Broadcasting in Odd Graphs *

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Abstract

Broadcasting refers to the process of information dissemination in a communication network whereby a message is to be sent from a single originator to all members of the network, subject to the restriction that a member may participate in only one message transfer during a given time unit. In this paper we present a family of broadcasting schemes over the odd graphs, O_{n+1}. It is shown that the broadcast time of O_{n+1}, $b(O_{n+1})$, is bounded by $2n$. Moreover, the conjecture that $b(O_{n+1}) = 2n$ is put forward, and several facts supporting this conjecture are given.

Keyword(s): Broadcasting; Odd Graphs; Networks

1 Introduction

The *Broadcasting* problem on a network, also called OTA (One-To-All), is the process of disseminating a message from one initial node to all others in the network. The message is known by this initial node, called the *originator* node, but not by any other one in the net (see [4], [7]). The process must be done as quickly as possible following a predefined model which conforms to the following rules: one unit of time is needed to make a call from one node to another, only one call can be done by the same node in a unit of time and calls are made only between adjacent nodes. A *broadcast scheme* is a formal description of this process.

We represent a communication network as a connected undirected graph $G = (V, E)$, where the node set V represents the set of processors and the edge

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set E represents the set of bidirectional communication lines between processors (see [1]).

Given a connected graph G and a vertex $u \in V$, the *broadcast time of u*, denoted by $b(u)$, is the minimum number of time units required to complete broadcasting from u. The *broadcast time b(G) of the graph G* is the maximum broadcast time of any vertex u of G. The problem of determining the value $b(G)$ is known to be NP-HARD (see [8]). Nevertheless the values of $b(G)$ are known for many common interconnection networks (like grids and hypercubes).

It is easy to see that, after t units of time, the maximum number of members which may have received the message is 2^t, including the originator. This fact implies that the minimum number of time units required to broadcast a message in a set of n members is $\lceil \log_2 n \rceil$. A *minimal broadcast network* is a communication network with n nodes in which a message can be broadcasted in $\lceil \log_2 n \rceil$ time units, regardless of the originator.

In this paper, we study $b(G)$ when G is an odd graph and we compare this value with $\lceil \log_2 |V| \rceil$, where $|V|$ is the order of the graph G. One important reason to consider this family of graphs is its vertex-transitivity, which allows each node in the associated network to execute the same communication software. Odd graphs have been used for many applications in computer systems. There are some results on analysis of odd graphs for building communication architecture for large multiprocessor systems (see [5]). The topology defined by odd graphs admits simple distributed routing algorithms. A complete analysis of odd graphs in [5] shows that the resulting networks based on it possess many good features which makes them competitives with various well-known architectures. Furthermore, these graphs have a remarkable property of partitioning into symmetrical regions, which is similar to the one discovered for *bisectional networks* in [6].

This paper is organized as follows: In the next section we recall the definition and some properties of the odd graphs (see [3], [9]). A family of broadcasting protocols is defined in Section 3. This family makes use of the symmetries of odd graphs. Some results on the broadcasting time of these protocols are given, in particular a sharp bound is found. In Section 4 the minimality in time of a particular protocol belonging to the above family is conjectured. Finally some facts supporting this conjecture are given.

2 Odd graphs

An *odd graph*, denoted by O_{n+1}, has as its set of vertices the n-subsets of a $(2n+1)$-set X; for instance the set $X = \{1, 2, \cdots, 2n+1\}$ can be considered. Two vertices $U, W \subset X$ are adjacent iff $U \cap W = \emptyset$. Therefore, O_{n+1} is a regular graph with order $\binom{2n+1}{n}$ and $\Delta = n+1$. The Figure 1 shows O_4.

The elements of the set X are called colours. The adjacency rule in O_{n+1} allows us the labelling of every edge with the unique colour of X which is not in its end-vertices. This fact, in turn, leads to the identification of the paths by the colours of their edges, see [3]. We note that for each colour of X there exist

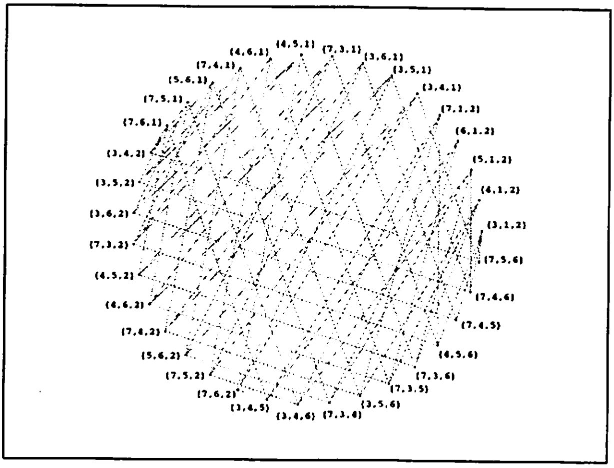

Figure 1: O_4

exactly $\binom{2n}{n}$ edges of O_{n+1} labelled with this colour.

Some well-known properties of O_{n+1} follow from this colouring:

- **P1** If a path of even length coloured by

$$c_1\, d_1\, c_2\, d_2\, \cdots c_k\, d_k,$$

leads from a vertex U to a vertex W, then

$$W = [U \cup' \{c_i\}] \backslash (\cup' \{d_i\}),$$

where $\cup'$ means that repeated elements are considered as many times as they appear.

- **P2** Analogously, if a path of odd length $2k + 1$ coloured by

$$c_1\, d_1\, c_2\, d_2\, \cdots c_k\, d_k\, c_{k+1},$$

joins a vertex U with a vertex W, then

$$W = [(X - U) \cup' \{d_i\}] \backslash (\cup' \{c_i\}).$$

By avoiding repetitions of the c_i's and d_i's we get the following result about shortest paths:

- **P3** Let U and W be two vertices of O_{n+1} and let $m = |U \cap W|$. Then, the shortest paths of even length between U and W have length $2(n - m)$ and the shortest paths of odd length between U and W have length $2m + 1$.

The following known facts about O_{n+1} are trivial consequences of P3:

- **P4** The shortest odd cycle length of O_{n+1} is $2n+1$. Moreover from P2 it follows that cycles of length $2n+1$ are those using the $2n+1$ colours of X.

- **P5** The distance between two nodes U and W such that $|U \cap W| = m$ is

$$d(U,W) = min\{2m+1, 2(n-m)\} = \begin{cases} 2m+1 & \text{if } m < \frac{n}{2}, \\ 2(n-m) & \text{if } m \geq \frac{n}{2}. \end{cases}$$

- **P6** The diameter of O_{n+1} is

$$D(O_{n+1}) = \max\{d(U,W) \,/ U, W \in V(O_{n+1})\} = n.$$

- **P7** O_{n+1} is distance-transitive (see [2], [3]). In particular, it is vertex-transitive.

From now on we call odd (even) paths those paths with odd (even) length value.

3 A Family of Broadcasting Protocols in Odd Graphs

We can choose any vertex of O_{n+1} as the originator of a protocol, since the odd graphs are vertex-transitive. We take $A = \{1, 2, \ldots, n\}$ as originator.

Let B be a vertex of O_{n+1} with $|A \cap B| = m$. From the sets A and B a classification of elements of X can be made as follows:

$$A \setminus B = \{a_1, a_2, \cdots, a_{n-m}\}, \quad A \cap B = \{\alpha_1, \alpha_2, \cdots, \alpha_m\},$$
$$B \setminus A = \{b_1, b_2, \cdots, b_{n-m}\}, \quad X \setminus (A \cup B) = \{\beta_1, \beta_2, \cdots, \beta_{m+1}\}.$$

From P1, P2 it follows that the path coloured by $b_1\, a_1\, b_2\, a_2 \cdots b_{n-m}\, a_{n-m}$ is an even shortest path from A to B, and $\beta_1\, \alpha_1\, \beta_2\, \alpha_2 \cdots \beta_m\, \alpha_m\, \beta_{m+1}$ is an odd shortest path from A to B (see [9]).

So the same vertex B can be attained from the originator by two different vertex-disjoint paths, an even and an odd shortest path respectively. We have used this fact to generate our family of protocols on O_{n+1}.

From now on, we assume, without loss of generality, that

$$a_1 < a_2 < \cdots < a_{n-m}, \quad \alpha_1 < \alpha_2 < \cdots < \alpha_m,$$
$$b_1 < b_2 < \cdots < b_{n-m}, \quad \beta_1 < \beta_2 < \cdots < \beta_{m+1}.$$

Definition 1 *Let $\mathcal{P}$ be the family of broadcasting protocols over O_{n+1} satisfying the following two rules:*

[R1] *The message is disseminated on the network from the originator $A = \{1, 2, \ldots, n\}$ only through paths coloured by $x_1\, y_1\, x_2\, y_2\, x_3\, y_3 \cdots$ where $x_i \in X \setminus A$, $y_i \in A$ and satisfying*

$$x_1 < x_2 < x_3 < \cdots,$$
$$y_1 < y_2 < y_3 < \cdots.$$

Given a path γ as in [R1] with final vertex B, we call a vertex C a γ-adjacency of B if the path obtained by adding the edge $\{B, C\}$ to γ satisfies [R1].

[R2] *Let B be a vertex of O_{n+1} informed from A at time t, through a path γ as in [R1]. Let B', B'' be two γ-adjacencies of B uninformed at time less than or equal to t. The vertex B cannot inform B' sooner than B'' if the colour of the edge $\{B, B'\}$ is greater than the colour of the edge $\{B, B''\}$.*

It is obvious that $\mathcal{P} \neq \emptyset$ because we can reach any vertex $N \neq A$ from A through exactly two different paths, odd and even respectively, and satisfying the rule [R1]. We remark that [R1] implies that the paths used are the shortest ones within the same parity class.

Proposition 1 *There exists a unique protocol $P_0 \in \mathcal{P}$ such that any vertex B which is informed at time t through a path γ and has uninformed γ-adjacencies informs one of these nodes at time $t+1$. Moreover, the time which the vertex B is informed according to P_0, $t_0(B)$, satisfies*

$$t_0(B) = \begin{cases} -n + \alpha_m + \beta_{m+1} & \text{if } B \text{ is informed through an odd path,} \\ -n + a_{n-m} + b_{n-m} & \text{if } B \text{ is informed through an even path.} \end{cases}$$

In any case

$$t_0(B) = min\{-n + \alpha_m + \beta_{m+1}, -n + a_{n-m} + b_{n-m}\}$$

and

$$\max\{t_0(B)/B \in V(O_{n+1}\} = 2n.$$

Proof. First, we will prove that the expression of $t_0(B)$ is true when the protocol P_0 is well defined.

Let B be an informed vertex through a path γ according to [R1] in time t. Assume that γ is an odd path. Then

$$\gamma = \beta_1 \, \alpha_1 \cdots \beta_m \, \alpha_m \, \beta_{m+1}.$$

Denote by $B_1, B_1', B_2, B_2', \cdots, B_m, B_m', B_{m+1} = B$ the vertices of the path γ in the same order as they have been informed. Since P_0 informs the γ-adjacencies of minimum colour, we have

$$t_0(B_1) = \beta_1 - n, \quad t_0(B_1') - t_0(B_1) = \alpha_1.$$

Similarly, for all $1 \leq i \leq m$ we have $t_0(B_{i+1}) - t_0(B_i') = \beta_{i+1} - \beta_i$ and for all $1 \leq i \leq m-1$ we have $t_0(B_{i+1}') - t_0(B_{i+1}) = \alpha_{i+1} - \alpha_i$.

So it follows that

$$\begin{aligned} t_0(B) &= \beta_1 - n + \beta_2 - \beta_1 + \cdots + \beta_{m+1} - \beta_m \\ &\quad + \alpha_1 + \alpha_2 - \alpha_1 + \cdots + \alpha_m - \alpha_{m-1} \\ &= -n + \beta_{m+1} + \alpha_m. \end{aligned}$$

On the other hand, if γ is an even path, we can proceed in the same form and we obtain

$$t_0(B) = -n + a_{n-m} + b_{n-m}.$$

If we prove the existence of the protocol P_0, then the uniqueness is obvious. To prove the existence, it is sufficient to see that if a vertex B is reached in the same time through the two paths and it has γ-adjacencies, then all the γ-adjacencies of B are already informed.

Now we will see that if B is a vertex informed at the same time t through the two paths

$$\begin{aligned} \gamma_{even} &= b_1\, a_1\, b_2\, a_2 \cdots b_{n-m}\, a_{n-m}, \\ \gamma_{odd} &= \beta_1\, \alpha_1\, \beta_2\, \alpha_2 \cdots \beta_m\, \alpha_m\, \beta_{m+1}, \end{aligned}$$

then all the γ-adjacencies of B are already informed.

The equality of times through the two paths implies

$$t_0(B) = -n + \alpha_m + \beta_{m+1} = -n + a_{n-m} + b_{n-m}.$$

We know that $\max\{\alpha_m, a_{n-m}\} = n$ and $\max\{\beta_{m+1}, b_{n-m}\} = 2n+1$. By the above identity of times, the values α_m and β_{m+1} (resp. a_{n-m} and b_{n-m}) can not take the maximum values simultaneously. We must study the following cases:

- $\alpha_m = n$ and $b_{n-m} = 2n+1$. Then B has no γ-adjacencies.
- $\beta_{m+1} = 2n+1$ and $a_{n-m} = n$. From the vertex B, following paths according to the rule [R1] we have two different types of γ-adjacencies:

  1. Let B' be a γ_{even}-adjacency of B, then the colour of the edge $\{B, B'\}$ is β_i for some $b_{n-m} < \beta_i$. The two paths connecting A and B' are

     $$\begin{aligned} &\beta_1\, \alpha_1\, \beta_2\, \alpha_2 \cdots \beta_{i-1}\, \alpha_{i-1}\, \beta_{i+1}\, \alpha_i \cdots \beta_{m+1}\, \alpha_m, \\ &b_1\, a_1\, b_2\, a_2 \cdots b_{n-m}\, a_{n-m}\, \beta_i. \end{aligned}$$

     So $t_0(B') = \min\{-n + \alpha_m + \beta_{m+1}, -n + a_{n-m} + \beta_i\}$.
     As

     $$-n + \alpha_m + \beta_{m+1} = -n + a_{n-m} + b_{n-m} < -n + a_{n-m} + \beta_i$$

     we have $t_0(B') = -n + \alpha_m + \beta_{m+1} = t_0(B)$, and B' has already been informed.

  2. Let B' be a γ_{odd}-adjacency of B and let $a_i \in A$ be the colour of the edge $\{B, B'\}$ with $\alpha_m < a_i$. The two paths connecting the vertices A and B' are

     $$\begin{aligned} &\beta_1\, \alpha_1\, \beta_2\, \alpha_2 \cdots \beta_m\, \alpha_m\, \beta_{m+1}\, a_i, \\ &b_1\, a_1 \cdots b_i\, a_{i+1}\, b_{i+1} \cdots b_{n-m-1}\, a_{n-m}\, b_{n-m}. \end{aligned}$$

     Then $t_0(B') = \min\{-n + a_i + \beta_{m+1}, -n + a_{n-m} + b_{n-m}\}$. By $\alpha_m < a_i$ we have

     $$-n + \alpha_m + \beta_{m+1} = -n + a_{n-m} + b_{n-m} < -n + \beta_{m+1} + a_i.$$

     Then $t_0(B') = t_0(B)$ and B' is an already informed vertex.

By the above proved equality of times, for all vertices B we have $t_0(B) \leq -n+n+(2n+1) = 2n+1$. If $\alpha_m+\beta_{m+1} = 3n+1$ (resp. $a_{n-m}+b_{n-m} = 3n+1$), then $-n + a_{n-m} + b_{n-m} \leq 2n-1$ (resp. $-n + \alpha_m + \beta_{m+1} \leq 2n-1$). So it follows that $t_0(B) \leq 2n$. As $t_0(B) = 2n$, if $B = \{1, 2, \cdots, n-1, 2n\}$, then $\max\{t_0(B)/B \in V(\bar{O}_{n+1})\} = 2n$. □

Figure 2 shows the broadcasting scheme P_0 for the graph O_4. The vertices are grouped by columns, according to the time needed to get the message to them from the originator vertex $\{1,2,3\}$.

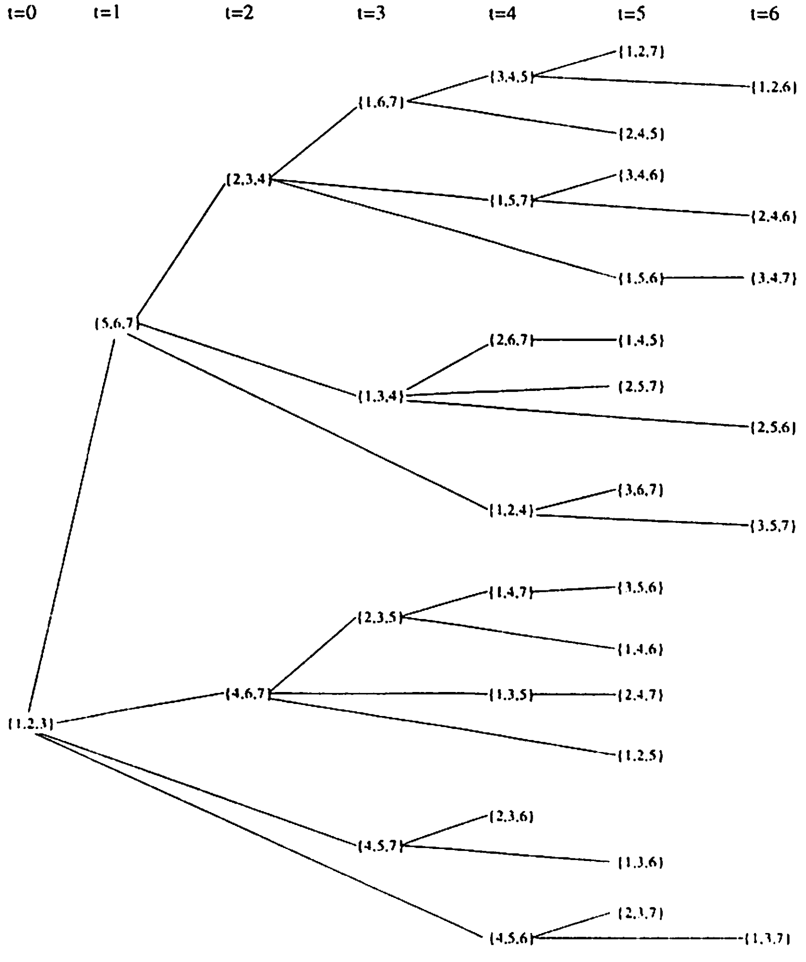

Figure 2: The protocol P_0 over O_4

Corollary 1. $b(O_{n+1}) \leq 2n$.

Let $V_{2n-i} = \{B \in O_{n+1} / t_0(B) = 2n - i\}$, for any $0 \leq i \leq 2n$. We denote by $V_{2n-i,2}$ the elements of V_{2n-i} which are reached at the same time $t = 2n - i$ through two different paths according to P_0. Let $v_{2n-i} = |V_{2n-i}|$ and $v_{2n-i,2} = |V_{2n-i,2}|$. Note that $\binom{k}{m} = 0$ for any integer value $m < 0$.

Proposition 2 *With the above notation, we have:*

i) $v_{2n-i,2} = \binom{2n-2i-2}{n-2i-1}$. *In particular,* $v_{2n-i,2} > 0$ *if and only if* $i \leq \lfloor\frac{n-1}{2}\rfloor$.

ii) $v_{2n} = v_{2n,2}$.

iii) If $i \leq n-1$, *then*

$$v_{2n-i} = \binom{2n-2i-2}{n-2i-1} + 2\sum_{j=n-i-1}^{n-2}\binom{2n-i-2}{j} + 2\sum_{j=2n-2i-1}^{2n-i-2}\binom{j}{j-n+1}.$$

iv) If $n \leq i \leq 2n-1$, *then* $v_{2n-i} = 2^{2n-i-1}$.

Proof.
i) Vertices receiving the message through two different paths at the same time $t = 2n - i$ are those of the two following options

option	α_m	β_{m+1}	a_{n-m}	b_{n-m}
(a)	n	$2n-i$	$n-i-1$	$2n+1$
(b)	$n-i-1$	$2n+1$	n	$2n-i$

Option (a) includes the vertices B with $n-i, \cdots, n, 2n-i+1, \cdots 2n+1 \in B$ and $n-i-1$, $2n-i \notin B$. The number of cases of this type is the same as the number of subsets from $\{1, \cdots, n-i-2, n+1, \cdots, 2n-i-1\}$ with $n-2i-2$ elements, that is $\binom{2n-2i-3}{n-2i-2}$. Similarly the number of cases of option (b) is $\binom{2n-2i-3}{n-2}$. So

$$v_{2n-i,2} = \binom{2n-2i-2}{n-2i-1}.$$

ii) It is easy to see that $v_{2n} = v_{2n,2}$.

iii) Vertices in V_{2n-i} are those satisfying $\min\{\alpha_m + \beta_{m+1}, a_{n-m} + b_{n-m}\} = 3n - i$. The cases related to $V_{2n-i,2}$ have been already calculated in *i*). Now we compute those cases such that $\alpha_m + \beta_{m+1} \neq a_{n-m} + b_{n-m}$. As $n \in \{\alpha_m, a_{n-m}\}$ and $2n+1 \in \{\beta_{m+1}, b_{n-m}\}$, we distinguish four more subcases:

Subcase 1. $\alpha_m = n$, $\beta_{m+1} = 2n+1$, so

$$(\alpha_m, \beta_{m+1}, a_{n-m}, b_{n-m}) = (n, 2n+1, n-k, 2n-i+k)$$

with $1 \leq k \leq i$, and we obtain the number of these subcases

$$\binom{2n-i+k-1-(k+1)}{n-(k+1)} = \binom{2n-i-2}{n-k-1}, \quad 1 \leq k \leq i.$$

Subcase 2. $a_{n-m} = n$, $b_{n-m} = 2n+1$, so

$$(\alpha_m, \beta_{m+1}, a_{n-m}, b_{n-m}) = (n-k, 2n-i+k, n, 2n+1)$$

with $1 \leq k \leq i$ and the number of these subcases is

$$\binom{2n-i-2}{n-(2+i-k)} = \binom{2n-i-2}{n+k-2-i}, \quad 1 \leq k \leq i.$$

Note that the number of subcases 1 is the same as the number of subcases 2, and their sum is

$$2 \sum_{j=n-i-1}^{n-2} \binom{2n-i-2}{j}.$$

Subcase 3. $\alpha_m = n$, $b_{n-m} = 2n+1$. We distinguish two more subcases, (3a) and (3b). Subcase (3a) corresponds to the vertices of the form

$$(\alpha_m, \beta_{m+1}, a_{n-m}, b_{n-m}) = (n, 2n-k, n-i-1, 2n+1),$$

with $0 \leq k \leq i-1$. The total number of these vertices is

$$\binom{2n-i-k-3}{n-i-k-2}, \quad 0 \leq k \leq i-1.$$

Subcase (3b) corresponds to those vertices of the form

$$(\alpha_m, \beta_{m+1}, a_{n-m}, b_{n-m}) = (n, 2n-i, n-k, 2n+1)$$

with $1 \leq k \leq i$. Its total number is

$$\binom{2n-i-k-2}{n-i-k-1}, \quad 1 \leq k \leq i.$$

Note that the number of (3a) subcases is the same as those of (3b).

Subcase 4. $a_{n-m} = n$, $\beta_{m+1} = 2n+1$. We distinguish two subcases, (4a) and (4b), as above. Subcase (4a) corresponds to the vertices of the form

$$(\alpha_m, \beta_{m+1}, a_{n-m}, b_{n-m}) = (n-i-1, 2n+1, n, 2n-k),$$

with $0 \leq k \leq i-1$. The total number of these vertices is

$$\binom{2n-i-k-3}{n-2}, \quad 0 \leq k \leq i-1.$$

Vertices of subcase (4b) are those of the form

$$(\alpha_m, \beta_{m+1}, a_{n-m}, b_{n-m}) = (n-k, 2n+1, n, 2n-i),$$

with $1 \leq k \leq i$. The total number of these vertices is

$$\binom{2n-i-k-2}{n-2}, \quad 1 \leq k \leq i.$$

The number of subcases of (4a) is the same as in (4b).

Adding all the possibilities in subcase 3 and subcase 4 we obtain

$$2\sum_{k=0}^{i-1}\binom{2n-i-k-2}{n-1} = 2\sum_{j=2n-2i-1}^{2n-i-2}\binom{j}{j-n+1}.$$

iv) It is a consequence of the fact that for $t \leq n$ all informed vertices have uninformed γ-adjacencies.□

The following situation appears in our protocol P_0: There exist different vertices B_1, B_2 such that they inform the same vertex B at time t. A suitable prearrangement of vertices would avoid this situation according to [R1] and [R2]. So a new protocol of the family $\mathcal{P}$ is obtained and it improves the values v_{2n-i}. The following Proposition assures that the time of this new protocol is the same as the previous P_0.

Proposition 3 *The time of any protocol $P \in \mathcal{P}$ is at least $2n$.*

Proof.
Protocol P_0 can only be improved in $\mathcal{P}$ if the possibility of informing a vertex B simultaneously through two different paths is avoided. In this case, the number of vertices which reduce the time needed to be informed is at most $\sum_{i>0} v_{2n-i,2}$, by Proposition 2. It is easy to see that this value is at most $v_{2n} = v_{2n,2}$. □

4 Conclusions

In this article we have obtained a good bound on the broadcasting time in odd graphs. This fact follows from the time needed to broadcast O_{n+1} by P_0, $2n$ units of time. Note that P_0 doubles the number of informed vertices in each step until the time $t = n+1$. Probably P_0 is a minimal time protocol in O_{n+1}. However it is difficult to prove that any other protocol defined in O_{n+1} will take more than $2n$ time units (and P_0 would be optimal in this sense). The following Proposition and Table reinforce this conjecture.

Proposition 4 *The time of the protocol P_0 over O_{n+1} is asymptotically equivalent to*

$$\left\lceil \log_2 \binom{2n+1}{n} \right\rceil$$

as $n \to \infty$.

Proof.
By Stirling's identity we have the following asymptotic equivalences as $n \to \infty$

$$\binom{2n+1}{n}^{1/2n} \sim \left(\frac{1}{\sqrt{2\pi n}}\right)^{\frac{1}{2n}} \left(\frac{2n+1}{n+1}\right)^{\frac{3}{4n}} \left(\frac{(2n+1)^2}{n^2+n}\right)^{\frac{1}{2}} \sim 2.$$

So $2n \sim \log_2 \binom{2n+1}{n}$ as $n \to \infty$. □

The following Table shows the variation of the expression $2n - \lceil \log_2 \binom{2n+1}{n} \rceil$ for different values of n.

n	$2n - \lceil \log_2 \binom{2n+1}{n} \rceil$
$n \leq 3$	0
$4 \leq n \leq 19$	1
$20 \leq n \leq 80$	2
$81 \leq n \leq 314$	3
$315 \leq n \leq 1302$	4
$1303 \leq n \leq 5213$	5
$5214 \leq n \leq 20859$	6
$n \geq 20860$	≥ 7

Note that for $n = 20859$ the number of vertices of O_{n+1} is about 1.8×10^{12556}.

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