The mapping $\overline{C} : G \longrightarrow C(\overline{G})$, a new graph theoretic map

Christian Lopez* and Ute A. Mueller
Department of Mathematics
Edith Cowan University
Mt. Lawley, WA 6050
AUSTRALIA

Abstract

The cycle graph $C(H)$ of a graph H is the edge intersection graph of all induced chordless cycles of H. We investigate iterates of the mapping $\overline{C} : G \to C(\overline{G})$ where C denotes the map that associates to a graph its cycle graph. We call a graph G vanishing under $\overline{C}$ if $\overline{C}^n(G) = \emptyset$ for some n, otherwise G is called $\overline{C}$-persistent. We call a graph G expanding under $\overline{C}$, if $\left|\overline{C}^n(G)\right| \longrightarrow \infty$, as $n \longrightarrow \infty$. We show that the lowest order of a $\overline{C}$-expanding graph is 6 and determine the behaviour under $\overline{C}$ of some special graphs, including trees, null graphs, cycles and complete bipartite graphs.

1 Introduction

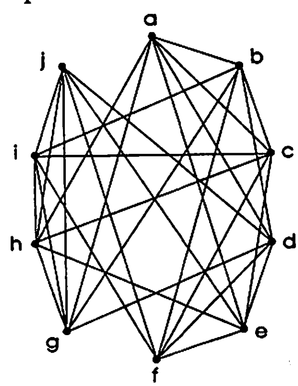
We consider finite, simple undirected graphs only. For definitions and notation not given here refer to [3]. Let $G = (V_G, E_G)$ be a graph. A graph $H = (V_H, E_H)$ is a *subgraph* of G, if $V_H \subseteq V_G$ and $E_H \subseteq E_G$. The graph H is an *induced subgraph* of G, denoted by $H \sqsubseteq G$, if H is an edge-maximal subgraph of G. An n-cycle is a sequence of adjacent vertices $v_0v_1 \dots v_n$ with $v_0 = v_n$ and $v_i \neq v_j$ for all other indices i and j. A *chord* of an n-cycle is an edge joining non-consecutive vertices of the cycle. A cycle is said to be *chordless* if it has no chords. We denote a chordless cycle on n vertices by c_n. We adopt the following notations: N_k denotes the graph consisting of k isolated vertices, P_n the path with n vertices, and $W_n \cong c_n + N_1$ denotes the wheel of order $n+1$. The empty graph will be denoted by $\emptyset$. We write $a \not\sim b$ if vertex a is not adjacent to vertex b.

*Present address: Department of Mathematics, University of the Philippines, Diliman, Quezon City 1101, Philippines

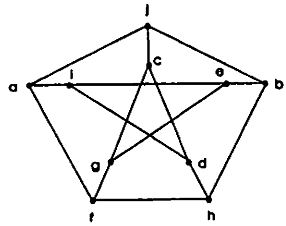
The *cycle graph* $C(G)$ of a graph G was introduced in [2] as the graph whose vertices are the chordless cycles of G with two of them adjacent in $C(G)$, if the corresponding cycles in G have an edge in common. Thus the cycle graph of an acyclic graph is the empty graph $\emptyset$. The operator C has been extensively studied in the literature ([1, 2, 7]) and graphs have been classified in accordance to their behaviour under iterates of the operator C. In [7] the question regarding the relationship between $C(G)$ and $C(\overline{G})$ was posed. In this paper we introduce the composite operator $\overline{C}$ defined by $\overline{C}(G) = C(\overline{G})$ and determine the behaviour of some classes of graphs under iterates of $\overline{C}$. In analogy with [2] we say that a graph G is $\overline{C}$*-vanishing* if there is a positive integer n such that $\overline{C}^n(G) = \emptyset$; G is called $\overline{C}$*-persistent*, if $\overline{C}^n(G) \neq \emptyset$ for all n; if moreover $\left|V(\overline{C}^n(G))\right| \longrightarrow \infty$ as $n \longrightarrow \infty$, then G is $\overline{C}$*-expanding*.

The image of a graph G under $\overline{C}$ is determined by the chordless cycles of $\overline{G}$. A vertex in $\overline{C}(G)$ corresponds to an induced chordless cycle in $\overline{G}$ and two vertices of $\overline{C}(G)$ are adjacent if the corresponding chordless cycles of $\overline{G}$ have at least one edge in common.

If $G = N_5$, then $\overline{C}(G) = \overline{C}(N_5) = C(\overline{N_5}) = C(K_5)$. The only chordless cycles of K_5 are 3-cycles. Denoting the vertex set of K_5 by $\{1, 2, 3, 4, 5\}$ the chordless cycles of K_5 are $a = 1231$, $b = 1241$, $c = 1251$, $d = 1341$, $e = 1351$, $f = 1451$, $g = 2342$, $h = 2352$, $i = 2452$ and $j = 3453$, and so $\overline{C}(N_5)$ is the following graph:



Its complement is the Petersen graph:



The Petersen graph has 12 induced chordless 5-cycles and 10 induced chordless 6-cycles. They are *A=ajcdia, B=jcgebj, C=beidhb, D=hdcgfh, E=fgeiaf, F=bhfgeb, G=hfaidh, H=fajcgf, I=ajbeia, J=jbhdcj, K=ajbhdia, L=jbhfgcj, M=bhfaieb, N= hfajcdh, O= fajbegf, P=ajcgeia, Q=jbeidcj, R=bhdcgeb, S=hfgeidh, T=faidcgf, U= ajbhfa* and *V=cdiegc*. Then in $\overline{C}^2(N_5)$ we have the following non-adjacency relations: $A \nsim F$, $B \nsim G$, $C \nsim H$, $D \nsim I$, $E \nsim J$, and $U \nsim V$. Thus $\overline{\overline{C}^2(N_5)} \cong 6P_2 \cup N_{10}$ and so $\overline{\overline{C}^2(N_5)}$ is acyclic. Thus $\overline{C}^3(N_5) = \emptyset$.

We denote the complement of the chordless cycle c_k by s_k and call it the *k-pointed star*. It has been shown in [4] that $\overline{C}(G)$ is the graph whose vertices are the induced k-pointed star subgraphs of G, $3 \leq k \leq |V(G)|$. Two vertices in $\overline{C}(G)$ are adjacent if the corresponding subgraphs s_k and s_ℓ in G have at least one pair of non-adjacent vertices in common. Thus the operator $\overline{C}$ is a subgraph defined operator (see [5]). The operator $\overline{C}$ satisfies the following properties (called axioms 3a and 3b in [5]);

Property 1 If $X, Y \subseteq G \sqsubseteq H$ and $x, y \in V(\overline{C}(H))$, then $x, y \in V(\overline{C}(G))$. Moreover, $xy \in E(\overline{C}(H))$ implies $xy \in E(\overline{C}(G))$.

Property 2 Let $G \sqsubseteq H$. Then $x \in V(\overline{C}(G))$ implies $x \in V(\overline{C}(G))$ and $xy \in E(\overline{C}(G))$ implies $xy \in E(\overline{C}(H))$.

An immediate consequence is the following heredity property (see Corollary 9.6 in [5])

Proposition 1 *If H is an induced subgraph of G, then $\overline{C}^n(H)$ is an induced subgraph of $\overline{C}^n(G)$ for every positive integer n.*

Proposition 1 provides a first step towards classifying graphs with regard to the behaviour of graphs under iterates of $\overline{C}$:

Proposition 2 *Every induced subgraph of a $\overline{C}$-vanishing graph is $\overline{C}$-vanishing. Every graph containing a $\overline{C}$-expanding induced subgraph is $\overline{C}$-expanding.*

2 The minimum order of a $\overline{C}$-expanding graph

In this section we establish that the minimum order of a $\overline{C}$-persistent graph is 6. We first show that all graphs of order 5 are $\overline{C}$-vanishing and then consider the iterated behaviour of N_k under $\overline{C}$.

Lemma 3 *All graphs of order at most* 5 *are $\overline{C}$-vanishing.*

Proof. By Proposition 2 it suffices to show that all graphs of order 5 are $\overline{C}$-vanishing. Straightforward calculation shows that for a graph G of order 5 and size at least 2 its image under $\overline{C}$ is isomorphic to one of $\emptyset$, N_1, N_2, P_2, P_3, c_3, K_4 and the graph consisting of K_4 and exactly one common edge with a cycle c_3 . Therefore $\overline{C}^2(G) = \emptyset$. For N_5 we have $\overline{C}^2(N_5) = \overline{6P_2 \cup N_{10}}$ so that $\overline{C}^3(N_5) = \emptyset$. Finally, $\overline{C}(P_2 \cup N_3)$ is an induced subgraph of $\overline{C}(N_5)$ and so $\overline{C}^3(P_2 \cup N_3) = \emptyset$. ■

In [6] the cycle graph of the complete graph K_n on n vertices was shown to be isomorphic to the uniform intersection graph $G(n,3,2)$, the graph whose vertex set consists of all 3-subsets of a set of cardinality n where two vertices are adjacent if they have precisely two elements in common. We therefore have $\overline{C}(N_k) = G(k,3,2)$. We will use this result to show that $\overline{C}^2(N_k)$ contains at least $k+2$ pair-wise non-adjacent vertices.

Proposition 4 *For each $k \geq 6$, the graph N_{k+2} is an induced subgraph of $\overline{C}^2(N_k)$.*

Proof. For each $k \geq 6$ we will exhibit $k+2$ 3-cycles in $\overline{G(k,3,2)}$ no two of which have an edge in common. Define for $0 \leq \ell \leq k-1$ the following 3-subsets of $\{1,2,\ldots,k\}$

$$\begin{aligned} A_1(\ell) &= \{(1+\ell)\text{mod}k, (2+\ell)\text{mod}k, (3+\ell)\text{mod}k\} \\ A_2(\ell) &= \{(1+\ell)\text{mod}k, (4+\ell)\text{mod}k, (5+\ell)\text{mod}k\} \\ A_3(\ell) &= \{(2+\ell)\text{mod}k, (4+\ell)\text{mod}k, (6+\ell)\text{mod}k\} \end{aligned}$$

and denote by $a_i(\ell)$ the vertex corresponding to $A_i(\ell)$ in $G(k,3,2)$. We associate to each set $A_i(\ell)$ a set of distances between consecutive vertices on the cycle c_k: If $A_i(\ell) = \{\alpha_i(\ell),\ \beta_i(\ell),\ \gamma_i(\ell)\}$ then the corresponding set is $\{d_{c_k}(\alpha_i(\ell),\beta_i(\ell)),\ d_{c_k}(\beta_i(\ell),\gamma_i(\ell)),\ d_{c_k}(\gamma_i(\ell),\alpha_i(\ell))\}$, where $d_{c_k}(u,v)$ denotes the length of the shortest path between the vertices u and v of c_k. For fixed i the set is independent of ℓ and will be denoted by t_i.

We have $t_1 = \{1,\ 1,\ 2\}$, while the remaining sets depend on the number of vertices of N_k:

$$t_2 = \begin{cases} \{3,\ 1,\ 2\}, k=6 \\ \{3,\ 1,\ 3\}, k=7 \\ \{3,\ 1,\ 4\}, k \geq 8 \end{cases} \text{, and } t_3 = \begin{cases} \{2,\ 2,\ 2\}, k=6 \\ \{2,\ 2,\ 3\}, k=7 \\ \{2,\ 2,\ 4\}, k \geq 8 \end{cases}$$

Thus the distance sets t_i and t_j differ in at least one element for $i \neq j$. Thus

$$A_i(\ell) \neq A_j(\ell),\ i \neq j$$

In fact, since the distance from the first element to the second element in $A_i(\ell)$ is different to the distance between the first element and second element in $A_j(\ell)$, we have

$$|A_i(\ell) \cap A_j(\ell)| \leq 1$$

and so the 3-cycles $a_i(\ell)$ and $a_j(\ell)$ do not have any common edge. Also we have by construction that

$$A_1(\ell) \cap A_2(\ell) \cap A_3(\ell) = \emptyset$$

Thus the set $S(\ell) = \{a_1(\ell), a_2(\ell), a_3(\ell)\}$ is the vertex set of a 3-cycle in $G(k,3,2)$. We can think of the set $A_i(\ell+1)$ as obtained from the set $A_i(\ell)$ by a rotation on c_k of one unit a clock-wise direction. Thus the sets $A_i(\ell)$, $i = 1,2,3$ and $\ell = 0,1,\ldots k-1$ have the following properties

$$\begin{aligned} A_1(\ell) &\neq A_1(m),\ \ell \neq m \\ A_2(\ell) &\neq A_2(m),\ \ell \neq m \\ A_3(\ell) &\neq A_3(m),\ \ell \neq m,\ k \geq 7 \\ A_3(\ell) &= A_3(\ell+2),\ k = 6 \end{aligned}$$

Since the distance sets t_i and t_j are not equal,

$$A_i(\ell) \neq A_j(m),\ \ell \neq m,\ i \neq j$$

Moreover, for $\ell \neq n$ the sets $S(\ell)$ and $S(n)$ satisfy

$$|S(\ell) \cap S(n)| \leq 1$$

and so the sets $S(\ell)$, $\ell = 0,\ldots k-1$ correspond to k pair-wise non-adjacent vertices in $\overline{C}^2(N_k)$.

For $k \geq 7$ the 3-cycles with vertex sets $\{\{1,2,4\}, \{1,3,5\}, \{3,4,6\}\}$ and $\{\{4,5,7\}, \{2,3,5\}, \{2,4,6\}\}$ do not have a common edge and also do not share any edge with $S(0),\ldots,S(k-1)$. For $k = 6$ the 3-cycles with vertex sets $\{\{1,2,5\}, \{2,3,6\}, \{1,3,4\}\}$ and $\{\{2,4,5\}, \{1,3,5\}, \{1,4,6\}\}$ do not have a common edge. They also do not have any edge in common with $S(0),\ldots,S(5)$. It thus follows that $N_{k+2} \sqsubseteq \overline{C}^2(N_k)$. ■

We now prove that the minimum order of a $\overline{C}$-expanding null graph is 6.

Proposition 5 *The null graph N_k of order k is $\overline{C}$-vanishing for $k < 6$ and $\overline{C}$-expanding for $k \geq 6$.*

Proof We have already shown in Lemma 3 that N_5 is $\overline{C}$-vanishing and so N_k is $\overline{C}$-vanishing for $k < 6$. From Proposition 1 it follows that N_{k+2m} is an induced subgraph of $\overline{C}^{2m}(N_k)$ for $m \geq 1$ and $k \geq 6$. Thus we have with Proposition 1 that $G(k+2m, 3, 2) = \overline{C}(N_{k+2m}) \sqsubseteq \overline{C}^{2m+1}(N_k)$ and hence $|V(\overline{C}^{2m+1}(N_k))| \geq \binom{k+2m}{3}$ and $|V(\overline{C}^{2m}(N_k))| \geq k+2m$ for each $m \geq 0$ and each $k \geq 6$. Thus, N_k is $\overline{C}$-expanding for $k \geq 6$. ■

An immediate consequence of Proposition 5 is

Corollary 6 *Every graph with vertex independence number at least* 6 *is* $\overline{C}$*-expanding.*

We conclude this section by determining the maximum size for a graph of order 6 to be $\overline{C}$-expanding

Proposition 7 *All graphs of order* 6 *and size at most* 3 *are* $\overline{C}$*-expanding, those of size greater than* 3 *are* $\overline{C}$*-vanishing.*

Proof. A straightforward calculation shows that N_6 is an induced subgraph of $\overline{C}^2(P_2 \cup N_4)$, $\overline{C}^2(2P_2 \cup N_2)$, $\overline{C}^2(c_3 \cup N_3)$, $\overline{C}^2(K_{1,3} \cup N_2)$ and $\overline{C}^2(P_3 \cup P_2 \cup N_1)$ and that $2P_2 \cup N_2 \sqsubseteq \overline{C}^2(3P_2)$, $K_{1,3} \cup N_2 \sqsubseteq \overline{C}^2(P_4 \cup N_2)$, and $3P_2 \sqsubseteq \overline{C}^2(P_3 \cup N_3)$. Thus it follows from Proposition 2 that the graphs of order 6 and size ≤ 3 are $\overline{C}$-expanding.

Graphs of order 6 and size ≤ 8 have at most 5 chordless cycles, so that their complements are $\overline{C}$-vanishing. If G is a connected graph of order 6 and size 6, then its image under $\overline{C}^2$ is one of $\emptyset$, N_1, N_2, or $P_2 \cup N_3$ and so they are $\overline{C}$-vanishing. Graphs of order 6 and size 6 with more than one connected component either contain an isolated vertex or are one of $2c_3$ or the disjoint union of a 4-cycle with one exactly one chord e, denoted by $c_4 + e$ and P_2. We denote this graph by $(c_4 + e) \cup P_2$. The complements of the graphs with an isolated vertex have at most 4 cycles, so that these graphs are $\overline{C}$-vanishing, and $\overline{C}^2(2c_3) = \overline{C}^2((c_4 + e) \cup P_2) = \emptyset$.

The images under $\overline{C}^2$ of graphs of size 5 can be shown to be $\emptyset$, N_1, $\overline{6P_2 \cup N_{10}}$ or the graph of order 6 consisting of one wheel W_4 and one cycle c_3 which have exactly one edge of c_4 in W_4 in common. Thus they are $\overline{C}$-vanishing. Finally, the images of graphs of size 4 under $\overline{C}^2$ are $\overline{P_2 \cup N_4}$, $\overline{P_2 \cup N_3}$, $\overline{6P_2 \cup N_{10}}$, K_9, c_3, W_5, K_7, K_4, and $2c_3$ and so in each case the third iterated image under $\overline{C}$ vanishes. ■

We now also have

Corollary 8 *Any graph that contains a graph of order* 6 *and size at most* 3 *as an induced subgraph is* $\overline{C}$*-expanding.*

3 Behaviour of Special Graphs under Iterates of $\overline{C}$

We may now determine the minimum order for cycles, paths, trees and bipartite graphs to be $\overline{C}$-expanding. As we will see below, the minimum order is 7 for trees and cycles, while for complete bipartite graphs it is 8, unless one of the partite sets has cardinality 1.

Proposition 9 *The path P_n of order n is $\overline{C}$-vanishing for $n < 7$ and $\overline{C}$-expanding for $n \geq 7$.*

Proof. By Proposition 7 and Lemma 3 every path of order at most 6 is $\overline{C}$-vanishing. Since $P_7 \sqsubseteq P_n$ for $n \geq 7$, it suffices to show that P_7 is $\overline{C}$-expanding. We let $P_7 = 1234567$, then $a = \{1, 3, 5\}$, $b = \{1, 3, 6\}$, $c = \{1, 3, 7\}$, $d = \{1, 4, 6\}$, $e = \{1, 4, 7\}$, $f = \{1, 5, 7\}$, $g = \{2, 4, 6\}$, $h = \{2, 4, 7\}$, and $i = \{2, 5, 7\}$ are 3-pointed stars and $m = \{\overline{23}, \overline{56}\}$ is a 4-pointed star in P_7 and so they are vertices in $\overline{C}(P_7)$. The following non-adjacency relations hold in $\overline{C}(P_7)$: $a \nsim d, e, g, h$; $b \nsim e, f, g, i$; $c \nsim d, g, i, m$; $d \nsim h, m$; $e \nsim g, i$; $f \nsim g$; $g \nsim i$. Thus $\{a, d, h\}$, $\{a, e, g\}$, $\{b, e, i\}$, $\{b, f, g\}$, $\{c, d, m\}$, and $\{c, g, i\}$ are six 3-pointed stars in $\overline{C}(P_7)$ any two of which do not have two vertices in common. Therefore $N_6 \sqsubseteq \overline{C}^2(P_7)$ and so P_7 is $\overline{C}$-expanding by Proposition 2. ■

As an immediate consequence we have

Corollary 10 *Every connected graph of diameter at least 6 is $\overline{C}$-expanding.*

Corollary 8 and Proposition 2 allow us to classify each tree of order at least 7 as $\overline{C}$-expanding:

Proposition 11 *Each tree of order at least 7 is $\overline{C}$-expanding.*

Proof. Since each tree of order greater than 7 contains a tree of order 7 as an induced subgraph, we need only establish that trees of order 7 are $\overline{C}$-expanding. The graph $K_{1,6}$ contains N_6 as an induced subgraph. There are three trees of order 7 with degree sequence 1112223. Two of these are obtained from $P_6 = 123456$ by adding a seventh vertex and joining it with an edge either to vertex 5 or to vertex 4. The last one is obtained from $P_5 = 12345$ by attaching P_2 to vertex 3. Call the trees T_1, T_2 and T_3 respectively. Then T_1 contains $P_4 \cup N_2$ as an induced subgraph, T_2 contains $P_3 \cup P_2 \cup N_1$ and T_3 contains $3P_2$.

The trees with degree sequence 1111224 are obtained from $P_5 = 12345$ by the addition of two vertices both of which are either adjacent to vertex 3 or to vertex 4. The first of these trees contains $2P_2 \cup N_2$ as an induced subgraph, the second contains $P_3 \cup N_3$.

The tree with degree sequence 1111125 contains $P_2 \cup N_4$ and the trees with degree sequence 1111233 contains $P_3 \cup P_2 \cup N_1$ and $K(1,3) \cup N_2$ respectively. Finally the tree with degree sequence 1111134 contains $P_3 \cup N_3$ as an induced subgraph.

Thus all trees of order 7 are $\overline{C}$-expanding by Corollary 8. ■

The proposition below shows that the minimum order of a $\overline{C}$-expanding cycle is the same as that of a path.

Proposition 12 *The chordless cycle c_n of order n is $\overline{C}$-vanishing for $n < 7$ and $\overline{C}$-expanding for $n \geq 7$.*

Proof It is a consequence of Propositions 3 and 8 that cycles of order at most 6 are $\overline{C}$-vanishing. Since P_7 is an induced subgraph of c_n for $n \geq 8$, it follows that c_n is $\overline{C}$-expanding for $n \geq 8$.

It remains to show that c_7 is $\overline{C}$-expanding. Let $c_7 = 12345671$. Then $a = \{1, 3, 5\}$, $b = \{1, 3, 6\}$, $c = \{1, 4, 6\}$, $d = \{2, 4, 6\}$, $e = \{2, 4, 7\}$, $f = \{2, 5, 7\}$, $g = \{3, 5, 7\}$, $h = \{\overline{12}, \overline{45}\}$, $m = \{\overline{23}, \overline{67}\}$, $n = \{\overline{34}, \overline{67}\}$ and $p = \{\overline{45}, \overline{71}\}$ are vertices in $\overline{C}(c_7)$ with the following non-adjacency relation: $a \not\sim c, d, e, f, m, n$; $b \not\sim d, e, f, g, h, p$; $c \not\sim e, f, m$; $d \not\sim f, g, p$; $e \not\sim g$; $f \not\sim n$; $g \not\sim h$. Hence, the elements of each of the six sets A, B, C, D, E and F below are 3-pointed stars in $\overline{C}(c_7)$ and so are vertices in $\overline{C}^2(c_7)$:

$$
\begin{aligned}
A &= \{\ \{a,c,e\}, \{a,d,f\}, \{b,d,p\}\ \} \\
B &= \{\ \{a,c,e\}, \{a,f,n\}, \{b,d,g\}\ \} \\
C &= \{\ \{a,c,m\}, \{a,f,n\}, \{b,d,f\}\ \} \\
D &= \{\ \{a,c,m\}, \{a,d,f\}, \{b,d,g\}\ \} \\
E &= \{\ \{a,c,f\}, \{b,d,p\}, \{b,e,g\}\ \} \\
F &= \{\ \{a,c,f\}, \{b,d,f\}, \{b,g,h\}\ \}
\end{aligned}
$$

These six sets correspond to 3-pointed stars in $\overline{C}^2(c_7)$. Furthermore, they are vertices in $\overline{C}^3(c_7)$ which form N_6 since they are pair-wise non-adjacent. Therefore, we have shown that $\overline{C}^3(c_7)$ is $\overline{C}$-expanding by Proposition 2, and so c_7 is $\overline{C}$-expanding. ■

Since the wheel $W_n = N_1 + c_n$ we have

Corollary 13 *The wheel W_n is $\overline{C}$-vanishing for $n \leq 6$ and $\overline{C}$-expanding for $n \geq 7$.*

Finally, we classify a complete bipartite graph as either $\overline{C}$-vanishing or $\overline{C}$-expanding based on the cardinalities of its partite sets.

Proposition 14 *Let $K_{m,n}$ be a complete bipartite graph with partite sets V_1 and V_2 such that $|V_1| = m$ and $|V_2| = n$, $m, n \geq 1$, $m \leq n$. Then $K_{m,n}$ is $\overline{C}$-vanishing for $m = 1, 2$ and $1 \leq n \leq 5$; $m = n = 3$; and $m = 3$, $n = 4$; and $\overline{C}$-expanding for all other values of m and n.*

Proof. Without loss of generality assume that $m \leq n$. We note that $K_{m,n} \cong N_m + N_n$ and so $\overline{C}(K_{m,n}) \cong \overline{C}(N_m + N_n) \cong C\left(\overline{N_m + N_n}\right) \cong C(K_m \cup K_n) \cong \overline{C}(N_m) \cup \overline{C}(N_n)$. Therefore $K_{m,n}$ is $\overline{C}$-expanding by Proposition 5 if either m or n is at least six and $K_{m,n}$ is $\overline{C}$-vanishing for $m = 1, 2$ and $1 \leq n \leq 5$.

Since $\overline{C}(K_{3,4}) = N_1 \cup K_4$, and hence $\overline{C}^2(K_{3,4}) = C(K_{1,4}) = \emptyset$, the graphs $K_{3,3}$ and $K_{3,4}$ are $\overline{C}$-vanishing.

The complete bipartite graphs $K_{3,5}$, $K_{4,4}$, $K_{4,5}$ and $K_{5,5}$ remain to be considered.

We observe that $\overline{\overline{C}(K_{3,5})} = \overline{C(K_3 \cup K_5)} = \overline{N_1 \cup \overline{C}(N_5)} = N_1 + \overline{\overline{C}(N_5)}$. Let the vertex set of K_5 be $\{1,2,3,4,5\}$. Then the vertices of $\overline{C}(N_5) \cong C(K_5)$ are $a = \{1,2,3\}$, $b = \{1,2,4\}$, $c = \{1,2,5\}$, $d = \{1,3,4\}$, $e = \{1,3,5\}$, $f = \{1,4,5\}$, $g = \{2,3,4\}$, $h = \{2,3,5\}$, $i = \{2,4,5\}$ and $j = \{3,4,5\}$. They have the following non-adjacency relations: $a \nsim f,i,j$; $b \nsim e,h,j$; $c \nsim d,g,j$; $d \nsim h,i$; $e \nsim g,i$; and $f \nsim g,h$. Letting x denote the vertex corresponding to N_1, it follows that $xdhx$, $xeix$, $xafx$, $xbjx$, $hbegfh$ and $aidcja$ are chordless cycles in $\overline{\overline{C}(K_{3,5})}$ no two of which have an edge in common so that they form six pair-wise non-adjacent vertices in $\overline{C}^2(K_{3,5}) = C(\overline{\overline{C}(K_{3,5})})$. Therefore, $\overline{C}^2(K_{3,5})$ contains N_6 as an induced subgraph. It follows that $K_{3,5}$, $K_{4,5}$ and $K_{5,5}$ are $\overline{C}$-expanding.

Finally we consider $K_{4,4}$. We have $\overline{C}(K_{4,4}) = C(2K_4) = 2K_4$ and so $\overline{C}^2(K_{4,4}) = C(K_{4,4})$. Let the partite sets of $K_{4,4}$ be $V_1 = \{1,2,3,4\}$ and $V_2 = \{5,6,7,8\}$. All chordless cycles of $K_{4,4}$ are 4-cycles. They include $a = 15261$, $b = 15271$, $c = 15281$, $d = 37483$, $e = 35463$, $f = 35473$, $g = 35483$, $h = 36473$ and $i = 36483$. These cycles correspond to vertices in $\overline{C}^2(K_{4,4})$ with the following non-adjacency relations: $a \nsim d,e,f,g,h,i$; $b \nsim d,e,f,g,h,i$; $c \nsim d,e,f,g,h,i$; $d \nsim e$; $f \nsim i$; and $g \nsim h$. Hence, the elements of the six sets below are 3-pointed stars in $\overline{C}^2(K_{4,4})$ and so are vertices in $\overline{C}^3(K_{4,4})$:

$$\begin{aligned}
A &= \{\ \{a,f,i\}\ ,\ \{a,g,h\}\ ,\ \{a,d,e\}\ \} \\
B &= \{\ \{b,f,i\}\ ,\ \{b,g,h\}\ ,\ \{b,d,e\}\ \} \\
C &= \{\ \{c,f,i\}\ ,\ \{c,g,h\}\ ,\ \{c,d,e\}\ \} \\
D &= \{\ \{a,f,i\}\ ,\ \{b,g,h\}\ ,\ \{c,d,e\}\ \} \\
E &= \{\ \{b,f,i\}\ ,\ \{c,g,h\}\ ,\ \{a,d,e\}\ \} \\
F &= \{\ \{c,f,i\}\ ,\ \{a,g,h\}\ ,\ \{b,d,e\}\ \}
\end{aligned}$$

While each of A, B and C intersects each of D, E and F in exactly one element, A, B and C are pair-wise disjoint; and D, E and F are pair-wise disjoint. Thus, they form six pair-wise non-adjacent vertices in $\overline{C}^4(K_{4,4})$. Therefore, N_6 is an induced subgraph of $\overline{C}^4(K_{4,4})$ and so $K_{4,4}$ is $\overline{C}$-expanding. ■

The graphs we have considered above have all been seen to be either $\overline{C}$-expanding or $\overline{C}$-vanishing. There is no fixed point of $\overline{C}$ in the range of these graphs. We end with an open problem: Are there any $\overline{C}$-persistent graphs which are not $\overline{C}$-expanding?

References

[1] Y. Egawa, M. Kano, & E.L. Tan, On cycle graphs. *Ars Combinatoria 32* (1991), 91-113.

[2] S.V. Gervacio, Cycle graphs. *Proceedings of the First Southeast Asian Graph Theory Colloquium.*(eds. K.M. Koh & H. P. Yap). Springer-Verlag, Singapore (1984) , 279-293.

[3] F. Harary, *Graph Theory,* Addison Wesley, Reading (1969).

[4] C. Lopez, On the relationship between a graph and the cycle graph of its complement, M.Sc. Thesis, Edith Cowan University, Perth, Australia(1995)

[5] E. Prisner, *Graph Dynamics,* Pitman Research Notes in Mathematics 338, Longman, Harlow (1995)

[6] E.L. Tan, *On the cycle graph of a graph and inverse cycle graphs.* Doctoral dissertation, University of the Philippines, Diliman, Quezon City, Philippines (1987).

[7] E.L. Tan, Some solved and unsolved problems in cycle graphs. *Proceedings of the JSPS Workshop, Tokyo, Japan 15-16 August 1990,* (eds. J. Akiyama, M-J. P. Ruiz & T. Saito), Mathematics Department, Ateneo de Manila University (1993), 81-89