# The kth Lower Multiexponent of Tournament Matrices

Bolian Liu  Wang Yan
(Department of Mathematics
South China Normal University
Guangzhou,P.R.China)

**Abstract**

In this paper we investigate the kth lower multiexponent $f(n,k)$ for tournament matrices.

It was proved that $f(n,3)=2$ if and only if $n \geq 11$.Thus the conjecture in [2] is disproved.Further we obtain a new sufficient condition for $f(n,k)=1$.

# 1 INTRODUCTION

An $n \times n$ Boolean matrix $A$ is called primitive if there exists some positive integer $t$ such that $A^t > 0$.Such a least positive integer $t$ is called the exponent of $A$.

As we know,a directed graph $D=(V,E)$ defined by a $(0,1)$ matrix $A=(a_{ij})$ consists of $n$ vertices $1,2,...,n$ such that an arc$(i,j)$ goes from $i$ to $j$ if and only if the entry $a_{ij}$ of $A$ is one.$A$ is called the adjacency matrix of $D$,while $D$ is called the associated digraph of $A$.If $A$ is primitive,then its associate digraph $D(A)$ is a primitive graph.It is well known that $D$ is primitive if and only if $D$ is strongly connected and $\gcd(r_1,r_2,...,r_s)=1$,where $L(D)=\{r_1,r_2,...,r_s\}$ is the set of distinct lengths of the directed cycles of $D$.

$f(D,k), 1 \leq k \leq n$,is called the kth lower multiexponent for a primitive directed graph $D$ of order $n$ if there exists a set $X$ of $k$ vertices of $D$ such that for each vertex $i$ of $D$,there is a walk of length $f(D,k)$ from some vertex of $X$ to $i$.Equivalently $f(D,k)$ is a least positive integer $t$ such that $A^t(D)$ has a $k \times n$ submatrix without zero column.

In [1], R.A.Brualdi and Bolian Liu first introduced $f(D,k)$ and found the upper bound of $f(G,k)$ for primitive simple graphs.In [2], Bolian Liu investigated $f(T_n,k)$ for primitive tournaments of order $n$.

A tournament $T_n$ is a directed graph $D$ such that each pair of distinct vertices $i$ and $j$ is joined by exactly one of the arcs$(i,j)$ or $(j,i)$ and no vertex is joined

ARS COMBINATORIA 56(2000), pp. 257-262

to itself by an arc.A tournament matrix $M_n$ is a matrix that is the adjacency matrix of a tournament $T_n$.

In 1967,J.W.Moon and N.J.Pullman ([3]) proved that a tournament $T_n$ is primitive if and only if $n \geq 4$ and $T_n$ is strongly connected.

In this paper we will consider only primitive tournament $T_n(n \geq 4)$.

Let

$$f(n,k) := MAX_{T_n} f(T_n,k), k = 1,2,...,n$$

where $MAX$ is taken over all the primitive tournament $T_n(n \geq 4)$.

According to the above definition,we have

$$f(n,k_1) \geq f(n,k_2) \quad if \quad k_1 \leq k_2. \tag{1}$$

In [2],Bolian Liu proved the following

**THEOREM A([2])**

$$f(n,k) = \begin{cases} 3 & k = 1 \\ 2 & k = 2 \\ 1 \text{ or } 2 & 3 \leq k < 2 + \lfloor \frac{1}{4}(n+1) \rfloor \\ 1 & k \geq 2 + \lfloor \frac{1}{4}(n+1) \rfloor \end{cases}$$

and conjectured

$$f(n,k) = \begin{cases} 3 & k = 1 \\ 2 & k = 2 \\ 1 & k \geq 3 \end{cases} \qquad n \geq 4 \qquad \text{(B.L.conjecture)}$$

In fact,let $n$ and $k$ be integers such that $3 \leq k \leq n$ and $n \geq 4$,we say a subtournament $T_k$ of a tournament $T_n$ is dominating if every vertex of $T_n$ loses to some nodes of $T_k$.It follows that a primitive tournament $T_n$ with the property that $f(T_n,k) = 1$ is equivalent to $T_n$ having at least one dominating subtournament $T_k$.

In this paper we show that $f(n,3) = 2$ if and only if $n \geq 11$.Hence B.L.conjecture in [2] is disproved.In [4],E. and G.Szekeres mentioned $S_k$ tournaments and proved an inequality $f(k) \geq (k+2)2^{k-1} - 1$.With that result,we know that if $4 \leq n \leq (k+1)2^{k-2} - 2$,then any strong tournament of order $n$ has one dominating $T_k$,which means

$$f(n,k) = 1 \quad if \quad 4 \leq n \leq (k+1)2^{k-2} - 2 \quad (k \geq 3).$$

More precisely we shall prove that

$$f(n,k) = 1 \quad if \quad 4 \leq n \leq k \cdot 2^{k-1} - 2 \quad (k \geq 3)$$

The result improves the above conclusion.

## 2 MAIN RESULTS

Let $T = (V, E)$ be a tournament whose set of vertices is $V$,$|V| = n$,and whose set of arcs is $E$.

For $i \in V(T)$,

$$N^+(i) := \{j|(i,j) \in E, j \in V\},$$

$$N^-(i) := \{j|(j,i) \in E, j \in V\}$$

$N^+(i)$ is also called the neighbourhood of $i$.Clearly,$N^+(i) \cup N^-(i) \cup \{i\} = V(T)$, for each $i \in V(T)$,and $|N^+(i)|+|N^-(i)| = n-1$.where $|S|$ denotes the cardinality of the set $S$.Let

$$\Delta^+(T) := \max\{\left|N^+(i)\right|, i \in V\},$$

$$\delta^-(T) := \min\{\left|N^-(i)\right|, i \in V\}$$

clearly $\Delta^+(T) \geq \frac{1}{n}\left(\begin{array}{c} n \\ 2 \end{array}\right) = \frac{n-1}{2}$.and $\Delta^+(T)+\delta^-(T) = n-1$.If $T$ is strong,then $\delta^-(T) > 0$.

For a subgraph $T'$ of $T$, $N^+_{T'}(i) := N^+(i) \cap V(T'), i \in V(T')$.

To prove Theorem 2.2,we first present a Lemma.

**Lemma 2.1.**For $k \geq 3$,let $T$ be a tournament and $k \leq |V(T)| \leq 2k-2$.If $T$ has no transmitter(that is there has no vertex,say $u$,$\Delta^+(u) = |V(T)|-1$),then there exists a subset of $V(T)$,say $X$,such that $\cup_{i \in X} N^+(i) = V(T)$ and $|X| \leq k$.Furthermore,for the set $S^* = V(T) - X$,there exists a vertex $v \in V(T)$, $N^+(v) \supset S^*$.

**Proof.**Let $v_1$ be a vertex with maximum outdegree.With the fact that $T$ has no transmitter,we know that $N^-(v_1)$ is nonempty.Since $\Delta^+(T) \geq \frac{|V(T)|-1}{2}$,we have

$$\left|N^-(v_1)\right| \leq |V(T)| - \frac{|V(T)|-1}{2} - 1,$$

i.e.

$$\left|N^-(v_1)\right| \leq \frac{|V(T)|-1}{2}.$$

Now $k \leq |V(T)| \leq 2k-2$,so $1 \leq |N^-(v_1)| \leq k-2$.Clearly $v_1 \in \cup_{u \in N^-(v_1)} N^+(u)$.Since $T' = T[N^-(v_1)]$ has at most one transmitter,surely we can add one vertex $u(u \in V(T))$ to $N^-(v_1) \cup \{v_1\}$ such that the set $N^-(v_1) \cup \{v_1, u\}$ whose neighbourhood union is $V(T)$.Thus $N^-(v_1) \cup \{v_1, u\}$ is the set we required.It is obvious that $|N^-(v_1) \cup \{v_1, u\}| \leq k-2+2 = k$.

Clearly $v_1 \in V(T)$ and $N^+(v_1) \supset V(T) - (N^-(v_1) \cup \{v_1, u\})$.Hence the lemma holds. □

Now we establish the following

**Theorem 2.2.** Let $T$ be a strong ournament of order $n(n \geq 4)$. If for some integer $k \geq 3, n-2 \geq \Delta^+(T) \geq n - k \cdot 2^{k-2} + 1$, then there exists a subset $X$ of $V(T)$ with $|X| = k$ such that $\cup_{v \in X} N^+(v) = V(T)$.

**Proof.** Let $v_1$ be a vertex with the largest outdegree in $T$. Then

$$1 \leq |N^-(v_1)| \leq k \cdot 2^{k-2} - 2.$$

Let $T_1$ denote $T \setminus (N^+(v_1) \cup \{v_1\})$. Clearly $V(T_1) = N^-(v_1)$.

If $|N^-(v_1)| \leq 2$, by $\Delta^+(T) \leq n-2$, then there exist two vertices whose neighbourhood union contains $V(T_1)$. Thus this theorem holds.

If $|N^-(v_1)| > 2$. Let $v_2$ be a vertex in $T_1$ whose outdegree(as a vertex in $T_1$) is maximal. Then

$$0 \leq |N^-_{T_1}(v_2)| \leq \frac{1}{2}(k \cdot 2^{k-2} - 2 - 1)$$

Hence $0 \leq |N^-_{T_1}(v_2)| \leq k \cdot 2^{k-3} - 2$, and for any vertex $u \in N^-_{T_1}(v_2)$, $N^+_{T_1}(u)$ contains $v_2$. Let $T_2$ denote $T_1 \setminus (N^+_{T_1}(v_2) \cup \{v_2\})$. Take a vertex with the largest outdegree in $T_2$, say $v_3$,

. . . . . . . . . .

And so on. Continuing the process we take $k-2$ vertices, say $v_1, v_2, \ldots, v_{k-2}$.

If there exists a subscript $i, 2 \leq i \leq k-2$, such that $\left|N^-_{T_{i-1}}(v_i)\right| \leq 2$, then we have $\left|\{v_1, v_2, \ldots, v_i\} \cup N^-_{T_{i-1}}(v_i)\right| \leq 2 + k - 2 = k$. It is not difficult to verify that this theorem holds. Otherwise let $T_{k-2}$ denote $T_{k-3} \setminus (N^+_{T_{k-3}}(v_{k-2}) \cup \{v_{k-2}\})$. Then

$$3 \leq |V(T_{k-2})| \leq k \cdot 2^{k-(k-1)} - 2,$$

We consider the following two cases.

Case 1. If $T_{k-2}$ contains a transmitter, then the theorem holds.

Case 2. In this case, $T_{k-2}$ has no transmitter. From the hypotheses, there is a $j$ such that $3 \leq j \leq k$ and $j \leq |V(T_{k-2})| \leq 2j - 2$, so by Lemma 2.1, we can find an $X$ with the desired properties such that $|X| \leq j \leq k$. Now we consider the subset $X$.

If $\cup_{u \in X} N^+(u) = V(T)$, since $|X| \leq k$, we can get a k-vertex subset $X'$ by adding $k - |X|$ vertices to $X$, then $X'$ is the required set. If $\cup_{u \in X} N^+(u) \neq V(T)$, then there exists a vertex $w \notin V(T_{k-2})$ with the property that $N^+(w) \supset X$. Hence $N^+(w) \cup N^+(v) \supset V(T_{k-2}) \cup \{v_{k-2}\}$, then the $k$ vertices $v_1, v_2, \ldots, v_{k-3}, v_{k-2}, v, w$ are required. The theorem holds.

This completes the proof of Theorem 2.2. □

By Theorem 2.2, clearly we have

**Corollary 2.3.** If $T_n$ is a strong tournament of order $n$ with

$$4 \leq n \leq k \cdot 2^{k-1} - 2 \quad (k \geq 3),$$

then $T_n$ has a dominating subtournament $T_k$.It follows that

$$f(n,k) = 1 \quad if \quad 4 \leq n \leq k \cdot 2^{k-1} - 2 \quad (k \geq 3).$$

In fact,Theorem 2.2 provides a means of finding out a dominating subtournament $T_k$ of every strong tournament $T_n$ with $4 \leq n \leq k \cdot 2^{k-1} - 2$ $(k \geq 3)$.

According to Corollary 2.3,we have

$$f(n,3) = 1 \qquad (4 \leq n \leq 10) \tag{2}$$

Furthermore for $n \geq 11$,we have

**Theorem 2.4.** $f(n,3) = 2 \quad for \quad n \geq 11$.

**Proof.** By Theorem A and (1),we have $f(n,3) \leq 2 (n \geq 11)$.Now we show that $f(n,3) \geq 2(n \geq 11)$.

Let $Q_{11}$ denote the (strong) tournament with vertices $1, 2, ..., 11$ in which arc$(i,j)$ is present if and only if $j - i$ in which arc$(i,j)$ is a quadratic residue modulo 11,clearly it is feasible.To verify that $Q_{11}$ has no dominating $T_3$,it is only necessary to consider the 55 3-cycles in $Q_{11}$:and even these don't need to be considered separately.For,every arc of $Q_{11}$ is similar to every other arc of $Q_{11}$ under the automorphism group of $Q_{11}$.So we need only examine the three 3-cycles containing any given arc of $Q_{11}$,then it is easy to check that $Q_{11}$ has no dominating $T_3$.

Next we let $T_n$ denote the tournament obtained from $Q_{11}$ by replacing vertex 1,say,of $Q_{11}$ by a transitive tournament $R_{n-10}$ and then adding arcs between all vertices of $R_{n-10}$ and the remaining vertices $i(2 \leq i \leq 11)$ of $Q_{11}$ that have the same orientations as the original arcs between vertices 1 and $i$.It is easy to see that this $T_n$ is strong and that if $Q_{11}$ has no dominating $T_3$,then $T_n$ doesn't either.It follows that $f(n,3) \geq 2(n \geq 11)$.

Hence the Theorem holds. □

By Theorem 2.4 and (2),we know that

$$f(n,3) = 2 \quad \text{if and only if} \quad n \geq 11.$$

Thus we disprove B.L.conjecture in [2].

According to Corollary 2.3,we know that if $f(n,4) > 1$ then $n \geq 31$.

In fact,P. Erdös had shown the following result([5]):

Let $n$ and $k$ be integers such that $n \geq 4$ and $4 \leq k \leq n$.If $n/\log n \geq k \cdot 2^k$,then there exists some strong $T_n$ having no dominating $T_k$.

From the above,we know that if $n/\log n \geq k \cdot 2^k (n \geq 4, 4 \leq k \leq n)$,then $f(n.k) = 2$.

**Acknowledgement**

The authors would like to thank the referees for their helpful comments and suggestions.Especially,the proof of Theorem 2.4 is greatly improved due to the referees' help.It is their kind help that leads to this Theory.

# References

[1] R.A.Brualdi and Bolian Liu,Generalized Exponents of Primitive Directed Graphs,J. Graph Theory,14(1990)4:483—499.

[2] Bolian Liu,Multiexponent of Tournament Matrices,ARS.Combinatoria 41(1995),269—277.

[3] J.W.Moon and N.J.Pullman,On the Powers of Tournament Matrices,J.Comb.Theory 3(1965)290-293.

[4] E.and G.Szekeres,On a Problem of Schütte and Erdös,Math.Gazette 49(1965)290-293.

[5] P. Erdös,On a problem in Graph Theorey,Math.Gazette 47(1963)220-223.