# Further Results on Independence in Direct-Product Graphs

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**Abstract:** For a graph $G$, let $\alpha(G)$ and $\tau(G)$ denote the independence number of $G$ and the matching number of $G$, respectively. Further, let $G \times H$ denote the direct product (also known as Kronecker product, cardinal product, tensor product, categorical product and graph conjunction) of graphs $G$ and $H$. It is known that $\alpha(G \times H) \geq \max\{\alpha(G) \cdot |H|, \alpha(H) \cdot |G|\}$ $=: \underline{\alpha}(G \times H)$ and that $\tau(G \times H) \geq 2 \cdot \tau(G) \cdot \tau(H) =: \underline{\tau}(G \times H)$. It is shown that an equality/inequality between $\alpha$ and $\underline{\alpha}$ is independent of an equality/inequality between $\tau$ and $\underline{\tau}$. Further, several results are presented on the existence of a complete matching in each of the two connected components of the direct product of two bipartite graphs. Additional results include an upper bound on $\alpha(G \times H)$ that is achievable in certain cases.

**Key words**: Independence number, matching number, direct product, complete matching

## 1 Introduction and Preliminaries

For a graph $G$, let $\alpha(G)$ and $\tau(G)$ denote the *independence number* of $G$ and the *matching number* of $G$, respectively. Further, let $G \times H$ denote the *direct product* (defined below) of graphs $G$ and $H$. This paper continues an earlier study [6] on $\alpha(G \times H)$ and $\tau(G \times H)$, and presents several new results.

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Independence number and matching number are prominent graph invariants having applications in areas such as information theory, extremal graph theory, optimization problems, VLSI layout, processor scheduling and network flows. Also, graph parameters such as rank, vertex covering number, edge covering number, chromatic number and chromatic index are related to them. While determination of $\alpha(G)$ is, in general, NP-hard [1], that of $\tau(G)$ is possible in polynomial time [2, 12]. For a product graph, it is economical to obtain these invariants via factor graphs, since problem size is much smaller in the factors than in the product. This observation is particularly interesting in view of a recent efficient algorithm for factoring a graph with respect to the direct product [4].

By a graph is meant a finite, simple and undirected graph. Unless indicated otherwise, graphs are also connected and have at least two vertices. Let $G$ stand also for $V(G)$. For $X \subseteq V(G)$, $\langle X \rangle$ denotes the subgraph induced by $X$. Let $P_m$ (resp. $C_m$) denote a path (resp. a cycle) on $m$ vertices.

Let $G$ be a graph. The *independence number* $\alpha(G)$ of $G$ is defined to be the largest number of mutually nonadjacent vertices in $G$. By a *matching* in $G$ is meant a set of edges, no two of which share a common vertex. The *matching number* $\tau(G)$ is defined to be the size of a largest matching. $G$ is said to have a *perfect matching* if $\tau(G) = \frac{1}{2}|G|$. For any undefined terms, see the recent monograph [5] that contains a wealth of information on product graphs.

A *complete matching* in a bipartite graph is a matching that includes every vertex of the smaller partite set. Not every bipartite graph has a complete matching.

**Definition 1** *Let $G = (V \cup W, E)$ be a bipartite graph with $|V| \leq |W|$. A set of $r$ vertices in $V$ is collectively incident on, say, $q$ vertices of $W$. The maximum value of the number $r - q$ taken over all values of $r = 1, 2, \cdots$ and all subsets of $V$ is called the deficiency $\delta(G)$ of $G$.* □

Note that $\delta(K_{m,n}) = m - n$, where $m \leq n$. The following statements are equivalent with respect to a bipartite graph $G = (V \cup W, E)$ with $|V| \leq |W|$: (i) $G$ has a complete matching, (ii) $\delta(G) \leq 0$, (iii) $\tau(G) = |V|$, and (iv) $\alpha(G) = |W|$, cf. Hall's marriage theorem.

The *direct product* $G \times H$ of graphs $G$ and $H$ is a graph with $V(G \times H) = V(G) \times V(H)$ and $E(G \times H) = \{\{(u,x),(v,y)\} : \{u,v\} \in E(G)$ and $\{x,y\} \in E(H)\}$. This product is commutative and associative in a natural way, and is distributive with respect to edge-disjoint union of graphs. It is variously known as *Kronecker product, cardinal product, tensor product, categorical product* and *graph conjunction*, and is the most natural graph product with applications in areas such as automata theory [3] and multiprocessor systems [9]. It is not difficult to see that certain computational

Table 1: $\alpha$ vs. $\underline{\alpha}$ and $\tau$ vs. $\underline{\tau}$

| $G \times H$ | $\alpha = \underline{\alpha}$? | $\tau = \underline{\tau}$? | Reference |
|---|---|---|---|
| $P_{2i} \times P_n / C_{2i} \times K_{1,n} / K_{2i} \times C_{2j}$ | Yes | Yes | Jha et al [6] |
| $K_{1,m} \times K_{1,n} / C_{2i+1} \times C_{2j+1}$<br>$m, n \geq 2$ and $i, j \geq 1$ | Yes | No | Jha et al [6] |
| $(K_m + K_n + e) \times (K_m + K_n + e)$<br>$m$ and $n$ of the same parity<br>$m, n \geq 3$ and $m \neq n$ | No | Yes | Remark following Lemma 2.3 |
| $(K_{1,m} + v) \times (K_{1,n} + v)$<br>$m, n \geq 3$ | No | No | Remark following Theorem 3.1 |

arrays [10] and diagonal networks [11] are representable by means of this product.

It is known that (i) $G \times H$ is bipartite if and only if $G$ or $H$ is bipartite, (ii) $G \times H$ is connected if and only if $G$ or $H$ is nonbipartite, (iii) If $G = (V \cup W, E)$ and $H = (X \cup Y, F)$ are both bipartite, then $G \times H$ consists of two connected components having vertex sets $(V \times X) \cup (W \times Y)$ and $(V \times Y) \cup (W \times X)$, respectively, and (iv) If $G$ and $H$ are bipartite graphs, one of which admits an automorphism that swaps the two colors, then the two components of $G \times H$ are isomorphic to each other [7].

It is easy to derive the following lower bounds:

1. $\alpha(G \times H) \geq \max\{\alpha(G) \cdot |H|, \alpha(H) \cdot |G|\} =: \underline{\alpha}(G \times H)$.

2. $\tau(G \times H) \geq 2 \cdot \tau(G) \cdot \tau(H) =: \underline{\tau}(G \times H)$.

Table 1 shows that an equality/inequality between $\alpha$ and $\underline{\alpha}$ is independent of an equality/inequality between $\tau$ and $\underline{\tau}$. Certain additional remarks appear below.

- $\alpha = \underline{\alpha}$ holds for each of $P_m \times P_n$, $C_m \times P_n$, $C_m \times C_n$ [6], and $K_m \times K_n$ [8]. However, for every graph $G$ and every natural number $i$, there exists a graph $H_i$ such that $\alpha(G \times H_i) > \underline{\alpha}(G \times H_i) + i$ [6].

- $\tau = \underline{\tau}$ necessarily holds if each of $G$ and $H$ contains a perfect matching. However, graphs exist for which there is an arbitrarily large gap between $\tau$ and $\underline{\tau}$. In particular, $\tau(K_{1,n} \times K_{1,n}) = n + 1$ while $\underline{\tau}(K_{1,n} \times K_{1,n}) = 2$.

The following statements are equivalent with respect to a bipartite graph $G$ having an independent set $I$ and a matching $M$: (i) $I$ is a largest independent set and $M$ is a largest matching, and (ii) $|I| + |M| = |G|$.

Section 2 deals with bounds on $\alpha(G \times H)$. An upper bound $\overline{\alpha}$ is presented that is shown to be achievable in many cases. That the lower bound $\underline{\alpha}$ may not be correct even for the product of certain trees is proved next.

Section 3 studies complete matching in the two components of the product of bipartite graphs. Existence of a complete matching in each of $G$ and $H$ ensures the existence of a complete matching in only one of the two components of $G \times H$. However, if one of $G$ and $H$ has a perfect matching, then the existence/non-existence of a complete matching in the other graph is inherited by each component of $G \times H$. Further, for each bipartite graph $G$, there exists an $r$ such that each component of $G \times K_{1,r}$ contains a complete matching.

# 2 Bounds on $\alpha(G \times H)$

It is easy to see that if $G$ is a graph, and $G_1, \cdots, G_r$ are subgraphs that constitute a vertex decomposition of $G$, then $\alpha(G) \leq \sum_{i=1}^{r} \alpha(G_i)$. This fact and the following lemma lead to an upper bound on $\alpha(G \times H)$ that appears in Theorem 2.2.

**Lemma 2.1** *If $m \geq n \geq 2$, then $K_m \times K_n$ admits of a vertex decomposition into $m$ cliques, each isomorphic to $K_n$.*

**Proof.** That the claim is true of $K_m \times K_2$ is easy. In what follows, let $m \geq n \geq 3$, and let $V(K_n) = \{0, \ldots, n-1\}$. For $0 \leq j \leq m-1$, each of $\{(i+j, i) : 0 \leq i \leq n-1\}$ induces a clique $K_n$ in $K_m \times K_n$. (Here $i+j$ is modulo $m$.) The resulting $m$ cliques constitute a vertex decomposition of $K_m \times K_n$. □

It follows that $\alpha(K_m \times K_n) \leq \max\{m, n\}$. Also, $\alpha(K_m \times K_n) \geq \underline{\alpha}(K_m \times K_n) = \max\{m, n\}$, and hence $\alpha(K_m \times K_n) = \max\{m, n\}$.

**Theorem 2.2** *If $G$ and $H$ are graphs, and $Q_1, \ldots, Q_m$ (resp. $R_1, \ldots, R_n$) are vertex-disjoint cliques in $G$ (resp. $H$), all different from $K_1$, then*

$$\begin{aligned}\alpha(G \times H) \leq \; & |G| \cdot |H| - (\textstyle\sum |Q_i| \cdot |R_j|)_{1 \leq i \leq m;\, 1 \leq j \leq n} \\ & + (\textstyle\sum max\{|Q_i|, |R_j|\})_{1 \leq i \leq m;\, 1 \leq j \leq n}.\end{aligned}$$

**Proof.** Let $G$, $H$, $Q_1, \ldots, Q_m$ and $R_1, \ldots, R_n$ be as stated. Note that $\{Q_i \times R_j : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$ constitutes a collection of $m \cdot n$ subgraphs of $G \times H$ that are mutually vertex-disjoint. By Lemma 2.1, each $\langle Q_i \times R_j \rangle$ contains $\max\{|Q_i|, |R_j|\}$ vertices that are mutually non-adjacent. Thus $\langle \{Q_i \times R_j : 1 \leq i \leq m \text{ and } 1 \leq j \leq n\} \rangle$ contains at most $(\sum \max\{|Q_i|, |R_j|\})_{1 \leq i \leq m;\, 1 \leq j \leq n}$ vertices that are mutually non-adjacent. Consequently, a largest independent set of $G \times H$ is of cardinality at most

$|G| \cdot |H| - (\sum |Q_i| \cdot |R_j|)_{1 \le i \le m;\ 1 \le j \le n} + (\sum \max\{|Q_i|, |R_j|\})_{1 \le i \le m;\ 1 \le j \le n}$.
□

Let the least value of $|G| \cdot |H| - (\sum |Q_i| \cdot |R_j|)_{1 \le i \le m;\ 1 \le j \le n} + (\sum \max\{|Q_i|, |R_j|\})_{1 \le i \le m;\ 1 \le j \le n}$ be denoted by $\overline{\alpha}(G \times H)$, where the minimum is taken over all vertex-disjoint cliques in $G$ and $H$ satisfying the hypothesis of Theorem 2.2. Clearly, $\overline{\alpha}(G \times H)$ is an upper bound on $\alpha(G \times H)$. It is next shown that $\alpha = \overline{\alpha}$ holds in certain cases.

For $m, n \ge 3$, let $(K_m + K_n + e)$ be the graph obtainable from a $K_m$ and a vertex-disjoint $K_n$ by introducing an edge (i.e., a bridge) between a vertex of $K_m$ and a vertex of $K_n$. Let vertex set of this graph be given by $\{0, \cdots, m-1\} \cup \{m, \cdots, m+n-1\}$ where edge set consists of (i) all edges among $0, \cdots, m-1$, (ii) all edges among $m, \cdots, m+n-1$, and (iii) the bridge $\{m-1, m\}$. (Let $G^2$ stand for $G \times G$.)

**Lemma 2.3** *For* $m, n \ge 3$, $\alpha((K_m + K_n + e)^2) = \overline{\alpha}((K_m + K_n + e)^2)$.

**Proof.** Assume that $m \ge n \ge 3$, and check to see that $\overline{\alpha}((K_m + K_n + e)^2) = 3m + n$. That this upper bound is correct is seen from the existence of the following independent set of the same cardinality:

$\{(0, j) :\ 0 \le j \le m-1\} \cup \{(m, j) :\ 0 \le j \le m-1\}$
$\cup\{(i, m+1) :\ 0 \le i \le m+n-1\}$. □

**Remark:** Note that $\underline{\alpha}((K_m + K_n + e)^2) = 2(m+n)$. Consequently, $\alpha((K_m + K_n + e)^2) = \underline{\alpha}((K_m + K_n + e)^2)$ if and only if $m = n$. Also, if $m$ and $n$ are of the same parity, then the graph $(K_m + K_n + e)$ contains a perfect matching, and hence so does $(K_m + K_n + e)^2$. It follows that if $m, n \ge 3$, $m \ne n$ and $m, n$ are of the same parity, then the graph $(K_m + K_n + e)^2$ is such that $\alpha > \underline{\alpha}$ and $\tau = \underline{\tau}$.

Let $G = (V \cup W, E)$ be a bipartite graph, and let $n \ge 3$. Consider the graph $G \times K_n$ that is bipartite having partite sets $V \times K_n$ and $W \times K_n$. Letting $|G| = r$ and $\tau(G) = t$, it is clear that $\underline{\alpha}(G \times K_n) = \alpha(G) \cdot n = (r - t) \cdot n$. Also, since $G$ contains exactly $t$ vertex-disjoint $K_2$'s, it follows that $\overline{\alpha}(G \times K_n) = r \cdot n - 2 \cdot t \cdot n + t \cdot n = (r - t) \cdot n$. Consequently, $\alpha(G \times K_n) = (r - t) \cdot n = \alpha(G) \cdot n$, and hence $\tau(G \times K_n) = t \cdot n = \tau(G) \cdot n$. Thus the graph $G \times K_n$ is such that each of $\underline{\alpha}$ and $\overline{\alpha}$ is achievable, and if $n$ is even, then $\tau = \underline{\tau}$ holds. The reader may further check to see that $G \times K_n$ contains a complete matching if and only if $G$ contains a complete matching.

It is next observed that if $m, n \ge 3$ and $m$ is odd, then $\alpha(C_m \times K_n) = \underline{\alpha}(C_m \times K_n) = (m-1)n/2$. To see this, consider the following sequences of vertices for $0 \le j \le n-1$: $(0, a_0 + j), (1, a_1 + j), \ldots, (m-1, a_{m-1} + j)$, where $a_0 = a_2 = \cdots = a_{m-3} = 0$; $a_1 = a_3 = \cdots = a_{m-2} = 1$; and $a_{m-1} = 2$. Each induces a cycle of length $m$, and the resulting $n$ cycles constitute a

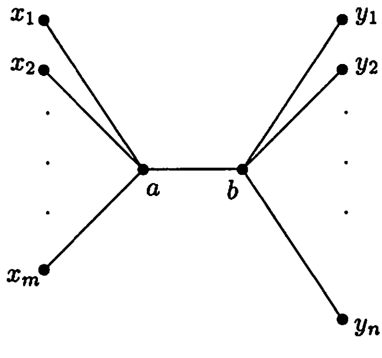


Figure 1: The tree $(K_{1,m} + K_{1,n} + e)$

vertex decomposition of $C_m \times K_n$. Thus $\alpha(C_m \times K_n) \le (m-1)n/2$. Also, $\underline{\alpha}(C_m \times K_n) = (m-1)n/2$.

**Remark:** Let $m$ and $n$ be both odd $\ge 3$, and $m \ge n$. It is known that $\alpha(C_m \times C_n) = (m-1)n/2$ [6]. Also, as shown above, $\alpha(C_m \times K_n) = (m-1)n/2$. Thus $C_m \times C_n$ is independence-insensitive to the introduction of a total of $mn(n-3)$ edges from $E(C_m \times K_n) \setminus E(C_m \times C_n)$. A similar statement holds with respect to each of the following: (i) $C_{2i} \times C_n$ and $C_{2i} \times K_n$, and (ii) $P_{2i+1} \times C_{2j+1}$ and $P_{2i+1} \times K_{2j+1}$.

Finally in this section, it is shown that $\alpha$ need not be equal to $\underline{\alpha}$ even for the product of two trees. Let $(K_{1,m} + K_{1,n} + e)$ be the tree that appears in Figure 1 and that is obtainable from a $K_{1,m}$ and a vertex-disjoint $K_{1,n}$ by introducing an edge between the "centers" of the two stars.

**Lemma 2.4** *$\alpha((K_{1,m} + K_{1,n} + e)^2) = \underline{\alpha}((K_{1,m} + K_{1,n} + e)^2)$ if and only if $m = n$.*

**Proof.** The two partite sets of $(K_{1,m} + K_{1,n} + e)$ are $\{b, x_1, \ldots, x_m\}$ and $\{a, y_1, \ldots, y_n\}$. Let $1 \le m \le n$. First consider the component

$$\langle (\{b, x_1, \ldots, x_m\} \times \{b, x_1, \ldots, x_m\}) \cup (\{a, y_1, \ldots, y_n\} \times \{a, y_1, \ldots, y_n\}) \rangle$$

of $(K_{1,m} + K_{1,n} + e)^2$ that has a total of $(m+1)^2 + (n+1)^2$ vertices. The set $I_1$ and the set $M_1$, respectively, constitute an independent set and a matching of this component:

$$\begin{aligned} I_1 &= (\{x_1, \cdots, x_m\} \times \{x_1, \cdots, x_m\}) \cup (\{y_1, \cdots, y_n\} \times \{y_1, \cdots, y_n\}) \\ &\quad \cup \{(a, y_i) : 1 \le i \le n\} \cup \{(y_i, a) : 1 \le i \le n\}. \\ M_1 &= \{\{(x_i, b), (a, y_i)\} : 1 \le i \le m\} \cup \{\{(b, x_i), (y_i, a)\} : 1 \le i \le m\} \\ &\quad \cup \{\{(b, b), (y_1, y_1)\}, \{(x_1, x_1), (a, a)\}\}. \end{aligned}$$

Note that $|I_1| = m^2 + n^2 + 2n$ and $|M_1| = 2m + 2$. Since $|I_1| + |M_1|$ is equal to the number of vertices in this component (that is bipartite), it follows that $I_1$ is a largest independent set and $M_1$ is a largest matching.

Next consider the other component

$$\langle(\{b, x_1, \ldots, x_m\} \times \{a, y_1, \ldots, y_n\}) \cup (\{a, y_1, \ldots, y_n\} \times \{b, x_1, \ldots, x_m\})\rangle$$

that has a total of $2(m+1)(n+1)$ vertices. $I_2$ and $M_2$ are an independent set and a matching, respectively:

$$\begin{aligned} I_2 &= (\{x_1, \cdots, x_m\} \times \{y_1, \cdots, y_n\}) \cup (\{y_1, \cdots, y_n\} \times \{x_1, \cdots, x_m\}) \\ &\quad \cup \{(a, x_i) : 1 \le i \le m\} \cup \{(b, y_i) : 1 \le i \le n\}. \\ M_2 &= \{\{(x_i, a), (a, x_i)\} : 1 \le i \le m\} \cup \{\{(b, y_i), (y_i, b)\} : 1 \le i \le n\} \\ &\quad \cup \{\{(b, a), (y_1, x_1)\}, \{(a, b), (x_1, y_1)\}\}. \end{aligned}$$

Note that $|I_2| = 2mn + m + n$ and $|M_2| = m + n + 2$. Since $|I_2| + |M_2|$ is equal to the total number of vertices in this component, $I_2$ is a largest independent set and $M_2$ is a largest matching.

Thus $\alpha((K_{1,m} + K_{1,n} + e)^2) = (m^2 + n^2 + 2n) + (2mn + m + n)$. Also, $\underline{\alpha}((K_{1,m} + K_{1,n} + e)^2) = (m+n) \cdot (m+n+2) = (m^2 + n^2 + 2n) + (2mn + 2m)$. Clearly, $\alpha = \underline{\alpha}$ if and only if $m = n$. □

Let $(2K_{1,m} + e)$ stand for $(K_{1,m} + K_{1,m} + e)$. The reader may check to see that if $m, p, q$ are integers such that $1 \le m \le min\{p, q\}$, then the graph $(2K_{1,m} + e) \times (K_{1,p} + K_{1,q} + e)$, that consists of two isomorphic components [7], is such that $\alpha = \underline{\alpha}$.

# 3 Complete Matching in the Product of Bipartite Graphs

Throughout this section, $G$ and $H$ are bipartite graphs, so $G \times H$ consists of two connected components, themselves bipartite.

**Theorem 3.1** *If each of $G$ and $H$ contains a complete matching, then one component of $G \times H$ contains a complete matching while the other component of $G \times H$ need not contain a complete matching.*

**Proof.** Let $G = (V \cup W, E)$ and $H = (X \cup Y, F)$ be bipartite graphs, where $|V| \le |W|$ and $|X| \le |Y|$. It is easy to see that if $M$ (resp. $M'$) is a matching in $G$ (resp. $H$) then $M \times M'$ (that is of size $2 \cdot |M| \cdot |M'|$) is a matching in $G \times H$ that is evenly divided between the two components. Based on this, existence of a complete matching in each of $G$ and $H$ implies existence of a complete matching in the component $\langle(V \times X) \cup (W \times Y)\rangle$ of $G \times H$.

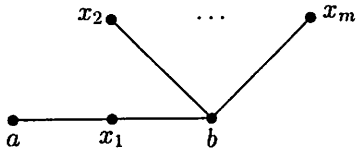


Figure 2: The tree $(K_{1,m} + v)$

On the other hand, the other component of $G \times H$ need not contain a complete matching. To see this, let $(K_{1,m} + v)$ be the tree that appears in Figure 2. It has $m + 2$ vertices with partite sets $X_0 = \{a, b\}$ and $X_1 = \{x_1, \cdots, x_m\}$. Similarly, let $(K_{1,n} + v)$ be the tree with partite sets $Y_0 = \{c, d\}$ and $Y_1 = \{y_1, \cdots, y_n\}$, where $c$ (resp. $d$) is analogous to $a$ (resp. $b$) in $(K_{1,m} + v)$. It is clear that each of $(K_{1,m} + v)$ and $(K_{1,m} + v)$ contains a complete matching. Assume that $3 \leq m \leq n$.

Now consider the component $\langle (X_0 \times Y_1) \cup (X_1 \times Y_0) \rangle$ of $(K_{1,m} + v) \times (K_{1,n} + v)$ that has a total of $2 \cdot (m + n)$ vertices. It is easy to check that

$$I = \{(a, y_i) : 1 \leq i \leq n\} \cup \{(b, y_i) : 2 \leq i \leq n\} \cup \{(x_i, c) : 2 \leq i \leq m\}$$

is an independent set of this component, and that

$$\begin{aligned} M = &\{\{(a, y_1), (x_1, c)\}, \{(a, y_2), (x_1, d)\}, \{(b, y_1), (x_2, c)\}\} \\ &\cup \{\{(b, y_i), (x_i, d)\} : 2 \leq i \leq m\} \end{aligned}$$

is a matching of this component. Note that $|I| = m + 2n - 2$ and $|M| = m + 2$. Since $|I| + |M|$ is equal to the number of vertices in this component, $I$ (resp. $M$) is a largest independent set (resp. largest matching). Since $|M|$ is strictly less than the cardinality of the smaller partite set, this component does not contain a complete matching. To conclude the proof, each of $P_{2i+1}$ and $P_{2j+1}$ contains a complete matching, and so does each component of $P_{2i+1} \times P_{2j+1}$ [6]. □

**Remark:** The trees $(K_{1,m} + v)$ and $(K_{1,n} + v)$ presented in the proof of Theorem 3.1 are such that $\alpha((K_{1,m} + v) \times (K_{1,n} + v)) = mn + m + 2n - 2 > \underline{\alpha}((K_{1,m} + v) \times (K_{1,n} + v)) = mn + 2n$. Also, $\tau((K_{1,m} + v) \times (K_{1,n} + v)) = m + 6 > \underline{\tau}((K_{1,m} + v) \times (K_{1,n} + v)) = 8$.

**Theorem 3.2** 1. *If $G$ contains a complete matching and $H$ contains a perfect matching, then each component of $G \times H$ contains a complete matching.*

2. *If $G$ does not contain a complete matching and $H$ contains a perfect matching, then each component of $G \times H$ does not contain a complete matching.*

**Proof.** (1) is left to the reader. For (2), it suffices to show that if $G$ does not contain a complete matching, then each component of $G \times K_{r,r}$ does not contain a complete matching, where $r \geq 1$.

Let $G = (V \cup W, E)$, where $|V| = m \leq |W| = n$, and let $\tau(G) = m - i$, so $\alpha(G) = n + i$, where $i \geq 1$. Observe that $\underline{\tau}(G \times K_{r,r}) = 2 \cdot (m - i) \cdot r$ and $\underline{\alpha}(G \times K_{r,r}) = 2 \cdot (n + i) \cdot r$. Since $\underline{\tau} + \underline{\alpha} = |G \times K_{r,r}|$, it follows that $\tau = \underline{\tau}$ and $\alpha = \underline{\alpha}$. The two components of $G \times K_{r,r}$ being isomorphic [7], the matching number of each component is equal to $(m - i) \cdot r$ that is strictly smaller than the cardinality of the smaller partite set of each component.□

It will follow from the result of Theorem 3.3 that "$H$ contains a perfect matching" appearing in the statement of Theorem 3.2(2) cannot be weakened to "$H$ contains a complete matching."

**Theorem 3.3** *For every bipartite graph $G$, there exists an integer $r$ such that each component of $G \times K_{1,r}$ contains a complete matching.*

**Proof.** Let $G = (V \cup W, E)$ be a bipartite graph, and let $G$ have a vertex decomposition into complete bipartite subgraphs $K_{m_1,n_1}, \cdots, K_{m_k,n_k}$ where $m_i, n_i \geq 1$. (The existence of such a decomposition may be proved by induction on the number of vertices of $G$.) It may be assumed that $V$ can be partitioned into $V_1, \cdots, V_k$, and $W$ can be partitioned into $W_1, \cdots, W_k$ such that $|V_i| = m_i$, $|W_i| = n_i$ and $\langle V_i \cup W_i \rangle$ induces $K_{m_i,n_i}$. Thus $|V| = m_1 + \cdots + m_k$ and $|W| = n_1 + \cdots + n_k$. Let $r = \max\{m_1, \cdots, m_k, n_1, \cdots, n_k\}$, and let $V(K_{1,r}) = \{0\} \cup \{1, \cdots, r\}$.

Now consider the graph $G \times K_{1,r}$ that consists of two components having vertex sets $(V \times \{0\}) \cup (W \times \{1, \cdots, r\})$ and $(W \times \{0\}) \cup (V \times \{1, \cdots, r\})$. It is clear that $(V \times \{0\})$ and $(W \times \{0\})$ are the smaller partite sets of the respective components.

The component $\langle (V \times \{0\}) \cup (W \times \{1, \cdots, r\}) \rangle$ is vertex-decomposable into $k$ subgraphs

$$\langle (V_1 \times \{0\}) \cup (W_1 \times \{1, \cdots, r\}) \rangle, \cdots, \langle (V_k \times \{0\}) \cup (W_k \times \{1, \cdots, r\}) \rangle$$

that are, respectively, isomorphic to $K_{m_1,n_1 r}, \cdots, K_{m_k,n_k r}$. It follows that the matching number of this component is at least $m_1 + \cdots + m_k$. Since this figure coincides with the cardinality of the smaller partite set, this component contains a complete matching. An analogous statement holds with respect to the other component. □

**Remark:** For some $r \geq 1$, consider the graph $(2K_{1,m} + e) \times K_{1,r}$ that consists of two isomorphic components [7]. The reader may check to see that each component contains a complete matching if and only if $r \geq m$. Thus the choice of the integer $r$ in the proof of Theorem 3.3 is tight.

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