THE EXPECTATION OF INDEPENDENT DOMINATION NUMBER OVER RANDOM BINARY TREES

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ABSTRACT. We derive a formula for the expected value $\mu(2n+1)$ of the independent domination number of a random binary tree with $2n+1$ vertices and determine the asymptotic behavior of $\mu(2n+1)$ as n goes to infinity.

1. INTRODUCTION

Let D be a digraph. A subset S of vertices of D is a *dominating set* of D if for each vertex v not in S there exists a vertex u in S such that (u, v) is an arc of D. A subset I of vertices of D is an *independent set* of D if no two vertices of I are joined by an arc in D. The *independence number* $\beta(D)$ of D is the number of vertices in any largest independent subset of vertices in D. An *independent dominating set* (or *kernel*) of D is an independent and dominating set of D. The *independent domination number* $\alpha'(D)$ of D is the number of vertices in any smallest independent dominating subset of vertices in D. Note that D might have no independent dominating sets as we can see in 3-cycles. For definitions not given here see [3].

There are n^{n-2} labeled trees T with n vertices. Let $\nu(n)$ denote the expected value of the independence number $\beta(T)$ over the set of such trees. A. Meir and J. W. Moon showed in [5] that

$$\nu(n) = \sum_{k=1}^{n} \binom{n}{k} \left(\frac{-k}{n}\right)^{k-1}$$

for $n = 1, 2, \ldots$, and that

$$\frac{\nu(n)}{n} \to .5671\cdots .$$

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They also showed in [6] that the expected independence number $\nu(2n+1)$ of binary trees with $2n+1$ vertices is

$$\nu(2n+1) \sim (.585786\cdots)(2n+1).$$

We want to do similar work for the expected independent domination number of binary trees.

A *binary tree* is an oriented rooted tree that consists either of a single vertex or is constructed from an ordered pair of smaller binary trees by joining their roots from a new vertex that serves as the root in the tree thus formed. The vertices are not labeled, although the root is distinguished from the remaining vertices, and two such trees are regarded as the same if and only if they have the same ordered pair of branches with respect to their roots. Notice that every vertex is incident with either zero or two arcs that lead away from the root; this fact implies that such trees must have an odd number of vertices. There are $\binom{2n}{n}/(2n+1)$ binary trees T with $2n+1$ vertices. Let $\mu(2n+1)$ denote the expected value of $\alpha'(T)$ over the set of such binary trees.

For any number n and positive integer k, $< n >_k$ denotes the falling factorial $< n >_k = n(n-1)\cdots(n-k+1)$ and $< n >_0 = 1$ for any n.

Our goal is to show that

$$\mu(2n+1) = \sum_{\substack{0 \le k \le n \\ k \text{ is even}}} (k+1)2^k \frac{< n >_k}{< 2n >_k}$$

for $n = 0, 1, 2, \ldots$, and that

$$\frac{\mu(2n+1)}{2n+1} \longrightarrow \frac{1}{2}$$

as $n \to \infty$.

2. Some Lemmas

An *oriented tree* is a tree in which each edge is assigned a unique direction and an *oriented forest* is defined analogously. J. von Neumann and O. Morgenstern showed [7] that every digraph without cycles has a unique independent dominating set, and hence so does every oriented tree. The proof was long and involved. However, for oriented trees (actually, for oriented forest), we have the following short algorithmic proof.

Lemma 1. *Every oriented tree T has a unique independent dominating set.*

Proof. It is sufficient to prove this theorem for oriented forests and so we shall state an algorithm which finds an independent dominating set for an oriented forest T. The algorithm begins by putting vertices with indegree zero into an independent dominating set. Next we remove the vertices that are already in the independent dominating set together with their out-neighbors to get a new oriented forest and repeat this process for the new oriented forest.

Algorithm: Let $T_1 = T$ be the given oriented forest and let $K_0 = \emptyset$. Put $i = 1$ and go to (1).

(1) Choose the set S_i of all vertices with indegree zero in the oriented forest T_i and let $K_i = K_{i-1} \cup S_i$.

(2) Let T_{i+1} be the oriented subforest of T_i induced by $V - N^+[K_i]$, where $N^+[K_i]$ denotes the union of the out-neighbors of K_i and K_i itself. If T_{i+1} is an empty digraph, let $K = K_i$ and stop. Otherwise, return to (1) putting $i = i + 1$.

Let T' be an oriented tree with n vertices. Then the average indegree of T' is

$$\Big(\sum_{v \in T'} indeg(v)\Big)/n = \frac{n-1}{n} < 1.$$

Thus there is a vertex v of T' with indegree zero. This implies that the algorithm terminates after finitely many steps.

First we want to prove that K is an independent dominating set of T. It is obvious that K is a dominating set of T. To show that K is an independent set, we let u and v be in K. Assume there is an arc between u and v, say, uv in T. Then, by (1), u and v cannot be chosen for K in the same step. If u were chosen for K in an earlier step than the step in which v was chosen, then v would not be in K. Therefore v must be chosen for K in an earlier step i than the step in which u is chosen for K. For this, u should have been deleted in an earlier step than step i. Thus u is not in K, which contradicts the fact that u is in K.

Next we want to show that T has a unique independent dominating set. Suppose that T has two distinct independent dominating sets K and L. Then any one of K and L cannot be a proper subset of the other. Otherwise, one of them contains an arc and cannot be independent. Let v_1 be a vertex in $K - L$. Then there is a vertex v_2 in $L - K$ that dominates v_1 and next there is a vertex $v_3 \neq v_1$ in $K - L$ that dominates v_2. Repeat this argument. Then we have a sequence $\{v_i\}$ of vertices such that $v_i \neq v_{i+2}$. Let j be the smallest integer such that $v_j = v_k$ for some $k < j$. Then $v_k = v_j, v_{j-1}, \dots, v_k$ is a cycle of length at least 3 in the underlying tree of T. This contradicts that T is an oriented tree. □

Let T be a binary tree. If we remove the root r of T, along with all arcs incident from r, we obtain a (possibly empty) ordered pair of disjoint binary trees, or 1-*branches*, whose roots were originally joined from r. Let y_{2n+1} denote the number of binary trees with $2n+1$ vertices. Then we know that $y_1 = 1$ and that

$$y_{2n+1} = \sum_{\substack{i+j=2n \\ i \text{ and } j \text{ are odd}}} y_i y_j \tag{2.1}$$

for $n \geq 1$. If we let

$$y = y(x) = \sum_{n=0}^{\infty} y_{2n+1} x^{2n+1}$$

be the ordinary generating function for binary trees, then it follows from equation (2.1) that

$$\begin{aligned} y &= x + x \sum_{n=1}^{\infty} \Big(\sum_{\substack{i+j=2n \\ i \text{ and } j \text{ are odd}}} (y_i x^i)(y_j x^j) \Big) \\ &= x(1+y^2) && (2.2) \\ &= \frac{1}{2x}[1 - (1-4x^2)^{1/2}] && (2.3) \\ &= \sum_{n=0}^{\infty} \frac{\binom{2n}{n}}{n+1} x^{2n+1}. && (2.4) \end{aligned}$$

This, of course, is a well-known argument (see [2]).

On the other hand, we can find the generating function y for binary trees using a slightly different approach. Let T be a binary tree with order at least 3 and root r, and let T_3 denote the binary subtree of T with 3 vertices and the same root r. If we remove T_3 of T, along with all arcs incident with vertices in T_3, we obtain an ordered 4-tuple (B_1, B_2, B_3, B_4) of disjoint binary trees, or *2-branches*, satisfying the following three conditions:

(i) Both B_1 and B_2 are either empty binary trees or both non-empty binary trees.

(ii) Both B_3 and B_4 are either empty binary trees or both non-empty binary trees.

(iii) The roots of B_1 and B_2 were originally joined from the left leaf of T_3 and the roots of B_3 and B_4 from the right leaf of T_3.

Now, using the same technique used to derive equation (2.2), we have

$$y = x + x^3(1 + 2y^2 + y^4) \tag{2.5}$$

which is equivalent to $y = x(1+y^2)$.

Lemma 2. *Let T be a binary tree. Then the independent domination number of T is one more than the sum of the independent domination numbers of all 2-branches of T.*

Proof. This follows immediately from the algorithm in Lemma 1. □

For $1 \le k \le 2n+1$, let $y_{2n+1,k}$ denote the number of binary trees with $2n+1$ vertices whose independent domination number is exactly k. Let

$$Y = Y(x,z) = \sum_{n=0}^{\infty}\Big(\sum_{k=1}^{2n+1} y_{2n+1,k}z^k\Big)x^{2n+1}.$$

It follows by a slight extension of the argument used to establish equation (2.5) that

$$Y = zx + zx^3(1 + 2Y^2 + Y^4). \tag{2.6}$$

The factor z is present in equation (2.6) because of Lemma 2. Here we note that $y = Y(x,1)$.

Lemma 3. *Let $\mu(2n+1)$ denote the expected value of the independent domination numbers of the y_{2n+1} binary trees with $2n+1$ vertices and define*

$$M = M(x) = \sum_{n=0}^{\infty} \mu(2n+1)y_{2n+1}x^{2n+1}.$$

Then we have

$$M = \frac{y}{1 - 4x^2y^2}.$$

Proof. It is easy to see that

$$M = M(x) = \sum_{n=0}^{\infty} \mu(2n+1)y_{2n+1}x^{2n+1} = Y_z(x,1).$$

If we differentiate both sides of equation (2.6) with respect to z, set $z = 1$, use the fact that equations (2.2) and (2.5) are equivalent, and solve for $Y_z(x,1)$, we find the required result. □

3. A Formula for $\mu(2n+1)$

We know that $M(x)$ is the ordinary generating function for the total sum of the independent domination numbers of binary trees. Therefore, using Maclaurin expansion of $M(x)$, we could find directly the expected value $\mu(2n+1)$ of the independent domination numbers of binary trees for small n. Actually, using (2.3), we have

$$M(x) = \frac{2x}{\sqrt{1-4x^2}(1+\sqrt{1-4x^2})(2-\sqrt{1-4x^2})}, \tag{3.7}$$

and routine use of *Mathematica* produces

$$\begin{aligned} M(x) &= x + x^3 + 6x^5 + 17x^7 + 66x^9 + 234x^{11} + 876x^{13} \\ &\quad +3265x^{15} + 12330x^{17} + 46766x^{19} + \cdots . \end{aligned}$$

Here is a table for $\mu(2n+1)$ and $\mu(2n+1)/(2n+1)$. The entries for $2n+1 \le 9$ were verified by drawing all of the diagrams for binary trees with up to 9 vertices.

TABLE 1. Values of $\mu(2n+1)$ and $\mu(2n+1)/(2n+1)$

$2n+1$	y_{2n+1}	$\mu(2n+1)y_{2n+1}$	$\mu(2n+1)$	$\frac{\mu(2n+1)}{2n+1}$
1	1	1	1/1=1.00	1
3	1	1	1/1=1.00	.3333
5	2	6	6/2=3.00	.6000
7	5	17	17/5=3.40	.4857
9	14	66	66/14=4.71	.5238
11	42	234	234/42=5.57	.5064
13	132	876	876/132=6.63	.5104
15	429	3265	3265/429=7.61	.5073
17	1430	12330	12330/1430=8.62	.5071
19	4862	46766	46766/4862=9.61	.5062

Furthermore, we can derive a reasonably explicit formula for $\mu(2n+1)$ as follows.

Theorem 4. *The expected value of the independent domination numbers of binary trees with* $2n+1$ *vertices is*

$$\mu(2n+1) = \sum_{\substack{0\le k\le n\\ k \text{ is even}}} (k+1)2^k \frac{<n>_k}{<2n>_k}.$$

Proof. The following identity appears in [8]:

$$\left(\frac{1-\sqrt{1-4x}}{2x}\right)^n = \sum_{k=0}^{\infty} \frac{n(2k+n-1)!}{k!(k+n)!}x^k \tag{3.8}$$

for integer $n \ge 1$. Using (2.3) and (3.8), we have

$$\begin{aligned} y(2xy)^n &= 2^n\left(\frac{1-\sqrt{1-4x^2}}{2x}\right)^{n+1} x^n \\ &= 2^n\left(\frac{1-\sqrt{1-4x^2}}{2x^2}\right)^{n+1} x^{2n+1} \\ &= 2^n\left(\sum_{k=0}^{\infty} \frac{(n+1)(2k+n)!}{k!(k+n+1)!}x^{2k}\right)x^{2n+1} \\ &= (n+1)2^n \sum_{k=n}^{\infty} \binom{2k+1-n}{k+1} \frac{x^{2k+1}}{2k+1-n}. \end{aligned}$$

Hence we have

$$\begin{aligned}
M(x) &= \frac{y}{1-4x^2y^2} \\
&= \sum_{m=0}^{\infty} y(2xy)^{2m} \\
&= \sum_{m=0}^{\infty} (2m+1)2^{2m} \sum_{k=2m}^{\infty} \binom{2k+1-2m}{k+1} \frac{x^{2k+1}}{2k+1-2m}. \qquad (3.9)
\end{aligned}$$

Therefore, by equating the coefficients of x^{2n+1} in both sides of (3.9), we have

$$\mu(2n+1)\frac{\binom{2n}{n}}{n+1} = \sum_{\substack{0\le k\le n \\ k \text{ is even}}} (k+1)2^k \frac{\binom{2n+1-k}{n+1}}{2n+1-k}$$

and hence

$$\mu(2n+1) = \sum_{\substack{0\le k\le n \\ k \text{ is even}}} (k+1)2^k \frac{<n>_k}{<2n>_k}.$$

This completes the proof. □

4. The Asymptotic Behavior of $\mu(2n+1)$

To determine the asymptotic behavior of $\mu(2n+1)/(2n+1)$, we need the following lemma, which is a slight modification of Theorem 2 in [1]; we omit the proof.

Lemma 5. *Let $A(u) = \sum_{n=0}^{\infty} a_n u^n$ and $B(u) = \sum_{n=0}^{\infty} b_n u^n$ be power series with radii of convergence $\rho_1 \ge \rho_2$, respectively. Suppose that $A(u)$ converges absolutely at $u = \rho_1$. Suppose that $b_n > 0$ for all n and that b_{n-1}/b_n approaches a limit b as $n \to \infty$. If $\sum_{n=0}^{\infty} c_n u^n = A(u)B(u)$, then $c_n \sim A(b)b_n$.*

Recall that our generating function $M(x)$ has alternate zero coefficients. To eliminate these, we substitute u for x^2 and define

$$M_*(u) = \sum_{n=0}^{\infty} \mu(2n+1) y_{2n+1} u^n.$$

Now we can state the main result of this paper.

Theorem 6. *The expected value of the independent domination numbers of binary trees with $2n+1$ vertices is*

$$\mu(2n+1) \sim \frac{1}{2}(2n+1).$$

Proof. It quickly follows from (3.7) that $M_*(u)$ becomes

$$M_*(u) = \frac{2}{\sqrt{1-4u}(1+\sqrt{1-4u})(2-\sqrt{1-4u})}.$$

Now we let

$$A(u) = \frac{2}{(1+\sqrt{1-4u})(2-\sqrt{1-4u})}$$

and let

$$B(u) = \frac{1}{\sqrt{1-4u}}.$$

Note that $A(u)$ can be rewritten as:

$$A(u) = \frac{2}{3}\Big(\frac{1-\sqrt{1-4u}}{4u} + \frac{2}{3+4u} + \frac{1}{3+4u}\sqrt{1-4u}\Big),$$

which has a power series expansion in u with radius of convergence $1/4$. Moreover, it is not too hard to see this power series converges absolutely at $u = 1/4$ using the fact that $\sqrt{1-4u}$ has a power series expansion in u with radius of convergence $1/4$ which converges absolutely at $u = 1/4$ (see, for example, p.426, [4]). On the other hand, we have

$$B(u) = \frac{1}{\sqrt{1-4u}} = \sum_{n=0}^{\infty}(-4)^n\binom{-\frac{1}{2}}{n}u^n$$

for $|u| < 1/4$. If we let

$$b_n = (-4)^n\binom{-\frac{1}{2}}{n},$$

it is easy to check that $b_{n-1}/b_n \to 1/4$ as $n \to \infty$ and that $b_n > 0$ for all n. Note that $M_*(u) = A(u)B(u)$. Therefore from Lemma 5 we have

$$\mu(2n+1)y_{2n+1} \sim A(1/4)b_n = b_n$$

and hence

$$\begin{aligned}\mu(2n+1) \quad &\sim \quad \frac{b_n}{y_{2n+1}} = (-4)^n\binom{-\frac{1}{2}}{n}\frac{n+1}{\binom{2n}{n}} = n+1 \\ &\sim \quad \frac{1}{2}(2n+1).\end{aligned}$$

This completes the proof. □

As we mentioned earlier, the expected independence number $\nu(2n+1)$ of binary trees with $2n+1$ vertices is

$$\nu(2n+1) \sim (.585786\cdots)(2n+1).$$

It is easy to see that $\alpha'(T) \leq \beta(T)$ for any binary tree T. Our result

$$\mu(2n+1) \sim (.5)(2n+1)$$

is consistent with these two facts.

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