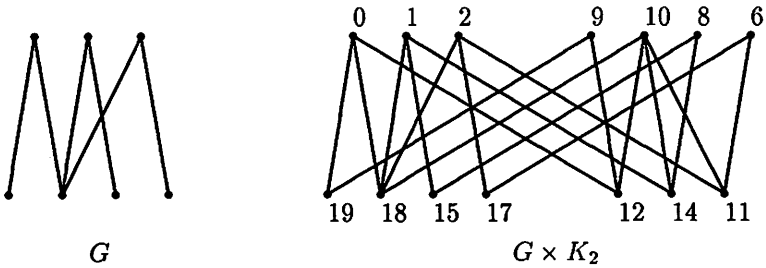


Figure 2: G does not have an α-valuation

any $d \geq 1$). We also show that if G is prestrong, then $G \times P_n$ has an α-valuation for all positive integers n. We give applications of these results. We also show that $K_{3,4}$ is prestrong, as is $K_{m,2}$ for any positive integer m. On the other hand, we show that if m and n are distinct odd integers, then $K_{m,n} \times K_2$ does not have a strong α-valuation.

Before we proceed we note that K_2, P_n and $K_{1,2n}$ are all prestrong (see Maheo's [10] strong α-valuation of Q_2, L_n and S_{2n}, respectively). Moreover, the α-labeling of $C_{4k} \times K_2$ given by Snevily [13] shows that C_{4k} is prestrong and $C_{4k} \times K_2$ has a strong α-valuation.

2 New Results

Lemma 1 *Let the graph G be prestrong, with notation as above, and let the valuation γ^* have the critical value λ^*. Then $\gamma(X_0) \subseteq [0, \lambda^*]$, $\gamma(Y_0) \subseteq [\lambda^* + 1 + t, l + t]$, $\gamma(Y_1) \subseteq [\lambda^* + 1, t - 1]$ and $\gamma(X_1) \subseteq [l + 1, \lambda^* + t]$.*

Proof. The first two containments follow from the fact that γ^* is an α-valuation with critical value λ^*. Now let $y_1 \in Y_1$. Since γ is a strong α-valuation we have $\gamma(y_0) - \gamma(y_1) \in [l + 1, l + s]$, where $y_0 = \pi(y_1) \in Y_0$. This means that $\gamma(y_0) - l - s \leq \gamma(y_1) \leq \gamma(y_0) - l - 1$. But since $\gamma(Y_0) \subseteq [\lambda^* + 1 + t, l + t]$, we have $\lambda^* + 1 = \lambda^* + 1 + t - l - s \leq \gamma(y_1) \leq l + t - l - 1 = t - 1$.

In the same way if $x_1 \in X_1$, then we have $\gamma(x_1) - \gamma(x_0) \in [l + 1, l + s]$, where $x_0 = \pi(x_1) \in X_0$. Then $\gamma(x_0) + l + 1 \leq \gamma(x_1) \leq \gamma(x_0) + l + s$. But $\gamma(X_0) \subseteq [0, \lambda^*]$, so $l + 1 \leq \gamma(x_1) \leq \lambda^* + l + s = \lambda^* + t$. ∎

Theorem 2 *Suppose the graph G is prestrong. Then $G \times P_n$ has an α-valuation for all positive integers n.*

Proof. We take the vertices of $G \times P_n$ to be of the form (v, k), where $v \in V(G)$ and $0 \leq k \leq n$. The edges of $G \times P_n$ are all edges $\{(x, k), (y, k)\}$, where $\{x, y\}$ is an edge of G and $0 \leq k \leq n$, along with the edges $\{(v, k), (v, k+1)\}$, where $v \in V(G)$ and $0 \leq k \leq n-1$. We extend the notation of the definition of a prestrong graph, so that $X_i = \{(x,i) : x \in X\}$ and $Y_i = \{(y,i) : y \in Y\}$, $0 \le i \le n$. Define A_i to be X_i if i is even and Y_i if i is odd. Likewise define B_i to be Y_i if i is even and X_i if i is odd. Then $G \times P_n$ is bipartite with vertex bipartition A, B, where $A = \bigcup_{i=0}^{n} A_i$ and $B = \bigcup_{i=0}^{n} B_i$.

Let γ be the valuation on the subgraph of $G \times P_n$ induced by $X_0 \bigcup Y_0 \bigcup X_1 \bigcup Y_1$ given in the definition of a prestrong graph. We define a valuation δ on $G \times P_n$ as follows. For $0 \le k \le n$ write $k = 2q + r$, where q and r are integers and $0 \le r < 2$. If $(a,k) \in A$ let $\delta(a,k) = \gamma(a,r) + qt$, while if $(b,k) \in B$ let $\delta(b,k) = \gamma(b,r) + (n-q-1)t$. We claim that δ is an α-valuation.

We start by showing that δ is one-to-one. First assume that k is even, so that $r = 0$. Then $\delta(A_k) = \gamma(X_0) + qt \subseteq [0, \lambda^*] + qt$ by the lemma. But $\delta(A_{k+1}) = \gamma(Y_1) + qt \subseteq [\lambda^* + 1, t-1] + qt$, and the elements of this interval are all larger than those of the first. Likewise $\delta(B_k) = \gamma(Y_0) + (n-q-1)t \subseteq [\lambda^* + 1 + t, l + t] + (n-q-1)t$, while $\delta(B_{k+1}) = \gamma(X_1) + (n-q-1)t \subseteq [l+1, \lambda^* + t] + (n-q-1)t$, and the elements of this interval are all smaller than those of the first.

Now suppose k is odd, so $r = 1$. Then $\delta(A_k) = \gamma(Y_1) + qt \subseteq [\lambda^* + 1, t - 1] + qt$, while $\delta(A_{k+1}) = \gamma(X_0) + (q+1)t \subseteq [0, \lambda^*] + (q+1)t$. Since the largest value in the first set is less that the smallest in the second, the sets are disjoint. Also $\delta(B_k) = \gamma(X_1) + (n-q-1)t \subseteq [l+1, \lambda^* + t] + (n-q-1)t$, while $\delta(B_{k+1}) = \gamma(Y_0) + (n-q-2)t \subseteq [\lambda^* + 1 + t, l + t] + (n-q-2)t$, and the largest element of this interval is less than the smallest of the first.

Since we have seen that as a function of k, $\delta(A_k)$ increases while $\delta(B_k)$ decreases, it only remains to show that every element of $\delta(A_n)$ is less than every element of $\delta(B_n)$. Again we need cases. Let $n = 2Q + R, 0 \le R < 2$. If n is even we have $\delta(A_n) = \gamma(X_0) + Qt \subseteq [0, \lambda^*] + Qt = [Qt, \lambda^* + Qt]$, and $\delta(B_n) = \gamma(Y_0) + (n-Q-1)t \subseteq [\lambda^*+1+t, l+t] + (Q-1)t = [\lambda^* + Qt + 1, l + Qt]$, so in this case we see that our new valuation δ has critical value $\lambda^* + Qt$. If n is odd the argument is somewhat different. In this case $n = 2Q + 1$, so $(n - Q - 1)t = Qt$. Thus the values of δ on $A_n \bigcup B_n$ are just Qt more than the values of γ on $Y_1 \bigcup X_1$. By the assumptions on γ the γ-edge labels on $X_1 \bigcup Y_1$ are exactly $[1, l]$. We see the critical value for δ is Qt more than the critical value of the α-valuation γ.

Now we look at edge labels. Note that from the definition of a prestrong graph, the edges between X_0 and Y_0 have γ-labels $[t+1, t+l]$, the edges between X_1 and Y_1 have γ-labels $[1, l]$, and the edges $\{(v,0),(v,1)\}$ have γ-labels $[l+1, t]$.

Now we compute the edge labels with respect to δ. First assume $k = 2q$ is even. Then the δ-labels on the edges of $A_k \bigcup B_k$ are the γ-labels on the edges of $X_0 \bigcup Y_0$ plus $(n-q-1)t - qt$, that is, $[t+1, t+l] + (n-2q-1)t = [1, l] + (n-k)t$. But if $k = 2q+1$ is odd, then the δ-labels of the edges of $A_k \bigcup B_k$ are the γ-labels on the edges of $X_1 \bigcup Y_1$ plus $(n-q-1)t - qt$, that is, $[1,l] + (n-2q-1)t = [1,l] + (n-k)t$ also.

Now we compute the δ-labels on edges between X_k and X_{k+1} and between Y_k and Y_{k+1}. If $k = 2q$ these are the γ-labels on $X_0 \bigcup X_1$ plus $(n-q-1)t - qt$, along with the γ-labels on $Y_0 \bigcup Y_1$ plus $(n-q-1)t - qt$. Together, these give $[l+1,t]+(n-2q-1)t = [l+1,t]+(n-k-1)t$. Likewise if $k = 2q+1$ these are the γ-labels on $X_0 \bigcup X_1$ plus $(n-(q+1)-1)t - qt$, along with the γ-labels on $Y_0 \bigcup Y_1$ plus $(n-q-1)t - (q+1)t$. Together these give $[l+1,t] + (n-2q-2)t = [l+1,t] + (n-k-1)t$ also.

Now if $0 \le k < n$ the δ-labels on edges from X_k to X_{k+1} and from Y_k to Y_{k+1} together with those on edges from X_k to Y_k are $([l+1,t] + (n-k-1)t) \bigcup ([1,l] + (n-k)t) = [(n-k-1)t + l + 1, (n-k)t] \bigcup [(n-k)t + 1, (n-k)t + l] = [(n-k-1)t + l + 1, (n-k)t + l]$. The union of these sets (working backwards from $k = n-1$) is

$$[l+1, l+t] \bigcup [l+t+1, l+2t] \bigcup \cdots \bigcup [l+(n-1)t+1, l+nt] = [l+1, l+nt].$$

Since the δ-labels of the edges from X_n to Y_n are $[1,l]$, δ is an α-valuation on $G \times P_n$. ∎

Lemma 2 *Let G_0 and G_1 be vertex disjoint graphs, each with n edges and $n+1$ vertices. Suppose G_i has an α-valuation γ_i with critical value $\lambda = \lfloor (n-1)/2 \rfloor$, $i = 0, 1$. Then the graph G^* formed by adding the edges $\{\gamma_0^{-1}(j), \gamma_1^{-1}(j)\}$, $0 \le j \le n$ to $G_0 \bigcup G_1$ has an α-valuation, and this is strong if $\gamma_1 \gamma_0^{-1}$ is an isomorphism.*

Proof. Let $V(G_0) = \{y_0, y_1, \ldots, y_n\}$ and $V(G_1) = \{z_0, z_1, \ldots, z_n\}$. Assume $\gamma_0(y_j) = \gamma_1(z_j) = j$, $0 \le j \le n$. We define a valuation δ on G^* by

$$\begin{aligned}
\delta(y_i) &= i, 0 \le i \le \lambda, \\
\delta(y_i) &= 2n + 1 + i, \lambda < i \le n, \\
\delta(z_i) &= 2n - i, 0 \le i \le n.
\end{aligned}$$

Note that $\{\delta(y_i) : 0 \le i \le \lambda\} = [0, \lambda]$, $\{\delta(z_i) : 0 \le i \le n\} = [n, 2n]$, and $\{\delta(y_i) : \lambda < i \le n\} = [2n + \lambda + 2, 3n + 1]$. Thus δ is one-to-one on $V(G^*)$ and has critical value $\lambda^* = 2n - \lambda - 1 = \delta(z_{\lambda+1})$, since $\delta(z_\lambda) - \delta(z_{\lambda+1}) = 1$.

Now $|\delta(z_i) - \delta(z_j)| = |i - j| = |\gamma_1(i) - \gamma_1(j)|$, and so the edges $\{z_i, z_j\}$ of G^* get the values $1, 2, \ldots, n$. Likewise if $i \le \lambda < j$, then $|\delta(y_i) - \delta(y_j)| = |2n + 1 + j - i| = 2n + 1 + |\gamma_0(i) - \gamma_0(j)|$, and so the edges $\{y_i, y_j\}$ of G^* get the values $2n+2, 2n+3, \ldots, 3n+1$.

Finally for $0 \le i \le \lambda$ the set of values of $|\delta(y_i) - \delta(z_i)| = 2(n-i)$ is

$$S = \{2(n-\lambda), 2(n-\lambda) + 2, \ldots, 2n\},$$

while for $\lambda < i \leq n$ the set of values of $|\delta(y_i) - \delta(z_i)| = 2i + 1$ is

$$T = \{2(\lambda+1)+1, 2(\lambda+1)+3, \ldots, 2n+1\}.$$

Recall that $\lambda = \lfloor (n-1)/2 \rfloor$. If n is odd, then the smallest numbers in S and T are $n+1$ and $n+2$, respectively, while if n is even this is reversed. In either case the values of $|\delta(y_i) - \delta(z_i)|$ are $[n+1, 2n+1]$, and so δ is an α-valuation.

Now with respect to δ we have $A = \{y_0, y_1, \ldots, y_\lambda, z_{\lambda+1}, z_{\lambda+2}, \ldots, z_n\}$ and $B = \{z_0, z_1, \ldots, z_\lambda, y_{\lambda+1}, y_{\lambda+2}, \ldots, y_n\}$. The graph G^* has $2n+n+1 = 3n+1$ edges. Thus if we set $s = n+1$ and $l = n$, then $|A| = |B| = s$ and $|E(G^*)| = 2l+s$. We define π to be $\gamma_1\gamma_0^{-1}$ on $V(G_0)$ and $\gamma_0\gamma_1^{-1}$ on G_1. Then π is a self-inverse map interchanging A and B, and the labels on the edges $\{y_i, \pi(y_i)\} = \{y_i, z_i\}$ are exactly the elements of $[n+1, 2n+1] = [l+1, l+s]$. Thus if $\gamma_1\gamma_0^{-1}$ is an isomorphism, then π is an automorphism and δ is a strong α-valuation. ∎

We call a bipartite graph *equitable* if in any partition of its vertices into two sets with no edges between the sets, the cardinalities of the two sets differ by at most one.

Theorem 3 *Let G be an equitable graph with n edges and $n+1$ vertices having an α-valuation. Then G is prestrong.*

Proof. Let G have the α-valuation γ with critical value λ. Since γ maps $V(G)$ onto $\{0, 1, \ldots, n\}$, it induces a bipartition of the vertices into sets with $\lambda+1$ and $n-\lambda$ vertices. Now $n - \gamma$ is an α-valuation interchanging A and B, so G has an α-valuation with $\lambda + 1 \leq n - \lambda$, or $\lambda \leq (n-1)/2$. Thus if the cardinalities of the sets in the bipartition differ by at most 1, then we can assume $\lambda = \lfloor (n-1)/2 \rfloor$.

Now we can think of $G \times K_2$ as the union of vertex-disjoint copies G_0 and G_1 of G, along with all edges $\{y, \pi(y)\}$, where π is an isomorphism of G_0 onto G_1 preserving the α-valuation. Then Lemma 1 gives a strong α-valuation δ of $G \times K_2$. Notice that if we define δ^* to be δ on $\{y_0, \ldots, y_\lambda\}$ and $\delta - (2n+1)$ on $\{y_{\lambda+1}, \ldots, y_n\}$, then δ^* is an α-valuation on G_0, and so G is prestrong. ∎

It is proved in [4] that if T is an equitable tree that has an α-valuation, then $T \times P_n$ has an α-valuation for all n. By using Theorems 1, 2 and 3 we get a generalization of this result.

Corollary 4 *If G is an equitable graph with one more vertex than edges and if G has an α-valuation, then $G \times Q_n$ has a strong α-valuation and $G \times P_n$ has an α-valuation for each positive integer n.*

3 Applications

If G is a graph, let G^- denote the graph formed by removing from G all vertices of degree 1, along with the edges incident with these vertices. In [11] Rosa calls any graph such that G^- is a path a *caterpillar*, and proves that each caterpillar has an α-valuation. Each caterpillar is a tree, and so is bipartite and has one more vertex than edges. Thus Corollary 4 applies to caterpillars.

It is possible for $G \times K_2$ to have a strong α-valuation, even though G itself does not even have an α-valuation. An example is shown in Figure 2.

Now we will consider graphs that are the vertex-disjoint unions of even cycles and equitable caterpillars. Let $P(v_0, \ldots, v_n)$ denote the path with edges $\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{n-1}, v_n\}$. An examination of Rosa's construction [11] of an α-valuation for caterpillars shows that the following statement holds.

Theorem 5 *Let G be a caterpillar with $G^- = P(v_0, \ldots, v_n)$. Then G has an α-valuation γ such that $\gamma(v_0) = 0$ and the edge with label 1 joins v_n and a vertex with degree 1.*

In [11], Rosa also shows that the cycle C_n has an α-valuation if and only if $n = 4k$ for some integer k. Since C_n has the same number of vertices as edges, in any graceful valuation γ exactly one integer in $[0, n]$ corresponds to no vertex. In the proof Rosa gives for C_{4k} this is $3k$; using the valuation $4k - \gamma$ instead would make it k. We denote the vertex-disjoint union of graphs G and H by $G \sqcup H$.

Theorem 6 *Let G be a caterpillar with $G^- = P(v_0, \ldots, v_n)$, where $n > 0$. Let $k + 1$ be the number of vertices of G an even distance from v_n. Then $C_{4k} \sqcup G$ has an α-valuation.*

Proof. By the previous theorem G has an α-valuation γ with critical value λ such that $\gamma(v_0) = 0$ and the edge with label 1 joins v_n and a vertex u of degree 1. Let q be the number of vertices of G.

First we treat the case when n is even. Then $\gamma(v_n) = \lambda = k$ and $\gamma(u) = k + 1$. We know that C_{4k} has an α-valuation δ such that for no vertex v is $\delta(v) = 3k$. The critical value for this valuation is $2k - 1$.

Now we define a valuation β on $C_{4k} \sqcup G$ as follows.

condition on v	$\beta(v)$	range
$\delta(v) \leq 2k - 1$	$\delta(v)$	$[0, 2k - 1]$
$\gamma(v) \leq k$	$\gamma(v) + 2k$	$[2k, 3k]$
$v = u$	$3k + q - 1$	$\{3k + q - 1\}$
$\gamma(v) > k + 1$	$\gamma(v) + 2k - 1$	$[3k + 1, 2k + q - 2]$
$\delta(v) \geq 2k$	$\delta(v) + q - 1$	$[2k + q - 1, 4k + q - 1] \setminus \{3k + q - 1\}$

Notice that β is one-to-one on $V(C_{4k} \sqcup G)$. The edge labels with respect to β are given in the following table.

edges	range
$E(C_{4k})$	$[1,4k]+q-1=[q,4k+q-1]$
$E(G)\backslash\{\{v_n,u\}\}$	$[2,q-1]-1=[1,q-2]$
$\{v_n,u\}$	$3k+q-1-(k+2k)=q-1$

Since C_{4k} has $4k+q-1$ edges, we see that β is an α-valuation for this graph, with critical value $\beta(v_n)=3k$.

Now suppose n is odd. Then $\gamma(u)=\lambda$, $\gamma(v_n)=\lambda+1$, and $k+1=q-(\lambda+1)$. Let ρ be an α-valuation for C_{4k} such that for no vertex v is $\rho(v)=k$. The critical value for this valuation is $2k$.

We define a valuation τ on $C_{4k} \sqcup G$ as follows.

condition on v	$\tau(v)$	range
$\rho(v)\leq 2k$	$\rho(v)$	$[0,2k]\backslash\{k\}$
$\gamma(v)<\lambda$	$\gamma(v)+2k+1$	$[2k+1,\lambda+2k]$
$v=u$	k	$\{k\}$
$\gamma(v)>\lambda$	$\gamma(v)+2k$	$[\lambda+1+2k,q-1+2k]$
$\rho(v)>2k$	$\rho(v)+q-1$	$[2k+q,4k+q-1]$

Notice that τ is one-to-one on $V(C_{4k} \sqcup G)$. The edge labels with respect to τ are given in the following table.

edges	range
$E(C_{4k})$	$[1,4k]+q-1=[q,4k+q-1]$
$E(G)\backslash\{\{v_n,u\}\}$	$[2,q-1]-1=[1,q-2]$
$\{v_n,u\}$	$\lambda+1+2k-k=\lambda+1+k=q-1$

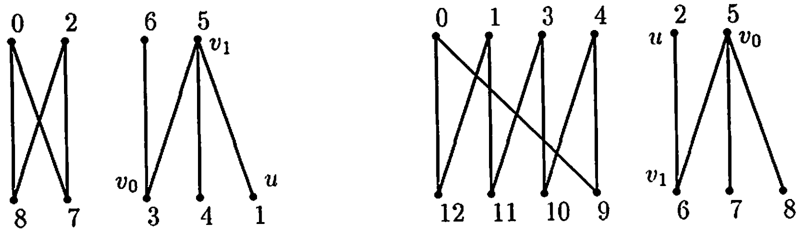
Since C_{4k} has $4k+q-1$ edges, we see that τ is an α-valuation for this graph, with critical value $\lambda+2k=\tau(v_n)-1$. ∎

Note that if n is odd, then reversing the way we number the path $P(v_0,\ldots,v_n)$ may change k, thus producing two different cycles whose vertex-disjoint union with G has an α-valuation. An example is shown in Figure 3.

Applying Theorems 1, 2, and 3 to the last result yields the following.

Theorem 7 *Let G be an equitable caterpillar so that $|V(G^-)|>1$, and let k be defined as in the previous theorem. Then $(C_{4k}\sqcup G)\times Q_m$ has a strong α-valuation and $(C_{4k}\sqcup G)\times P_m$ has an α-valuation for all positive integers m.*

The following shows that Theorem 6 does not exhaust the ways of getting graphs of the form $C_{4k}\sqcup G$ with strong α-valuations, where G is an equitable caterpiller.



$C_4 \bigsqcup G$ $\qquad$ $C_8 \bigsqcup G$

Figure 3: α-valuations of $C_4 \bigsqcup G$ and $C_8 \bigsqcup G$

Theorem 8 *Let k be a positive integer. Then there exist infinitely-many equitable caterpillars G such that $C_{4k} \sqcup G$ has an α-valuation.*

Proof. We know that C_{4k} has an α-valuation δ with critical value $2k$ such that $\delta(v) = k$ for no vertex v.

Let r be any integer such that $2r \geq k+1$. Let G be the caterpillar with vertices $u_1, u_2, \ldots, u_{4r}$ and edges $\{u_{2r-1}, u_j\}$ for $2r+1 \leq j \leq 4r$, edges $\{u_{4r}, u_j\}$ for $1 \leq j \leq 2r-2$, and the edge $\{u_{2r}, u_{2r+1}\}$. Note that G has $4r-1$ edges.

Now we define a valuation γ on $C_{4k} \bigcup G$ as follows. On C_{4k} we define γ to be δ on vertices v with $\delta(v) \leq 2k$, and to be $\delta + 4r - 1$ on the other vertices of C_{4k}. Notice that the set of values of γ on the vertices of C_{4k} is $[0, k-1] \bigcup [k+1, 2k] \bigcup [2k+4r, 4k+4r-1]$.

We define γ on G according to the following table.

condition on j	$\gamma(u_j)$	**range**
$1 \leq j \leq 2r-1$	$2k+j$	$[2k+1, 2k+2r-1]$
$j = 2r$	k	$\{k\}$
$2r+1 \leq j \leq 4r-k$	$k+6r-j$	$[2k+2r, k+4r-1]$
$4r-k+1 \leq j \leq 4r$	$2k-1+j$	$[k+4r, 2k+4r-1]$

Notice that γ is one-to-one on $V(C_{4k} \sqcup G)$. The edge labels with respect to γ are given in the following table.

edges	**label**	**range**
$E(C_{4k})$		$[4r, 4k+4r-1]$
$\{u_j, u_{4r}\}, 1 \leq j \leq 2r-2$	$4r-j-1$	$[2r+1, 4r-2]$
$\{u_{2r-1}, u_j\}, 2r+1 \leq j \leq 4r-k$	$4r-k+1-j$	$[1, 2r-k]$
$\{u_{2r-1}, u_j\}, 4r-k+1 \leq j \leq 4r$	$j-2r$	$[2r-k+1, 2r]$
$\{u_{2r}, u_{2r+1}\}$	$4r-1$	$\{4r-1\}$

From the table we see that the set of edge labels is $[1, 4k+4r-1]$. ∎

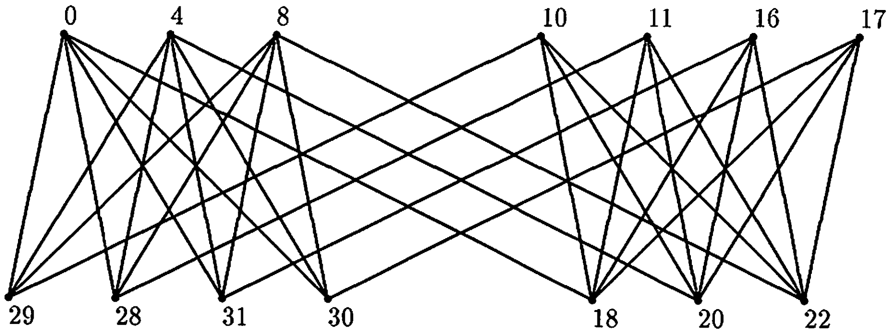


Figure 4: $K_{3,4} \times K_2$ is prestrong

4 Strong α-valuations of $K_{m,n} \times K_2$

In [2] induction proofs are given that $K_{3,3} \times Q_n$ and $K_{4,4} \times Q_n$ have α-valuations for all positive integers n. By using Maheo's Theorem 1 it would also suffice to show that $K_{3,3} \times K_2$ and $K_{4,4} \times K_2$ have strong α-valuations, which is the case. We show that $K_{3,4}$ is prestrong in Figure 4; it follows that $K_{3,4} \times Q_n$ has a strong α-valuation and $K_{3,4} \times P_n$ has an α-valuation for all positive integers n.

Theorem 9 *Let m be a positive integer. Then $K_{m,2}$ is prestrong, and so $K_{m,2} \times Q_n$ has a strong α-valuation and $K_{m,2} \times P_n$ has an α-valuation for all positive integers n.*

Proof. Let $G = K_{m,2} \times K_2$ have vertices $y_0, y_1, \ldots, y_{m+1}, z_0, z_1, \ldots, z_{m+1}$, with edges $\{y_i, y_j\}$ and $\{z_i, z_j\}$ for $i = 1, m+1$, and $1 \leq j \leq m$, and $\{y_i, z_i\}$ for $0 \leq i \leq m+1$. Our bipartition will be into $A = \{y_0, z_1, z_2, \ldots, z_m, y_{m+1}\}$ and $B = \{z_0, y_1, y_2, \ldots, y_m, z_{m+1}\}$. We define π on $V(G)$ by $\pi(y_i) = z_i$ and $\pi(z_i) = y_i$ for $0 \leq i \leq m+1$. (See Figure 1)

We define a valuation γ on $V(G)$ as in the following table.

vertex v	$\gamma(v)$	**range**
y_{m+1}	0	$\{0\}$
y_0	m	$\{m\}$
$z_i, 1 \leq i \leq m$	$3m - 2i + 2$	$[m+2, 3m]$
z_0	$3m+1$	$\{3m+1\}$
z_{m+1}	$3m+2$	$\{3m+2\}$
$y_i, 1 \leq i \leq m$	$5m+3-i$	$[4m+3, 5m+2]$

It is clear from the table that γ is one-to-one and that $v \in A$ if and only if $\gamma(v) \leq 3m$. Notice that G has $2(2m) + m + 2 = 5m + 2$ edges.

Now we compute the edge labels.

edge	label	range
$\{z_0, z_i\}, 1 \le i \le m$	$2i-1$	$\{1, 3, \ldots, 2m-1\}$
$\{z_{m+1}, z_i\}, 1 \le i \le m$	$2i$	$\{2, 4, \ldots, 2m\}$
$\{y_0, z_0\}$	$2m+1$	$\{2m+1\}$
$\{y_i, z_i\}, 1 \le i \le m$	$2m+i+1$	$[2m+2, 3m+1]$
$\{y_{m+1}, z_{m+1}\}$	$3m+2$	$\{3m+2\}$
$\{y_0, y_i\}, 1 \le i \le m$	$4m+3-i$	$[3m+3, 4m+2]$
$\{y_{m+1}, y_i\}, 1 \le i \le m$	$5m+3-i$	$[4m+3, 5m+2]$

From this table we can see that γ is an α-valuation. We take $s = m+2$ and $l = 2m$ in the definition of a strong α-valuation. Then $|A| = |B| = s$ and $|E(G)| = 5m+2 = 2l+s$. From the table we see that the labels of the edges $\{a, \pi(a)\}$ for $a \in A$ are indeed the elements of $[l+1, l+s] = [2m+1, 3m+2]$. Furthermore if we define γ^* to be γ on $\{y_0, y_m\}$ and $\gamma - t = \gamma - (3m+2)$ on $\{y_1, y_2, \ldots, y_m\}$, we easily check that γ^* is an α-valuation on the graph induced by $\{y_0, y_1, \ldots, y_{m+1}\}$. ∎

Theorem 10 *Let G be a connected bipartite graph with vertex partition X_1, X_2 such that the vertices of X_i all have the same degree, $i = 1, 2$, and $|X_1|$ and $|X_2|$ are distinct odd integers. Then $G \times K_2$ does not have a strong α-valuation.*

Proof. Let $X_1 = \{x_1, x_2, \ldots, x_m\}$ and $X_2 = \{x_{m+1}, x_{m+2}, \ldots, x_{m+n}\}$. Let $\deg(x_i) = a$, $1 \le i \le m$, and $\deg(x_i) = b$, $m < i \le m+n$. We will take $G^* = G \times K_2$ to have vertices $y_1, y_2, \ldots, y_{m+n}, z_1, z_2, \ldots, z_{m+n}$ and edges $\{y_i, y_j\}$ and $\{z_i, z_j\}$ whenever $\{x_i, x_j\}$ is an edge of G, along with edges $\{y_i, z_i\}$, $1 \le i \le m+n$.

Suppose that G^* has a strong α-valuation γ with automorphism π and critical value λ. Since G^* is connected the sets in the corresponding bipartition are uniquely determined as $A = \{y_1, \ldots, y_m, z_{m+1}, \ldots, z_{m+n}\}$ and $B = \{z_1, \ldots, z_m, y_{m+1}, \ldots, y_{m+n}\}$. Without loss of generality we assume that A contains the vertices v with $\gamma(v) \le \lambda$. Notice that G^* has $2(m+n)$ vertices and $2ma + m + n$ edges. If γ is a strong α-valuation we must have $s = m+n$ and $l = ma$. Notice that the degree of each vertex of G^* is 1 more than the degree of the corresponding vertex of G.

Now if we sum the labels of all the edges of G^* and use the fact that we have a graceful valuation we get

$$(a+1)\sum_{i=1}^{m}\gamma(z_i)+(b+1)\sum_{i=m+1}^{m+n}\gamma(y_i)-(a+1)\sum_{i=1}^{m}\gamma(y_i)-(b+1)\sum_{i=m+1}^{m+n}\gamma(z_i) =$$

$$\sum_{j=1}^{2ma+m+n} j = (2ma+m+n)(2ma+m+n+1)/2.$$

On the other hand if we sum the labels of the edges $\{v, \pi(v)\}$ for $v \in A$ and use the fact that we have a strong α-valuation we get

$$\sum_{i=1}^{m} \gamma(z_i) + \sum_{i=m+1}^{m+n} \gamma(y_i) - \sum_{i=1}^{m} \gamma(y_i) - \sum_{i=m+1}^{m+n} \gamma(z_i) =$$

$$\sum_{j=ma+1}^{ma+m+n} j = (m+n)(2ma+m+n+1)/2.$$

Subtracting $a+1$ times this equation from the previous one yields

$$(b-a)\sum_{i=m+1}^{m+n} \gamma(y_i) - (b-a)\sum_{i=m+1}^{m+n} \gamma(z_i) = (2ma+m+n+1)(ma-na)/2.$$

Now by counting edges in G we see that $ma = nb$. Since $m \neq n$, also $a \neq b$. Then $ma - na = nb - na = n(b-a)$. Thus dividing the last displayed equation by $b-a$ gives

$$\sum_{i=m+1}^{m+n} \gamma(y_i) - \sum_{i=m+1}^{m+n} \gamma(z_i) = (2ma+m+n+1)n/2.$$

This is a contradiction because the right side is not an integer when m and n are odd. ∎

Corollary 11 *If m and n are distinct odd integers, then $K_{m,n} \times K_2$ does not have a strong α-valuation.*

In [11] it is proved that if each vertex of a graph with a graceful valuation has even degree, then the number of edges of the graph must be congruent to 0 or 3 modulo 4. Thus if m and n are odd, the graph $K_{m,n} \times K_2$ cannot have a graceful valuation unless $2mn+m+n \equiv 0 \pmod 4$, that is, $m \equiv n \pmod 4$. The Corollary rules out strong α-valuations even in the latter case. The graph $G \times K_2$ in Theorem 9 need not have all even degrees. For example, X_1 might have 9 vertices of degree 10 in G, and X_2 15 vertices of degree 6 in G.

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The strong α-valuation for the graph on the right in Figure 2 was obtained using a computer program written by Michael Kenig. The authors thank Mr. Kenig for this contribution.

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