Cycles through specified vertices in 1-tough graphs

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Abstract

Bollobás, Brightwell [1] and independently Shi [3] proved the existence of a cycle through all vertices of degree at least $\frac{n}{2}$ in any 2-connected graph of order n. The aim of this paper is to show that the above degree requirement can be relaxed for 1-tough graphs.

1 Introduction

Bollobás and Brightwell [1] proved that if G is a graph of order n and W is a set of w vertices of degree at least d, then there is a cycle through at least $\lceil \frac{w}{\lceil \frac{n}{d} \rceil - 1} \rceil$ vertices of W. In the case $d \geq \frac{n}{2}$ this implies the existence of a cycle through all vertices of degree at least $\frac{n}{2}$. This special case was proved independently by Shi [3] for 2-connected graphs. In fact, Bollobás and Brightwell proved an Ore type result which can be read as

Theorem 1 [1] *Let G be a graph on n vertices and let $W \subseteq V_G$ such that each pair of non-adjacent vertices $u, v \in W$ satisfies $d(u) + d(v) \geq n$. If $|W| \geq 3$, then G contains a cycle through all vertices of W.*

In this paper we show that the above degree requirement can be relaxed for 1-tough graphs. A graph is called 1-*tough* if $t(G - S) \leq |S|$ for every subset S of V_G with $t(G - S) \geq 2$, where $t(G - S)$ denotes the number of components of $G - S$. To be able to state our result we need some definitions and notation.

For any subset S of V_G the subgraph induced by the vertices from S is denoted by $\langle S \rangle$. Let u and v be two non-adjacent vertices of G. Let

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$\psi(u,v)$ be the number of components of $\langle N(u)\rangle$ containing no neighbour of v, where $N(u)$ denotes the neighbourhood of u. Let $\alpha(u,v) = |N(u) \cap N(v)|$ and $\beta(u,v) = |\{x;\, x \notin N(u) \cup N(v),\, d(u,x) = 2 \text{ or } d(v,x) = 2\}|$, where $d(x,y)$ denotes the distance of x and y. We define, involving these terms, a graph invariant

$$\chi(u,v) = \mathrm{pos}(\min\{\psi(u,v), \psi(v,u)\} - 1) + \mathrm{pos}(\alpha(u,v) - \beta(u,v) - 1),$$

where $\mathrm{pos}(x) = \max\{0, x\}$. Notice that similar invariants were defined in [4, 5, 2] where several conditions for hamiltonicity were generalized. Our aim is to show that the degree-sum requirement from Theorem 1 can be decreased by $\chi(u,v)$ for 1-tough graphs. To see the significance of χ let us observe that in any graph of girth ≥ 5, $\psi(u,v) \geq \delta - 1$, where δ denotes the minimal degree of the graph. Thus $\chi(u,v) \geq \delta - 2$ for all pairs of non-adjacent vertices in such graph. Moreover, if there is a pair of non-adjacent vertices with "many" neighbours in common, then, usually, $\psi(u,v)$ is "small", but $\alpha(u,v) - \beta(u,v)$ can be "large". Thus χ may be well applicable to both sparse and dense graphs. Our main result is the following Theorem 2 the proof of which is given in the next section.

Theorem 2 *Let G be a 1-tough graph of order n and let $W \subseteq V_G$. Assume that for each pair of non-adjacent vertices $x, y \in W$ we have $d(x) + d(y) + \chi(x,y) \geq n$. Then, G contains a cycle through all vertices of W.*

Let W be a subset of V_G of a graph G. If we define for all $u \in W$ $\omega_W(u) = \min_{\substack{u \neq v \in W \\ uv \notin E_G}} \chi(u,v)$, then we obtain

Corollary 1 *Let $W \subseteq V_G$ of a 1-tough graph G. If for all $v \in W$, $d(v) \geq \frac{n-\omega_W(v)}{2}$, then G contains a cycle through all vertices of W.*

Proof. Let W be a set of vertices v of degree at least $\frac{n-\omega_W(v)}{2}$. If $|W| \leq 2$, then the statement follows from the 2-connectivity of G. If all vertices from W are adjacent to each other, then, trivially, there is a cycle containing all vertices of W. Thus, let x and y be two non-adjacent vertices from W. Then $d(x)+d(y) \geq \frac{n-\omega_W(x)}{2} + \frac{n-\omega_W(y)}{2} \geq \frac{n-\chi(x,y)}{2} + \frac{n-\chi(y,x)}{2} = n - \chi(x,y)$.

The following example shows that Corollary 1 is indeed stronger than the related result of Bollobás, Brightwell [1] and Shi [3]. We define for each $p \geq 6$ the graph G_p of order $2p^2+2p+3$ as follows: G_p consists of $p+1$ copies of K_{p+1} with the vertex sets $\{u_{1,i}, u_{2,i}, \ldots, u_{p+1,i}\}$, $(i = 1, 2, \ldots, p+1)$; of p^2 vertices $v_1, v_2, \ldots, v_{p^2}$; of another two vertices x, y; and all edges $v_j u_{k,l}$, $(j = 1, 2, \ldots, p^2)$, $(k = 2, 3, \ldots, p+1)$, $(l = 1, 2, \ldots, p+1)$; $xu_{1,i}$, $(i = 1, 2, \ldots, p+1)$; and yv_i, $(i = 1, 2, \ldots, p^2)$.

For a graph G_p, $p \geq 6$, let $W = V_{G_p} \setminus \{u_{1,1}, u_{1,2}, \ldots, u_{1,p+1}, x, y\}$. It is easy but time consuming exercise to observe that G_p is 1-tough and that $\omega_W(v_i) \geq 1$ for all $v_i \in W$ and $\omega_W(u_{i,j}) \geq 3$ for all $u_{i,j} \in W$. Thus by Corollary 1, there is a cycle through all vertices of W. But the related result of Bollobás, Brightwell [1] and Shi [3] cannot be applied because no vertex of G_p has degree $\geq p^2 + p + 2$. Note that using Theorem 1 one can ensure a cycle through all vertices of $\{v_j, u_{2,i}, \ldots, u_{p+1,i}\}$ for some i and j.

2 Proof of Theorem 2

Let $C = c_1 c_2 \ldots c_k c_1$ be a cycle of length k. If $u = c_i \in V_C$, then by u^+ (u^-) we denote the vertex c_{i+1} (c_{i-1}), where $c_{k+1} = c_1$ and $c_0 = c_k$. We say that C is *non-extendable* if for each edge uv of C there is no $u - v$ path internally disjoint from C. If P is a path and x and y are two vertices of P, then by $[x, y]_P$ we denote the unique subpath of P beginning at x and ending at y. Similarly, if C is a cycle and x and y are two its vertices, $x \neq y$, then by $[x\epsilon, y]_C$, where $\epsilon \in \{x^+, x^-\}$, we denote the unique subpath of C beginning at x, ending at y, and passing through ϵ. Finally, let P and Q be two internally disjoint paths with just one vertex in common—the last vertex of P is the first vertex of Q. Then by $P \circ Q$ we denote the path P followed by Q.

Proof of Theorem 2. If $|W| \leq 2$, then the result follows from the 2-connectivity of G. Thus let $|W| \geq 3$. Assuming the theorem is false, let $C = c_1 c_2 \ldots c_k c_1$ be a non-extendable cycle containing as many vertices from W as possible. Note that C contains at least two vertices of W. Let $H = V_G \setminus V_C$. Then there is at least one vertex, say c, from $H \cap W$. Since G is 2-connected, it follows that there is a path P connecting two vertices of C that is internally disjoint from C and contains c. Without loose of generality we may assume that $V_C \cap V_P = \{c_1, c_t\}$. Moreover, we may assume that C and P are chosen in such a way that:

(i) the path $[c_1, c]_P$ is as short as possible;

(ii) t is as small as possible with respect to (i).

It follows from (ii), that there is no edge cc_i for $1 < i < t$. By the choice of C, there is at least one vertex $c_s \in W \cap V_C$, where $1 < s < t$. If we take s as small as possible, it will hold that $c_i \notin W$ for $1 < i < s$.

Claim 1. *Under the above assumptions the following holds.*

(Φ) *If $u \in V_C$ and there is a $u - c$ path internally disjoint from C, then $u^+ c_s \notin E_G$, unless $u = c_1$.*

(Ψ) *For $u \in V_C$ there is no $u - u^+$ path internally disjoint from C.*

(Ω) *There is no $c_s - c$ path internally disjoint from C.*

To prove (Φ), assume there is such a $u - c$ path, say L, and the edge u^+c_s. If the path L is not internally disjoint from $[c_1, c]_P$, then let w denote the first vertex on the path L (passing through u towards c) which is also an inner vertex of $[c_1, c]_P$. Moreover, if $[u, w]_L$ is not internally disjoint from $[c, c_t]_P$ as well, let x denote the first vertex on the path $[u, w]_L$ (passing through u towards w) which is also an inner vertex of $[c, c_t]_P$ and let y denote the first vertex on the path $[w, u]_L$ (passing through w towards u) which is also an inner vertex of $[c, c_t]_P$.
We distinguish several cases according to u and L. Let $u \in \{c_2, c_3, \ldots, c_s\}$ ($u \in \{c_s^+, \ldots, c_k\}$). If L is internally disjoint from $[c_1, c]_P$, then, since $c_2, c_3, \ldots, u^- \notin W$ and $c \in W$, the cycle $[c_1, c]_P \circ [c, u]_L \circ [uu^+, c_1]_C$ ($[c_1, c]_P \circ [c, u]_L \circ [uu^-, c_s]_C \circ c_s u^+ \circ [u^+(u^+)^+, c_1]_C$) contains more vertices from W than C, a contradiction. If L is not internally disjoint from $[c_1, c]_P$ and $[u, w]_L$ is internally disjoint from $[c, c_t]_P$, then consider the cycle $F = [c_1, w]_P \circ [w, u]_L \circ [uu^+, c_1]_C$ ($F = [c_1, w]_P \circ [w, u]_L \circ [uu^-, c_s]_C \circ c_s u^+ \circ [u^+(u^+)^+, c_1]_C$). Obviously, it contains all the vertices from W that are on C. Moreover, the path $J = [w, c_t]_P$ is a path containing c which is internally disjoint from F and $V_F \cap V_J = \{w, c_t\}$. It is a matter of routine to observe that we may assume that F is non-extendable. But the path $[w, c]_J$ is shorter than the path $[c_1, c]_P$, contradicting (i). If L is not internally disjoint from $[c_1, c]_P$ and $[u, w]_L$ is not internally disjoint from $[c, c_t]_P$, then considering the cycle $F = [c_1, w]_P \circ [w, y]_L \circ [y, x]_P \circ [x, u]_L \circ [uu^+, c_1]_C$ ($F = [c_1, w]_P \circ [w, y]_L \circ [y, x]_P \circ [x, u]_L \circ [uu^-, c_s]_C \circ c_s u^+ \circ [u^+(u^+)^+, c_1]_C$) and the path $J = [w, y]_P$, or the path $J = [w, x]_P$ if x precedes y on $[c, c_t]_P$, we obtain again a contradiction with (i).

The part (Ψ) follows directly from the assumption that C is non-extendable. The part (Ω) has been proved, in fact, in (Φ) - the case $u = c_s$.

Consider the following sets; see Figure 1.

$$\begin{aligned}
X &= \{x;\ x \notin V_C,\ xc \in E_G\},\\
Y &= \{y;\ y \in V_C,\ y^-c \in E_G\} - \{c_2\} + \{c\},\\
Z &= \{z;\ \exists k \in H :\ ck, kz \in E_G,\ zc, zc_s \notin E_G,\ z \neq c\},\\
R &= \{r;\ r \in V_C,\ \exists l \in H :\ cl, lr^-, r^-c_s \in E_G,\ r^-c \notin E_G\} - \{c_2\},\\
U &= \{u;\ u \in V_C,\ uc \in E_G,\ u^-c, uc_s \notin E_G\},\\
T &= \{t;\ t \in V_C,\ d(t, c) \geq 3,\ d(t, c_s) \geq 3\}.
\end{aligned}$$

Claim 2. *The sets X, Y, Z, R, U, and T are pairwise disjoint.*

This follows immediately from their definitions for all pairs with the exceptions of Y, Z; Z, R; and U, R. So, assume there is a vertex $u \in Y \cap Z$. Since $u \in Y$, $u \in V_C$ and there is the edge $u^- c$. Similarly, since $u \in Z$, there is a vertex $k \in H$ and edges ck, ku in G. Thus the path $u^- \, c \, k \, u$ is internally disjoint from C, contradicting (Ψ). Let $u \in Z \cap R$. Then $u \in V_C$ and there exist k and l in H so that $ck, ku, cl, lu^- \in E_G$. If $k \neq l$, then the path $u \, k \, c \, l \, u^-$ is internally disjoint from C, contradicting (Ψ). If $k = l$, then $u \, k \, u^-$ is such a path. Finally, let $u \in U \cap R$. Since $u \in U$, $u \in V_C$ and the edge uc is in G. Since $u \in R$, there is a vertex $l \in H$ and edges $u^- l, lc$ in G. But now the path $u \, c \, l \, u^-$ is internally disjoint from C, again contradicting (Ψ). This proves the claim.

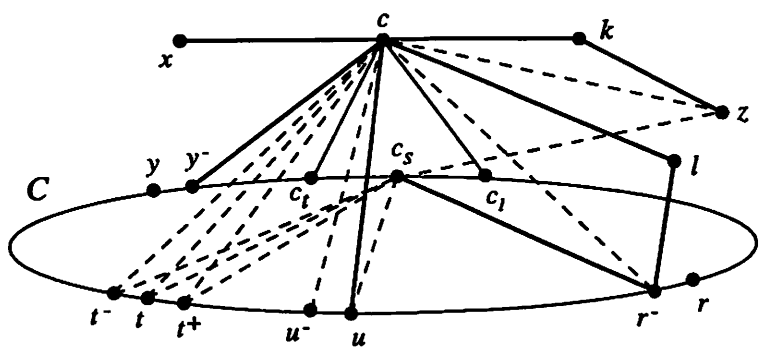

Figure 1:

Claim 3. *No neighbour of c_s is in $L = X \cup Y \cup Z \cup R \cup U \cup T$.*

This follows directly from their definitions for all these sets with the exceptions of X, Y, and R. Consider the case $c_s x$, $x \in X$, is an edge of G. Then the path $c_s \, x \, c$ is internally disjoint from C, contradicting (Ω). Let $c_s y$, $y \in Y$, be an edge of G. Note that, by assumptions, $y \neq c, c_2, c_s$. Since $y^- \neq c_1$, the edge $y^- c$ is a path internally disjoint from C, contradicting (Φ). Finally, let rc_s, $r \in R$, be an edge of G. Note that $r \neq c_2, c_s$. Then the path $r^- \, l \, c$ is internally disjoint from C, contradicting (Φ). Thus no neighbour of c_s is in L. This proves Claim 3.

In what follows we estimate the cardinalities of the six sets to obtain an upper bound for the degree of the vertex c_s.

Claim 4. *The sets X, Y, Z, R, U, and T satisfy the following:*

(1) $|X \cup Y| \geq d_H(c) + d_C(c) = d_G(c)$;

(2) $|Z \cup R \cup U| \geq pos(\psi(c, c_s) - 1)$;

(3) $|T| \geq pos(\alpha(c, c_s) - \beta(c, c_s) - 1)$.

(1) holds trivially. To prove (2), let m (l) denote the number of components of $\langle N(c) \rangle$ containing no neighbour of c_s and containing some (no) vertex of V_C. Obviously, $m + l = \psi(c, c_s)$.

First, consider the components containing some vertex from V_C. Let C_j, $1 \leq j \leq m$, be one of them. Let c_i be the vertex from $V_{C_j} \cap V_C$ with the smallest subscript i. The vertex c_{i-1} cannot be adjacent to c, otherwise it would belong to C_j and i would not be the smallest. It follows from the definition of C_j that $cc_i \in E_G$ and $c_s c_i \notin E_G$, thus c_i belongs to U. We can find for each such component a vertex from U, and so $|U| \geq m$.

Second, consider the components containing no vertex from V_C; say $D_1, \ldots, D_l$. Define a bipartite graph K with bipartition (A, B) as follows. The set A contains l vertices corresponding to the components D_s $(1 \leq s \leq l)$. The set B consists of vertices corresponding to the vertices in the set $N(\cup_{i=1}^{l} D_i) \setminus \{c\}$. We define the edge-set of K as: The edge ab, $a \in A$ and $b \in B$, is from K iff the corresponding vertex to b is from $N(D_a)$, where D_a is the corresponding component to the vertex a. By the well-known Hall's theorem, K contains a matching M that saturates every vertex in A. Indeed, if this is not the case, then, by Hall's theorem, there is a subset S of A such that $|N(S)| < |S|$. Without loss of generality we assume that S consists of vertices corresponding to $D_1, \ldots, D_j$ $(j \leq l)$. Now it follows from the construction of K that $|N(\cup_{i=1}^{j} D_i) \setminus \{c\}| < j$. If we delete all vertices from $N(\cup_{i=1}^{j} D_i) \cup \{c\}$, then G becomes disconnected with components $D_1, \ldots, D_j$, and at least one another component containing the vertex c_s. Indeed, since $c_s c \notin E_G$, the vertex $c_s \notin \cup_{i=1}^{l} D_i$ and, by (Ω), $c_s \notin N(\cup_{i=1}^{l} D_i)$. But this contradicts the fact that G is 1-tough. Thus there is such a matching M guaranteeing the existence of l vertices, say $v_1, \ldots, v_l$, from $N(\cup_{i=1}^{l} D_i) \setminus \{c\}$, one for each D_i; say the vertex v_i is adjacent to D_i $(i = 1, \ldots, l)$. If there is not the edge $c_s v_i$, then let v be the vertex of D_i adjacent to v_i. Thus there are edges cv, vv_i and are not edges cv_i (since v_i is not from D_i), and $c_s v_i$, and it follows that $v_i \in Z$. If there is the edge $c_s v_i$, then, by (Ω), the vertex $v_i \in V_C$. Since v_i is not from D_i, there is not the edge $v_i c$, and it follows that if $v_i \neq c_1$, then $v_{i+1} \in R$. We have observed that $|Z \cup R| \geq l - 1$. By Claim 2, $|Z \cup R \cup U| \geq \text{pos}(\psi(c, c_s) - 1)$.

To obtain a lower bound on the cardinality of the set T let $v_1, \ldots, v_{\alpha(c,c_s)}$ be the common neighbours of c and c_s. By (Ω), all these vertices lie on C.

Claim 4.1. *If there are two vertices $x_i, x_j \in V_C \setminus \{c_2, \ldots, c_{t-1}\}$ such that $i < j$ and $cx_i, c_s x_j \in E_G$, then a vertex $x_l \in V_C$ must exist such that*

$i < l < j$ and $cx_l, c_s x_l \notin E_G$.
If $x_i = c_1$, then, by assumptions $j \geq t$, and the claim follows (by taking $x_l = c_s$). Thus let $x_i \neq c_1$. Since the edge $x_i c$ is internally disjoint from C, by (Φ), we have $j - i \geq 2$. By (Ψ), the edge $x_{i+1}c \notin E_G$, and again by (Φ), the edge $x_{i+1}c_s \notin E_G$. We put $x_l = x_{i+1}$. Obviously, $i < i+1 < j$.

Since for $i = 2, \ldots, t-1$ we have $c_i c \notin E_G$, all vertices $v_1, \ldots, v_{\alpha(c,c_s)}$ belong to $V_C \setminus \{c_2, \ldots, c_{t-1}\}$. Recall that in our notation c_1 can be substituted by c_{k+1}. Now, by Claim 4.1, there are $\alpha(c, c_s)$ vertices on C that are adjacent neither to c nor c_s. At most $\beta(c, c_s)$ of them are of distance 2 either from c or c_s, and at most one of them is c_s. Thus there are at least $\mathrm{pos}(\alpha(c, c_s) - \beta(c, c_s) - 1)$ vertices in T. This proves Claim 4.

Now we are ready to estimate the upper bound of $d(c_s)$. It holds $d(c_s) \leq |V_G| - |\{c_s\}| - |X \cup Y| - |Z \cup R \cup U| - |T| \leq n - 1 - d(c) - \mathrm{pos}(\psi(c, c_s) - 1) - \mathrm{pos}(\alpha(c, c_s) - \beta(c, c_s) - 1) \leq n - 1 - d(c) - \chi(c, c_s)$. But the vertices c and c_s are non-adjacent and both are from the set W, a contradiction.

References

[1] Bollobás, B., Brightwell, G.: *Cycles through specified vertices*, Combinatorica **13**, (1993), 147-155.

[2] Polický, I.: *A sufficient condition for hamiltonian graphs*, to appear in Math. Slovaca.

[3] Shi, R.: *2-Neighborhoods and hamiltonian conditions*, J. Graph Theory **16**, (1992), 267-271.

[4] Stacho, L.: *Old hamiltonian ideas from a new point of view*, Mathematische Nachrichten, **186**, 1997, 251-258.

[5] Stacho, L.: *A sufficient condition guaranteeing large cycles in graphs*, Discrete Mathematics, **169**, 1997, 273-277.