NILPOTENT SQS-SKEINS WITH NILPOTENT DERIVED SLOOPS

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Abstract. In [5], Guelzow gave an example of semiboolean SQS-skeins of nilpotent class 2, all its derived sloops are Boolean "or of nilpotence class 1". In this paper, we give an example of nilpotent SQS-skein of class 2 whose derived sloops are all of nilpotence class 2. Guelzow [6] has also given a construction of semiboolean SQS-skeins of nilpotence class n whose derived sloops are all of class 1. As an extension result, we prove in the present paper the existence of nilpotent SQS-skeins of class n all of whose derived sloops are nilpotent of the same class n; for any positive integer n.

1. Introduction

An SQS-skein is an algebra $(T; q)$ with a ternary operation q satisfying the following identities:

$$q(x, y, z) = q(y, x, z) = q(z, x, y) \ , \ q(x, x, y) = y \ , \ q(x, y, q(x, y, z)) = z \ .$$

A Steiner quadruple system or simply a quadruple system is a pair (P, B) where B is a collection of 4-subsets of P called *blocks* such that any 3-subset of P is contained in exactly one block of B. The number $|P| = n$ is the *cardinality* of the Steiner quadruple system (P, B), which is denoted by SQS(n). An SQS(n) exists iff n 2 or 4 (mod 6) [3]. There is one to one correspondence between Steiner quadruple systems and SQS-skeins [7].

A semiboolean SQS-skein is an SQS-skein satisfying the identity :

$$q(x, u, q(u, y, z)) = q(y, u, q(u, x, z))$$

In [5], Guelzow gave an example of an SQS-skein to prove that the class of all semiboolean SQS-skeins is different from the class of all Boolean SQS-skeins. That example is a nilpotent SQS-skein of class 2 and of cardinality 16 whose derived sloops are all Boolean "of nilpotent class 1". A sloop is an algebra (L; . , 1) with a binary operation . and an identity element 1 satisfies the identities:

$$x . y = y . x \quad , \quad 1 . x \quad x \quad \text{and} \quad x . (x . y) = y \ .$$

Quackenbush has extensively studied the algebraic properties of sloops in [8]. The algebraic concept of nilpotency for Mal'cev variety or for modular variety can be found in [4] and [9]. The existences of nilpotent SQS-skeins and of nilpotent sloops of any class n are proved in [1] and [2].

2. A nilpotent SQS-skein all of whose derived sloops are nilpotent

We will give in this section an example of nilpotent class 2 whose derived sloops are all of nilpotence class 2. In fact, the following representation theorem of finite nilpotent SQS-skeins given by Guelzow [6] directed me to build this example on the vector space over $GF(2)$.

Theorem 1 [6]. *Let $\mathcal{S} = (S\,;\,q)$ be a finite SQS-skein of nilpotence class $k > 0$ and $\log_2 |S| = m$. Then there exists an m-dimensional vector space V over $GF(2)$ and a family of polynomials p_i over $GF(2)$; $1 \le i \le m$ such that $\mathcal{B} = (V\,;\,t)$ is isomorphic to $\mathcal{S}$ where $t(x, y, z)_i := x_i + y_i + z_i + p_i(x, y, z)$.*

This representation is given in more details in [6].

To consider our example, let P be a 4-dimensional vector space over $GF(2)$ and let q be a ternary operation on P defined by:

$$q\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}\right) = \begin{pmatrix} x_1 + y_1 + z_1 \\ x_2 + y_2 + z_2 \\ x_3 + y_3 + z_3 \\ x_4 + y_4 + z_4 + (x_3 + y_3 + z_3 + x_3y_3 + x_3z_3 + y_3z_3)\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ 1 & 1 & 1 \end{vmatrix} \end{pmatrix}$$

It can be easily verified that q satisfies the commutative and the general idempotent identities. To satisfy the Steiner identity, we have :

$q(A, B, q(A, B, X))_i = x_i \;:\; i = 1, 2, 3$ and

$$q(A, B, q(A, B, X))_4 = x_4 + (a_3 + b_3 + x_3 + a_3b_3 + a_3x_3 + b_3x_3)\begin{vmatrix} a_1 & b_1 & x_1 \\ a_2 & b_2 & x_2 \\ 1 & 1 & 1 \end{vmatrix} +$$

$$(a_3 + b_3 + x_3 + a_3b_3 + a_3x_3 + b_3x_3)\begin{vmatrix} a_1 & b_1 & a_1 + b_1 + x_1 \\ a_2 & b_2 & a_2 + b_2 + x_2 \\ 1 & 1 & 1 \end{vmatrix} = x_4$$

Then $\mathcal{D} := (P\,;\,q)$ is an SQS-skein. The projection π of P on to the first three components is an epimorphism. And one easily checks that $\mathcal{D}$ is not Boolean, then the center $\xi(\mathcal{D}) = \ker \pi$. I.e. $\mathcal{D}$ is nilpotent of class 2.

From the following theorem, we can say directly that all derived sloops of $\mathcal{D}$ are nilpotent of class at most 2.

Theorem 2 [2]. *Any derived sloop of nilpotent SQS-skein of class n is a nilpotent sloop of class at most n .*

To show that all derived sloops of $\mathcal{D}$ are nilpotent of class 2, it is enough to prove that all derived sloops are not Boolean. Then we have to show; for any vector A in P, there are three vectors X, Y, Z ∈ P satisfying

$$q(A, X, q(A, Y, Z)) \neq q(A, Y, q(A, X, Z)).$$

It is not easy to prove this inequality directly, for this reason we prove this inequality for each vector A separately in the following eight cases:

(1) For $A = \begin{pmatrix} 0 \\ 0 \\ 0 \\ i \end{pmatrix}$; i = 0,1 , take $X = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $Y = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $Z = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

(2) For $A = \begin{pmatrix} 1 \\ 0 \\ 0 \\ i \end{pmatrix}$; i = 0,1 , take $X = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $Z = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

(3) For $A = \begin{pmatrix} 0 \\ 1 \\ 0 \\ i \end{pmatrix}$; i = 0,1 , take $X = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $Z = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

(4) For $A = \begin{pmatrix} 0 \\ 0 \\ 1 \\ i \end{pmatrix}$; i = 0,1 , take $X = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$, $Y = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $Z = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

(5) For $A = \begin{pmatrix} 1 \\ 1 \\ 0 \\ i \end{pmatrix}$; i = 0,1 , take $X = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $Z = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

(6) For $A = \begin{pmatrix} 1 \\ 0 \\ 1 \\ i \end{pmatrix}$; i = 0,1 , take $X = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $Z = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

(7) For $A = \begin{pmatrix} 0 \\ 1 \\ 1 \\ i \end{pmatrix}$; i = 0,1 , take $X = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $Z = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

(8) For $A = \begin{pmatrix} 1 \\ 1 \\ 1 \\ i \end{pmatrix}$; i = 0,1 , take $X = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $Z = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

For all these cases the two sides of the first three components are equal.

I.e. $\quad q(A, X, q(A, Y, Z))_i = q(A, Y, q(A, X, Z))_i \quad$; i = 1, 2, 3.

But, the two sides of the forth component are not equal, where

L.H.S. = $q(A, X, q(A, Y, Z))_4$

$$= x_4 + y_4 + z_4 + (a_3 + y_3 + z_3 + a_3 y_3 + a_3 z_3 + y_3 z_3) \begin{vmatrix} a_1 & y_1 & z_1 \\ a_2 & y_2 & z_2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$+ (a_3 + x_3 + y_3 + z_3 + a_3 y_3 + a_3 z_3 + x_3 y_3 + x_3 z_3) \begin{vmatrix} a_1 & x_1 & y_1 + z_1 \\ a_2 & x_2 & y_2 + z_2 \\ 1 & 1 & 0 \end{vmatrix}$$

R.H.S. = $q(A, Y, q(A, X, Z))_4$

$$= x_4 + y_4 + z_4 + (a_3 + x_3 + z_3 + a_3 x_3 + a_3 z_3 + x_3 z_3) \begin{vmatrix} a_1 & x_1 & z_1 \\ a_2 & x_2 & z_2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$+ (a_3 + x_3 + y_3 + z_3 + a_3 x_3 + a_3 z_3 + x_3 y_3 + y_3 z_3) \begin{vmatrix} a_1 & y_1 & x_1 + z_1 \\ a_2 & y_2 & x_2 + z_2 \\ 1 & 1 & 0 \end{vmatrix}$$

By calculating the two sides in each case we can find in all eight cases that:

$$\text{L.H.S.} \neq \text{R.H.S.}$$

This means that there are four vectors A , X , Y & Z $\in$ P satisfying $q(A, X, q(A, Y, Z)) \neq q(A, Y, q(A, X, Z))$. Consequently, we have the SQS-skein $\mathfrak{Q} := (P ; q)$ is nilpotent of class 2 and all its derived sloops are also nilpotent of class 2 .

3. Nilpotent SQS-skeins all whose derived sloops of nilpotence class n

In [2]; the author gave a construction of an SQS-skein with a derived sloop, both are nilpotent of the same class n; for any positive integer n. In [6],Guelzow gave a construction of nilpotent SQS-skeins of class n, whose derived sloops are all Boolean "of nilpotence class 1", such class of SQS-skeins is known by semiboolean SQS-skeins. Our aim in this paper is to construct a nilpotent SQS-skein of class n, all of whose derived sloops are nilpotent of the same class n. In other words, we will generalize the example given in section 2.

The author has given in [1] a construction of nilpotent SQS-skeins of class n. This construction is generalized and reformulated by Guelzow [6] in the following theorem to a construction that is known by Generalized Doubling Construction.

Theorem 3 [6]. *Let $\tau_* = (T_*, q_*)$ be an SQS-skein and R be a set of 4-element subalgebras of τ_*. Suppose $T = T_* \times GF(2)$ and χ_R is the characteristic function of R, then $\tau = (T; q)$ is an SQS-skein, where*

$$q((x, i_x),(y, i_y),(z, i_z)) := (q_*(x, y, z), i_x + i_y + i_z + \chi_R + \langle x, y, z\rangle_{\tau_*}).$$

If τ_* is nilpotent of class n, then τ is nilpotent of class n or n + 1.

The author has given in [2] a construction of nilpotent sloops of class n .and this construction is reformulated in [2] by using the same sense of the above theorem to the generalized doubling construction of nilpotent sloops as in the following theorem.

Theorem 4 [2]. *Let $\mathcal{L}_* := (L_*; *, 1)$ be a sloop and R_* be a set of 3-element subalgebras of $\mathcal{L}_*$. If $\mathcal{L} = (L \times GF(2); \circ, (1, o))$ and χ_{R_*} is the characteristic function of R_*, then $\mathcal{L}$ is a sloop, where the binary operation $\circ$ is given by*

$$(x, i_x)\circ(y, i_y) := (x*y, i_x + i_y + \chi_{R_*}\langle x, y\rangle_{\mathcal{L}_*}).$$

Moreover, if $\mathcal{L}_*$ is nilpotent of class n, then $\mathcal{L}$ is nilpotent of class n or n + 1

The next theorem gives us the characteristic of the center of subdirectly irreducible SQS-skeins and of subdirectly irreducible sloops.

Theorem 5 [1][2]. *Let τ be a subdirectly irreducible SQS-skein or sloop. Then the unique atom of the congruence lattice of τ is the center $\xi(\tau)$ iff*
$| |x|\xi(\tau) | = 2$.

Now, we are able to prove our main step of mathematical induction in the following theorem.

Theorem 6. *If τ^* is a subdirectly irreducible SQS-skein of nilpotence class n whose derived sloops are all subdirectly irreducible and also of nilpotence class n, then there is an irreducible SQS-skein of nilpotence class n + 1 and all its derived sloops are subdirectly irreducible and of nilpotence class n + 1.*

Proof. τ^* is irreducible and of nilpotence class n, then we may assume that $\tau^* = (T^* ; q^*)$, where $T^* = \{x_0, x_1, \dots, x_{2^m-1}\}$ and the congruence lattice of τ^* has a unique atom $\theta^* = \xi(\tau^*)$. By theorem 5, θ^* can be assumed without losing the truth of the generality by $\theta^* = \bigcup_{\text{even } i=0}^{2^m-2} \{x_i, x_{i+1}\}^2$. Also each derived sloop $\mathcal{L}_i^* := (T^*; \circ_i, x_i)$ is subdirectly irreducible and of nilpotence class n, where $\circ_i$ is defined by $y \circ_i z := q^*(x_i, y, z)$, then $\xi(\mathcal{L}_i^*) = \theta^*$; $i = 0, 1, \dots, 2^m-1$.

By choosing the set $R := \{\{x_i, x_{i+1}, x_{i+2}, x_{i+3}\} : i = 0, 4, 8, \dots, 2^m - 4\}$ and applying theorem 3, we get $\tau := (T^* \times GF(2) ; q)$ is an SQS-skein, where q is defined as in theorem 3. I.e.

$q((x, i_x),(y, i_y),(z, i_z)) := (q^*(x, y, z), i_x + i_y + i_z + x_R \langle x, y, z\rangle_{\tau^*})$.

According to theorem 3, τ is nilpotent of class n or n + 1 and the kernel of the projection $\pi : T^* \times GF(2) \to T^*$ is a central congruence of τ. To prove that τ is subdirectly irreducible and of nilpotence class n + 1, it is enough; to prove that ker π is the unique atom of the congruence lattice of τ, according to thereom 5.

The congruence θ^* is extended in τ to the congruence

$$\theta = \bigcup_{\text{even } i=0}^{2^m-2} \{(x_i, 0),(x_i, 1),(x_{i+1}, 0),(x_{i+1}, 1)\}^2 .$$

By considering any block b of the set R, say $b = \{x_i, x_{i+1}, x_{i+2}, x_{i+3}\}$ and by assuming that the congruence lattice of τ has another atom δ, then θ would cover each of ker π and δ: It means that the class $[(x_i, 0)]\delta$ has two cases:

(1) $[(x_i, 0)]\delta = \{(x_i, 0),(x_{i+1}, 0)\}$ or

(2) $[(x_i, 0)]\delta = \{(x_i, 0),(x_{i+1}, 1)\}$.

And by choosing two distinct pairs (x_l, x_{l+1}) and (x_k, x_{k+1}) of θ^* such that $l \notin \{i, i+1, i+2, i+3\}$ and $q^*(x_i, x_{i+2}, x_l) = x_k$.

We will use the same tactic used in the proof of the construction of nilpotent SQS-skeins of class n in [1] to get the contradiction with the number of elements of $[(x_i, 0)]\delta$. For the first case (1), one can show that the pairs $((x_i, 0),(x_{i+1}, 0))$, $((x_l, 0),(x_{l+1}, 0))$, $((x_{i+2}, 0),(x_{i+3}, 1))$ are elements of δ, and then $((x_k, 0),(x_k, 1))$ or $((x_k, 0),(x_{k+1}, 1)) \in \delta$ that gives the contradiction $|[(x_k, 0)]\ \delta| > 2$.

For the second case (2), one can show that the pairs $((x_i, 1),(x_{i+1}, 0))$, $((x_{i+2}, 0),(x_{i+3}, 0))$, $((x_l, 0),(x_{l+1}, 0))$ are elements of δ, and then $((x_k, 1),(x_k, 0))$ or $((x_k, 1),(x_{k+1}, 0)) \in \delta$ that gives also the contradiction $|[(x_k, 0)]\ \delta| > 2$.

Therefore, the atom ker π is the unique atom of C(τ). I.e. τ is subdirectly irreducible and nilpotent of class n + 1.

It is easy to see that ker π is a central congruence of any derived sloop $\mathcal{L}_i := (T \times GF(2)\ ;\ \cdot_i,\ (x_i, 0))$ of τ, where $(y, i_y) \cdot_i (z, i_z) := q((x_i, 0), (y, i_y), (z, i_z))$. This means that

$$(y, i_y) \cdot_i (z, i_z) = (q(x_i, y, z),\ i_y + i_z + x_R\langle x_i, y, z\rangle_\tau *)$$
$$= (y \circ_i z,\ i_y + i_z + x_{R_i}\langle y, z\rangle_{\mathcal{L}_i}\ ),$$

where R_i is the set $\{\{y, z, w\} : \{x_i, y, z, w\} \in R\}$ i.e. R_i is the singleton set $\{\{x_{i+1}, x_{i+2}, x_{i+3}\}\}$.

Then according to theorem 4 or specially to the construction 1 in [2], the derived sloop $\mathcal{L}_i$ is a subdirectly irreducible and nilpotent of class n + 1;

for any $i = 0, 1, \ldots, 2^m - 1$. This completes the proof.

Theorem 7. *For any positive integer n, there is nilpotent SQS-skein of class n, in which every derived sloop is also of nilpotence class n.*

Proof. From the given example in section 2, we have a subdirectly irreducible SQS-skein of cardinality 2^4, in which every derived sloop is also irreducible and nilpotent of class 2. According to theorem 6 and by the principal of mathematical induction, we get directly the structure of the required SQS-skein.

Corollary 8. *There is a nilpotent SQS-skeins τ of class n which its derived sloops are all of nilpotence class n, in which τ and all it derived sloops have the same center and the same central series.*

Proof. The given example in section 2, $\mathcal{D}$ is an SQS-skein of nilpotence class 2 satisfying the required properties of the corollary. And by theorem 6, we can construct an example of nilpotent SQS-skein of class 3 and any of its derived sloops is also of class 3.

Namely, the SQS-skein $\tau := (P \times GF(2)\ ;\ q)$, where q is defined as in theorem 6. The center $\xi(\tau)$ = ker π is the same center of any derived sloop $\mathcal{L}_i$ i.e. $\xi(\tau) = \xi(\mathcal{L}_i) = ker\ \pi$, where π is the projection of $P \times GF(2)$ on to P. We have the center $\xi(\tau/ker\ \pi) \cong \xi(\mathcal{D})$ = the center of any derived sloop of $\mathcal{D}$; if the center $\xi(\tau/ker\ \pi)$ denoted by $\xi_2(\tau)/ker\ \pi$. then the central series of τ and of any of its derived sloop $\mathcal{L}_i$ ($i = 0, 1, \ldots, 2^{n+2}$) is

$\Delta := 0 \subseteq \xi_1(\tau) := \xi(\tau) = \xi(\mathcal{L}_i) \subseteq \xi_2(\tau) \subseteq 1 := \nabla$.

By repeating the same process, we get our required structure.

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