Improved bounds for the product of the domination and chromatic numbers of a graph

Y. Caro
Department of Mathematics
School of Education
University of Haifa - ORANIM
Tivon
Isreal 36006

Y. Roditty
School of Mathematical Sciences
Tel-Aviv University
Ramat-Aviv, Tel-Aviv
Isreal 69978

ABSTRACT. In this note we solve almost completely a problem raised by Topp and Volkmann [7] concerning the product of the domination and the chromatic numbers of a graph.

Graphs in this note are finite, without loops and multiple edges. We will use the notation of [5].

The *k-domination number* of a graph G, denoted by $\gamma_k(G)$, is equal to

$$\min\{|S|: \text{every vertex of } V\backslash S \text{ is adjacent to at least } k \text{ vertices of } S\}.$$

If $k = 1$, then $\gamma_k = \gamma$, where γ, is the domination number of a graph. Domination in graphs is a well-known area of graph theory, with a large number of papers and a recent volume of Discrete Mathematics is devoted to it (see [3]).

Let (see [4] and [7]) $B_{n,\delta}$, be the smallest integer B such that for every graph G on n vertices and minimum degree δ, the inequality $\gamma(G)\chi(G) \leq B$ holds. Furthermore, let $R_{n,\delta}$ the smallest integer R such that for every δ-regular graph with n vertices, the inequality $\gamma(G)\chi(G) \leq R$ holds.

In [7] the problem of estimating $B_{n,\delta}$ was restricted to connected graphs only, and it was proved there that for $\delta \geq 2$,

$$\frac{\delta}{(\delta+1)^2}n^2 \leq B_{n,\delta} \leq \frac{\delta}{(\delta-1)}(n+1)^2.$$

Our result, theorem 0.1 below, shows that both estimates are poor for large δ.

Let $|V(G)| = n$, and $\delta = \delta(G)$, where $\delta(G) = \min\{\deg(v): v \in V(G)\}$.

Before establishing our result, recall the well-known theorem due to Lovasz,

Theorem A. (Lovasz (see [1], [2])): *If* $\delta \geq 1$ *then,*

$$\gamma_1(G) \leq \frac{n(1+\log(\delta+1))}{\delta+1}.$$

Using Theorem A we prove:

Theorem 0.1.

1. *There exists a constant* $0 < c < 1$ *such that*
$$\frac{cn^2(1+\log(\delta+1))}{\delta+1} \leq B_{n,\delta} \leq \frac{n^2(1+\log(\delta+1))}{\delta+1}.$$

2. $$R_{n,\delta} \leq n(1+\log(\delta+1)) \leq n(1+\log n).$$

Proof: Recall that $\chi(G) \leq \Delta(G)+1$. Thus, by Theorem A, we have

$$\gamma_1(G)\cdot\chi(G) \leq \frac{n(1+\log(\delta+l))}{\delta+1}n = \frac{n^2(1+\log(\delta+1))}{\delta+1}$$

proving the upper bound in (1).

If G is δ-regular, then, by the same argument,

$$\gamma_1(G)\cdot\chi(G) \leq \frac{n(1+\log(\delta+1))}{\delta+1}(\delta+1) = n(1+\log(\delta+1)) \leq n(1+\log n)$$

proving the upper bound in (2).

Construct the connected graph H as follows. Identify a vertex of $K_{\frac{n}{2}+1}$ with a vertex of the connected graph G with $\frac{n}{2}$ vertices constructed in [1]. Since

$$\gamma_1(G) \geq \frac{(1-o(1))\frac{n}{2}(1+\log(\delta+1))}{\delta+1},$$

and $\gamma_1(H) \geq \gamma_1(G)$ and $\chi(H) \geq \chi(K_{\frac{n}{2}+1}) = \frac{n}{2}+1$. Hence,

$$\gamma_1(H)\cdot\chi(H) \geq \frac{(1-o(1))n^2(1+\log(\delta+1))}{4(\delta+1)},$$

proving the lower bound in (1). This completes the proof of the theorem. □

References

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