On the Symmetry Groups of Hypergraphs of Perfect Cwatsets

Daniel K. Biss*
Department of Mathematics
Harvard University
Cambridge, MA 02138

October 1, 1997

Abstract

Cwatsets are subsets of $\mathbb{Z}_2^d$ which are nearly subgroups and which naturally appear in statistics and coding theory [8]. Each cwatset can be represented by a highly symmetric hypergraph [7]. We introduce and study the symmetry group of the hypergraph and connect it to the corresponding cwatset. We use this connection to establish structure theorems for several classes of cwatsets.

1 Introduction

Cwatsets are subsets of $\mathbb{Z}_2^d$ that were introduced by G.J. Sherman and M. Wattenberg [8]. They are generalizations of subgroups of $\mathbb{Z}_2^d$ and they appear naturally in statistics and in coding theory (see [1], [8]). In [8], using standard group-theoretic tools, some basic properties about cwatsets were proven, most notably the fact that they are projections into $\mathbb{Z}_2^d$ of subgroups of $\mathbb{Z}_2^d \rtimes_\phi S_d$, the semidirect product of $\mathbb{Z}_2^d$ by S_d. To date, one paper [7] has been published in response to [8]. In [7], perfect cwatsets were introduced. The hope was that any cwatset could be represented as a direct sum of perfect cwatsets; although this turned out not to be the case, it seems that studying perfect cwatsets could still be fruitful. Indeed, a one-to-one correspondence between perfect cwatsets and a certain type of hypergraphs was established, and the combinatorial properties of these hypergraphs were used to prove a number of statements about the orders of cwatsets.

*Please address all correspondence to: 4685 Heritage Woods Road, Bloomington, IN 47401.

After first defining morphisms of cwatsets and deriving several basic properties of the category of cwatsets (such as characterizing isomorphisms of cwatsets in Theorem 1), we attempt to unify the two earlier approaches by introducing a group-theoretic structure, the structure group of a cwatset, which is directly related to the hypergraph of a cwatset (see our Theorem 2). Theorem 3 then shows that every subgroup of the symmetric group S_d whose action on $\{1, 2, \ldots, d\}$ is transitive is a subgroup of the structure group of some perfect cwatset. Theorem 3 also provides a way of checking whether the subgroup is in fact the structure group of a perfect cwatset. Theorem 4 provides a decomposition of perfect cwatsets into irreducible cwatsets which are even simpler than the perfect cwatsets from [7], and Theorem 5 establishes a number of properties of this decomposition.

The results through Theorem 5 build a fairly specific group-theoretic and graph-theoretic structure; enough so that the standard literature on graph theory is now of great use in classifying cwatsets through their hypergraphs. By applying these results to graphs in Sections 6 and 7, we are able to completely classify a small class of cwatsets (see Theorem 6). Using these same facts about groups and graphs, and applying them to more general cases, we obtain existence of a number of cwatsets of various orders (see Theorems 7 and 8).

2 Preliminaries

Recall that $\mathbb{Z}_2^d$ is the group of binary strings of length d under the operation of digit-by-digit addition without carrying. For $\mathbf{b} \in \mathbb{Z}_2^d$ and $\sigma \in S_d$, we write as $\mathbf{b}^\sigma$ the permutation by σ of the digits of $\mathbf{b}$, i.e. if $\mathbf{b} = (a_1, a_2, \ldots, a_d)$ with $a_i \in \mathbb{Z}_2$, $\mathbf{b}^\sigma = (a_{\sigma^{-1}(1)}, a_{\sigma^{-1}(2)}, \ldots, a_{\sigma^{-1}(d)})$.

Let $\Omega = \{\mathbf{b}_1, \mathbf{b}_2, \ldots \mathbf{b}_n\}$ be a subset of $\mathbb{Z}_2^d$, and let $\sigma \in S_d$, the symmetric group of permutations on d elements. When we speak of Ω^σ, we mean $\{\mathbf{b}_1^\sigma, \mathbf{b}_2^\sigma, \ldots, \mathbf{b}_n^\sigma\}$.

Definition 1 [8] *A subset Ω of $\mathbb{Z}_2^d$ is a* cwatset *(of degree, or dimension, d) if for every $\mathbf{b} \in \Omega$, there exists a permutation $\sigma \in S_d$ such that $(\Omega + \mathbf{b})^\sigma = \Omega$.*

A subset of $\mathbb{Z}_2^d$ is a cwatset if and only if it is the projection of a subgroup of the semidirect product $\mathbb{Z}_2^d \rtimes_\phi S_d$ [8]; in fact, although it is never explicitly stated in [8], the proof of this fact produces a subgroup of $\mathbb{Z}_2^d \rtimes_\phi S_d$ which is the unique maximal group projecting onto Ω. We call this subgroup G^Ω. Here, for a permutation $\sigma \in S_d$ and $\mathbf{b} \in \mathbb{Z}_2^d$, we define $\phi(\sigma)(\mathbf{b}) = \mathbf{b}^\sigma$, the permutation of the digits of $\mathbf{b}$ by σ. Recall that the semidirect product $N \rtimes_\phi K$ of N and K, where $\phi : K \to \mathrm{Aut}(N)$, is the set $N \times K$ along with the operation $(n_1, k_1) \cdot (n_2, k_2) = [n_1 \cdot (\phi(k_1))(n_2), k_1 \cdot k_2]$. It follows that a cwatset of degree d must have order dividing $2^d d!$ (see [8]). The converse does not hold; that is, there exists k which divides $2^d d!$ and $k < 2^d$ for which there is no cwatset in $\mathbb{Z}_2^d$ of order k (for example, there is no cwatset of degree 5 and order 15 [7]). This result naturally brings up the question of exactly what orders of cwatsets can be constructed given a fixed degree. This question was the focus of [7] and motivates most of the constructions that will be used in our discussion.

It is often useful to represent a cwatset by a matrix whose rows are the elements of the cwatset. Hence, a permutation is simply an action on the columns.

For convenience, in the matrix representation of a cwatset, a column containing k 1's is called a *k-column*. Since the number of 1's in a column is invariant under a permutation of the rows, the definition of a k-column is independent of the matrix representation.

Definition 2 [7] *A cwatset of order n is called* perfect *if for some k, all of its columns are either k-columns or $(n-k)$-columns.*

We call $\max\{k, n-k\}$ the splitting number of Ω.

Definition 3 [7] *Let Ω be a perfect cwatset of degree d and order n with splitting number k strictly larger than $n/2$. Then Ω is called a (d,m)-cwatset if the number of k-columns is exactly m.*

Definition 4 [7] *A* hypergraph *is a set of labelled vertices and edges, in which any given edge and vertex are called either incident or not incident. (Observe that a graph is a hypergraph with the limitation that each edge is incident to exactly two vertices.)*

In [7], the hypergraph $H(\Omega)$ associated with a (d,m)-cwatset Ω is defined as follows: Each column represents a vertex, and each row represents an edge. An edge is incident to a vertex if the corresponding column is a k-column and the entry of the corresponding row in that column is 0, or if the column is an $(n-k)$-column with a 1 in the corresponding row.

Since there is a well-defined correspondence between the cwatsets, matrices, and hypergraphs that we will be considering, we may speak of the columns or rows of a hypergraph, or of the edges and vertices of a matrix or cwatset.

It is easy to check that the hypergraph of a (d,m)-cwatset of order n and splitting number k has d vertices, n edges, and each vertex is incident to $n-k$ edges. Furthermore, each edge is incident to exactly m vertices [7]. Consequently, the hypergraph of a $(d,2)$-cwatset is simply a graph, which makes $(d,2)$-cwatsets considerably easier to study.

We say that a hypergraph is *regular* if each vertex is incident to the same number of edges, and *uniform* if each edge is incident to the same number of vertices. Hence, every graph is uniform.

The hypergraph of each perfect cwatset has the following three properties [7]:

(i) It is regular.

(ii) It is uniform.

(iii) For any two edges e and f, there is a permutation on the vertices which sends edges onto edges and e onto f. (A graph with this property is called an *edge-symmetric* graph [3].)

Also, if a hypergraph has the properties (i), (ii), and (iii), then it represents a perfect cwatset [7].

Notice also that it is clear that if there is a (d,m)-cwatset Ω of order n, with splitting number k, then for every natural number μ there is a $(d\mu, m)$-cwatset Ω_μ of order $n\mu$ and splitting number $k + n(\mu - 1)$. Indeed, if we take $H(\Omega_\mu)$ to be μ disjoint copies of $H(\Omega)$, then it is clear that $H(\Omega_\mu)$ has properties (i), (ii), and (iii), and furthermore, it has $d\mu$ vertices and $n\mu$ edges, and each edge is still incident to m vertices, and each vertex is still incident to $n - k$ edges, so the splitting number becomes $n\mu - (n - k) = k + n(\mu - 1)$.

3 Morphisms of Cwatsets

None of the earlier work on cwatsets discusses any sorts of maps between cwatsets; thus cwatsets cannot be considered to form a category. In fact, there is not even a notion of what it means for two cwatsets to be isomorphic. The aim of this section is to endow the set of cwatsets with a category structure. Obviously, this structure would have to reflect both the additive and permutation structures that cwatsets carry.

Definition 5 *Let Ω_1 and Ω_2 be cwatsets, and G^{Ω_1} and G^{Ω_2} their corresponding subgroups of $\mathbb{Z}_2^d \rtimes_\rho S_d$; we define a* morphism $\varphi : \Omega_1 \to \Omega_2$ *to be a map making the following diagram commutative, where $\psi : G^{\Omega_1} \to G^{\Omega_2}$ is some group homomorphism, and the vertical maps π_1 are the projections. The set of all such φ is denoted $Mor(\Omega_1, \Omega_2)$.*

$$\begin{array}{ccc} G^{\Omega_1} & \xrightarrow{\ \psi\ } & G^{\Omega_2} \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_1} \\ \Omega_1 & \xrightarrow{\ \varphi\ } & \Omega_2 \end{array}$$

Notice that there may exist no such φ for a given ψ. In particular, if $(\mathbf{b}, \sigma)$ and $(\mathbf{b}, \sigma')$ are two elements of G^{Ω_1}, then $\varphi(\pi_1(\mathbf{b}, \sigma)) = \varphi(\mathbf{b}) = \varphi(\pi_1(\mathbf{b}, \sigma'))$, and $\pi_1(\psi(\mathbf{b}, \sigma)) = \pi_1(\psi(\mathbf{b}, \sigma'))$, so the first coordinate of $\psi(\mathbf{b}, \sigma)$ must not depend on σ (this is not surprising, since, after all, our goal is to define a function of $\mathbf{b}$). This is equivalent to the requirement that $\pi_1(\psi(\mathbf{0},\sigma)) = \mathbf{0}$ for all $(\mathbf{0},\sigma) \in G^{\Omega_1}$.

Proposition 1 *With morphisms defined as above, cwatsets form a category which contains a zero object, and for which kernels and images are defined.*

PROOF: It is elementary to show that cwatsets form a category: if $\Omega_1 = \Omega_2$, then $G^{\Omega_1} = G^{\Omega_2}$ and $\psi = \text{id}$ obviously induces the identity map on Ω_1, and if we have cwatsets Ω_1, Ω_2, and Ω_3, with corresponding groups G^{Ω_1}, G^{Ω_2}, and G^{Ω_3}, and maps $\varphi_1 : \Omega_1 \to \Omega_2$, $\varphi_2 : \Omega_2 \to \Omega_3$, $\psi_1 : G^{\Omega_1} \to G^{\Omega_2}$, and $\psi_2 : G^{\Omega_2} \to G^{\Omega_3}$, then $\psi := \psi_2 \circ \psi_1$ induces $\varphi := \varphi_2 \circ \varphi_1$, so we have a composition law.

The cwatset $\{0\}$ serves as a zero object (an object which is both initial and final). We also have kernels; indeed, if $\varphi : \Omega_1 \to \Omega_2$ is a morphism, then $\Omega_0 = \{\mathbf{b} \in \Omega_1 | \varphi(\mathbf{b}) = \mathbf{0}\}$ is a cwatset itself. Indeed, if $\mathbf{b}_0 \in \Omega_0$, there exists a permutation σ_0 with $\Omega_1^{\sigma_0} + \mathbf{b}_0 = \Omega_1$. We must show that $\Omega_0^{\sigma_0} + \mathbf{b}_0 = \Omega_0$. Let $\mathbf{b} \in \Omega_0$. Then $\mathbf{b}^{\sigma_0} + \mathbf{b}_0 \in \Omega_0$ because $\psi(\mathbf{b}^{\sigma_0} + \mathbf{b}_0, \sigma_0\sigma) = \psi((\mathbf{b}_0, \sigma_0) \cdot (\mathbf{b}, \sigma)) = \psi(\mathbf{b}_0, \sigma_0) \cdot \psi(\mathbf{b}, \sigma) = (\mathbf{0}, \tau_0) \cdot (\mathbf{0}, \tau) = (\mathbf{0}, \tau_0\tau)$, so $\Omega_0^{\sigma_0} + \mathbf{b}_0 \subset \Omega_0$ and hence $\Omega_0^{\sigma_0} + \mathbf{b}_0 = \Omega_0$. The injection $i : \Omega_0 \to \Omega_1$ can easily be seen to satisfy the universal property of the kernel.

We can also define the image of a map $\varphi : \Omega_1 \to \Omega_2$ by $\text{im}(\varphi) = \{\mathbf{b}_2 \in \Omega_2 | \exists \mathbf{b}_1 \in \Omega_1, \varphi(\mathbf{b}_1) = \mathbf{b}_2\}$. This is a cwatset, because if $\psi : G^{\Omega_1} \to G^{\Omega_2}$ is the map corresponding to φ on subgroups of the semidirect product, $\text{im}(\varphi) = \pi_1(\text{im}(\psi))$. □

Now that we have this category, perhaps the first reasonable issue to consider is that of what constitutes an isomorphism of cwatsets. That is, if $\psi : G^{\Omega_1} \to G^{\Omega_2}$ induces an isomorphism $\varphi : \Omega_1 \to \Omega_2$, what can be said about ψ? Certainly, ψ need not be a group isomorphism; indeed, the zero cwatset Ω_0^1 in $\mathbb{Z}_2^1$ and the zero cwatset Ω_0^2 in $\mathbb{Z}_2^2$ are certainly isomorphic, but $G^{\Omega_0^1} = \langle e \rangle$, the group of one element, whereas $G^{\Omega_0^2} \cong \mathbb{Z}_2$. However, note that in the cwatset $\Omega_0^2 = \{00\}$, the two columns are identical: we will soon see that all obstructions to making ψ an isomorphism are of this form.

Definition 6 *For a cwatset Ω, the* isotropy group *is the subgroup $I < S_d$ consisting of all elements σ such that for all* $\mathbf{b} \in \Omega$, $\mathbf{b}^\sigma = \mathbf{b}$.

Note that the isotropy subgroup of any cwatset Ω will be of the form $I \cong S_{m_1} \times \ldots \times S_{m_s}$, where the m_i form a partition of d, the number of columns of Ω, and each m_i corresponds to some set of m_i identical columns. The following theorem was asserted in an earlier version of this article; the author would like to thank C. Girod et al. [6] for pointing out an error in the proof and providing a correction.

Theorem 1 *For any cwatset Ω, the group $0 \times I = \{(0,\sigma)|\sigma \in I\}$ is normal in G^{Ω}. A morphism $\varphi : \Omega_1 \to \Omega_2$ of cwatsets is an isomorphism if and only if it arises from a homomorphism $\psi : G^{\Omega_1} \to G^{\Omega_2}$ which induces a quotient map $\overline{\psi} : G^{\Omega_1}/(0 \times I_1) \to G^{\Omega_2}/(0 \times I_2)$ that is an isomorphism.*

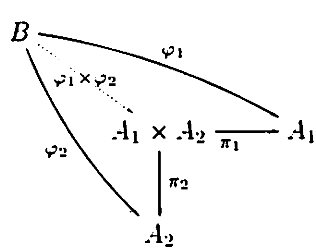
Notice that if $\psi : G^{\Omega_1} \to G^{\Omega_2}$ is an arbitrary map giving a morphism of cwatsets, then we do not necessarily have $\psi(0 \times I_1) \subset (0 \times I_2)$. For example, if $\Omega_1 = \{00, 11\}$ and $\Omega_2 = \{00, 01, 10, 11\}$, then $G^{\Omega_i} = \Omega_i \rtimes S_2$, but $I_1 = S_2$ whereas $I_2 = \{\mathrm{id}\}$. If ψ is the inclusion $\psi : G^{\Omega_1} \hookrightarrow G^{\Omega_2}$, then $\psi(I_1) = S_2 \not\subset I_2$. Hence, we cannot always define a quotient morphism $\overline{\psi} : G^{\Omega_1}/(0 \times I_1) \to G^{\Omega_2}/(0 \times I_2)$.

Recall that a functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is said to be fully faithful if for every pair of objects $A, B \in \mathcal{C}$, $\mathcal{F} : \mathrm{Mor}(A, B) \to \mathrm{Mor}(\mathcal{F}(A), \mathcal{F}(B))$ is an isomorphism, and that $\mathcal{F}$ is said to be essentially surjective if for every object $B \in \mathcal{D}$, there is an object $A \in \mathcal{C}$ with $\mathcal{F}(A) \cong B$. Furthermore, $\mathcal{F}$ induces an equivalence of categories if and only if $\mathcal{F}$ is fully faithful and essentially surjective. Now let $\mathcal{G}$ be the category whose objects are groups of the form G^{Ω} and whose morphisms are group homomorphisms which induce morphisms of cwatsets, where morphisms which induce the same map on cwatsets are identified.

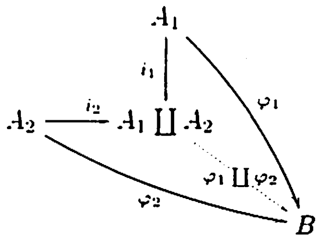
Corollary 1 *The functor $\mathcal{F}$ from the category of cwatsets to $\mathcal{G}$ given by $\mathcal{F}(\Omega) = G^{\Omega}$ is an equivalence of categories. Furthermore, two objects G^{Ω_1} and G^{Ω_2} in $\mathcal{G}$ are isomorphic if and only if there exist maps $\psi_1 : G^{\Omega_1} \to G^{\Omega_2}$ and $\psi_2 : G^{\Omega_2} \to G^{\Omega_1}$ with $\psi_1(0 \times I_1) \subset 0 \times I_2$, $\psi_2(0 \times I_2) \subset 0 \times I_1$, $(\psi_2 \circ \psi_1)^{-1}(0 \times I_1) = 0 \times I_1$, and $(\psi_1 \circ \psi_2)^{-1}(0 \times I_2) = 0 \times I_2$.*

This furnishes us a strong connection with group theory; it suggests that studying the category $\mathcal{G}$ might be of interest in future work concerning cwatsets.

We now give one last construction which will be of use later. Recall that if A_1, A_2, and $A_1 \times A_2$ are objects in a category with maps $\pi_1 : A_1 \times A_2 \to A_1$ and $\pi_2 : A_1 \times A_2 \to A_2$ such that for every object B with maps $\varphi_1 : B \to A_1$ and $\varphi_2 : B \to A_2$ there is a unique map $\varphi_1 \times \varphi_2 : B \to A_1 \times A_2$ making the diagram



commute, then $A_1 \times A_2$ is called the product of A_1 and A_2. Also, if $A_1 \coprod A_2$ is an object with maps $i_1 : A_1 \to A_1 \coprod A_2$ and $i_1 : A_2 \to A_1 \coprod A_2$ such that for every object B with maps $\varphi_1 : A_1 \to B$ and $\varphi_2 : A_2 \to B$ there is a unique map $\varphi_1 \coprod \varphi_2 : A_1 \coprod A_2 \to B$ making the diagram



commute, then $A_1 \coprod A_2$ is called the coproduct of A_1 and A_2. Lastly, in a category where for every pair A_1 and A_2, $A_1 \times A_2$ and $A_1 \coprod A_2$ exist and are equal (note that uniqueness of products and coproducts follows immediately from the definition), then the object $A_1 \times A_2$ is denoted $A_1 \oplus A_2$ and is called the direct sum of A_1 and A_2.

The following definition was the first step toward the decomposition theorem presented in [7]:

Definition 7 [7] *The direct sum $\Omega_1 \oplus \Omega_2$ of cwatsets $\Omega_1 = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{k_1}\}$ of dimension d_1 and $\Omega_2 = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_{k_2}\}$ of dimension d_2 is the $(d_1 + d_2)$-dimensional cwatset of order $k_1 k_2$ with elements of the form $\mathbf{b}_i\mathbf{c}_j$, where the first d_1 digits form an element of Ω_1, and the last d_2 digits form an element of Ω_2.*

It was suggested but not proven in [7] that any cwatset is the direct sum of perfect cwatsets. In particular, a computer search gave the counterexample $\{0000000000, 1100011000, 1010011110, 1001001111, 1000100011\}$. However, because of the combinatorial structure associated with perfect cwatsets, it seems that the study of perfect cwatsets could be of interest in its on right, and therefore, from now on we restrict our discussion to perfect cwatsets.

Notice that in our category, given cwatsets Ω_1 and Ω_2, it is easy to see that the maps $\pi_1 : \Omega_1 \oplus \Omega_2 \to \Omega_1$ and $\pi_2 : \Omega_1 \oplus \Omega_2 \to \Omega_2$ defined by $\pi_1(\mathbf{b}_i\mathbf{c}_j) = \mathbf{b}_i$ and $\pi_2(\mathbf{b}_i\mathbf{c}_j) = \mathbf{c}_j$ make $\Omega_1 \oplus \Omega_2$ into the product $\Omega_1 \times \Omega_2$, and the maps $i_1 : \Omega_1 \to \Omega_1 \oplus \Omega_2$ and $i_2 : \Omega_2 \to \Omega_1 \oplus \Omega_2$ defined by $i_1(\mathbf{b}) = \mathbf{b0}$ and $i_2(\mathbf{c}) = \mathbf{0c}$ make $\Omega_1 \oplus \Omega_2$ into the coproduct $\Omega_1 \coprod \Omega_2$, so $\Omega_1 \oplus \Omega_2$ really is the direct sum.

4 The Structure Group of a Cwatset

In [8], cwatsets were studied using primarily group-theoretic analysis. In [7], the focus shifted exclusively to hypergraphs. We will demonstrate that the two constructions are connected, using the notion of the structure group of a cwatset.

Recall that a perfect cwatset is a cwatset of order n each of whose columns have either k or $n-k$ 1's for some integer k. We define a perfect subset of $\mathbb{Z}_2^d$ to be a set of order n which, when arranged in matrix form, only has k-columns and $(n-k)$-columns. Hence, since the only combinatorial property of perfect cwatsets that was used in the construction of their hypergraphs was the fact that they were perfect, we can define the hypergraph of a perfect subset of $\mathbb{Z}_2^d$ in an analogous manner (since we still have k-columns and $(n-k)$-columns).

Lemma 1 *For any perfect set $\Omega \subset \mathbb{Z}_2^d$ and for every $\mathbf{b} \in \mathbb{Z}_2^d$, $H(\Omega) = H(\Omega + \mathbf{b})$.*

PROOF: Suppose, without loss of generality, that the ith column of Ω is a k-column. Pick an arbitrary $\mathbf{b} \in \mathbb{Z}_2^d$. If the ith place of $\mathbf{b}$ is a 0, then the ith column of $\Omega + \mathbf{b}$ is the same as the ith column of Ω and hence, by definition, is incident to the same rows. If the ith place of $\mathbf{b}$ is a 1, then the ith column of $\Omega + \mathbf{b}$ is an $(n-k)$-column, with 1's in exactly the same places as there were 0's in the ith column of Ω, so again, it is incident to the same rows. Since this is true for any column, the two hypergraphs are equal. □

Definition 8 *The group of automorphisms, $S(H)$, of a hypergraph H is the group of all permutations of the vertices which map edges onto edges.*

Note that $\sigma \in S(H(\Omega))$ if and only if $H(\Omega) = H(\Omega^\sigma)$.

Definition 9 *The structure group G_Ω of a cwatset Ω is*

$$\{\sigma \in S_d | \textit{there exists } \mathbf{b} \in \Omega \textit{ such that } \Omega^\sigma = \Omega + \mathbf{b}\}.$$

As we will now see, we are justified in calling this object a group; in fact, it is closely associated with the structure of both Ω and $H(\Omega)$.

Theorem 2 *For any cwatset Ω, $S(H(\Omega)) = G_\Omega$.*

PROOF: We first show that $G_\Omega \subset S(H(\Omega))$. Fix $\sigma \in S_d$, and suppose there exists $\mathbf{b} \in \Omega$ such that $\Omega + \mathbf{b} = \Omega^\sigma$. From Lemma 1, $H(\Omega) = H(\Omega + \mathbf{b})$, and clearly $H(\Omega + \mathbf{b}) = H(\Omega^\sigma)$, so $H(\Omega) = H(\Omega^\sigma)$; this means that $\sigma \in S(H(\Omega))$.

Next we prove that $S(H(\Omega)) \subset G_\Omega$. Suppose $\sigma \in S(H(\Omega))$. Then $H(\Omega) = H(\Omega^\sigma)$. Pick an arbitrary vertex of $H(\Omega)$. It corresponds to the ith column of Ω for some i. Also, the same vertex of $H(\Omega^\sigma)$ corresponds to the ith column of Ω^σ. Assume, without loss of generality (as in the proof of Lemma 1) that the ith column of Ω is a k-column. The same rows of Ω must be incident to the ith column as in Ω^σ, since their hypergraphs are equal. Hence, if the ith column of Ω^σ is a k-column, then it must be identical to the ith column of Ω. If the ith column of Ω^σ is an $(n-k)$-column, then its 1's must appear in exactly the same places as the 0's of the ith column of Ω, so that adding 1 to each entry in the column would produce the ith column of Ω. Therefore, if we let the ith place of $\mathbf{b} \in \mathbf{Z}_2^d$ to be 0 if the ith columns of Ω and Ω^σ are both k-columns or are both $(n-k)$-columns, and 1 if one of them is a k-column and the other is an $(n-k)$-column, then $\Omega + \mathbf{b} = \Omega^\sigma$. We will now show that $\mathbf{b} \in \Omega$, which would imply the desired result. Since Ω is a cwatset, $\mathbf{0} \in \Omega$, so $\mathbf{0} \in \Omega^\sigma = \Omega + \mathbf{b}$ and therefore $\mathbf{0} = \mathbf{a} + \mathbf{b}$ with $\mathbf{a} \in \Omega$. But $\mathbf{a} + \mathbf{b} = \mathbf{0}$ if and only if $\mathbf{a} = \mathbf{b}$; therefore $\mathbf{b} \in \Omega$. □

In fact, it is easy to see that G_Ω is the projection into S_d of G^Ω. Thus, the map $\Omega \mapsto G_\Omega$ defines a covariant functor from the category of cwatsets to the category of groups.

Since G_Ω acts transitively on the edges of Ω, the orbit of any edge is Ω itself. For any edge e, let I_e be the isotropy subgroup of G_Ω^i for e; that is, the subgroup of G_Ω which fixes e. Then the orbit of e is Ω, and the stabilizer of e is I_e, so $|\Omega| = \frac{|G_\Omega|}{|I_e|}$. Since, if Ω is perfect, $|\Omega|$ divides $|G_\Omega|$, which divides $|S_d| = d!$, we have

Proposition 2 *If Ω is a perfect cwatset in $\mathbf{Z}_2^d$, then $|\Omega|$ divides $d!$.* □

The divisibility result in [8] only stated that the order of Ω must divide $2^d d!$. Note however, that the new property does not hold for cwatsets that are not perfect. For example, the group $\mathbb{Z}_2^4$ is certainly a cwatset of order $2^4 = 16$ but 16 does not divide $24 = 4!$.

It does not seem reasonable to search for a complete classification of cwatsets based on their structure groups; it is easy to construct, even for fixed m, more than one distinct (d, m)-cwatset with the same structure group (for example, the graphs H_1 and H_2 on the vertices $\{1,2,3,4\}$ where H_1 has edges $\{1,3\}$ and $\{2,4\}$ and H_2 has edges $\{1,2\}$, $\{2,3\}$, $\{3,4\}$, and $\{4,1\}$). However, it would be of use to understand which groups can appear as structure groups of certain cwatsets.

If G is the structure group of a perfect cwatset, G acts both on the set of edges and the set of vertices of the hypergraph of the cwatset. We know that it acts transitively on these edges. We would now like to give some conditions on a subgroup of S_d for it to be the structure group of a cwatset.

Theorem 3 *Let $G < S_d$ give a transitive action on $\{1,2,\ldots,d\}$. Then for every $m < \frac{d}{2}$ there is a (d,m)-cwatset Ω such that $G < G_\Omega$.*

PROOF: Label the d vertices $\{1,2,\ldots,d\}$ and define e to be the edge $\{1,2,\ldots,m\}$. Let $H_{G,m}$ be the hypergraph whose edges are $\{e^\sigma | \sigma \in G\}$. It is obvious that $G < S(H_{G,m})$. We now show that $H_{G,m}$ is the hypergraph of some perfect cwatset Ω, and that this cwatset is a (d,m)-cwatset.

Notice first of all that since G acts transitively on the vertices, $|\{\sigma \in G|\sigma(i) = j\}|$ is independent of the choice of i and j Indeed, let i be a vertex, and let I_i be the isotropy subgroup for i; then for every j, $\{\sigma \in G \mid \sigma(i) = j\}$ is a coset of I_i, so its order is independent of the choice of j. Hence, we must only show that $|I_i| = |I_j|$ for every i and j. Fix i and j. Since G acts transitively, there exists $\sigma \in G$ such that $\sigma(i) = j$. Hence, $I_j = I_{\sigma(i)}$. But $\tau \in I_i$ if and only if $\tau(i) = i$, which occurs if and only if $\sigma\tau(i) = j$, which occurs if and only if $\sigma\tau\sigma^{-1}(j) = j$, meaning that $I_j = I_{\sigma(i)} = \sigma I_i \sigma^{-1}$, so $|I_i| = |I_j|$.

Because of this fact, $H_{G,m}$ is regular, for the number of edges incident to a vertex v is $|\{\sigma \in G|\sigma(i) = v, i \in \{1,2,\ldots,m\}\}| = m|I_i|$. Also, $H_{G,m}$ is uniform, because each edge is of the form $\{\sigma(1),\sigma(2),\ldots,\sigma(m)\}$ and hence has m elements (this also implies that Ω is a (d,m)-cwatset). Lastly, it is evident by construction that $H_{G,m}$ is edge-symmetric.

Therefore, $H_{G,m} = H(\Omega)$, so $G < S(H_{G,m}) = S(H(\Omega)) = G_\Omega$. □

Note that Ω is a minimal (d,m)-cwatset with $G < G_\Omega$ in the sense that if Ω' is a cwatset with $G < G_{\Omega'}$, and Ω and Ω' have at least one common edge, then $H_{G,m}$ is isomorphic to a subhypergraph of $H(\Omega')$.

Furthermore, if G is in fact the structure group of a (d,m)-cwatset one of whose edges is $\{1,2,\ldots,m\}$, then $H_{G,m}$ is exactly the hypergraph of that cwatset. This gives us an easy way of determining whether a fixed subgroup of S_d is the structure group of a (d,m)-cwatset: G is the structure group of a (d,m)-cwatset if and only if for some ordering of the vertices, $G = S(H_{G,m})$.

5 Properties of Sums of Hypergraphs

Clearly, we would prefer to find a necessary and sufficient condition which would determine whether a given subgroup of S_d corresponds to a cwatset, without using an algorithm like the one described in the previous section. We first define a new form of irreducible cwatset whose hypergraph has stronger symmetry properties.

Definition 10 *Two vertices u and v of a hypergraph H are* isomorphic *if there exists an automorphism of H mapping u onto v; we write $u \cong v$. If all of the vertices of a graph are isomorphic, we call the graph* point-symmetric.

For convenience we will denote an edge by the set of vertices to which it is incident.

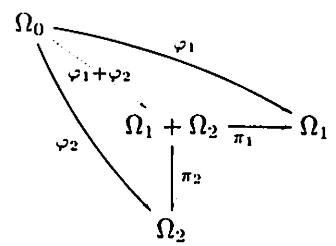
Definition 11 *A hypergraph H whose vertices and edges are $v_1, v_2, \ldots, v_d$ and $e_1, e_2, \ldots, e_n$ respectively, is a sum $H_1 + H_2$ of two hypergraphs H_1, with vertices $v_1, v_2, \ldots, v_l$ and edges $e_1^1, e_2^1, \ldots, e_n^1$ and H_2, with vertices $v_{l+1}, v_{l+2}, \ldots, v_d$ and edges $e_1^2, e_2^2, \ldots, e_n^2$ if $e_i^1 = e_i \cap \{v_1, v_2, \ldots, v_l\}$ and $e_i^2 = e_i \cap \{v_{l+1}, v_{l+2}, \ldots, v_n\}$. A sum of more than two hypergraphs is defined in an analagous manner.*

Translating this from the language of hypergraphs into the language of cwatsets, we find that a perfect cwatset Ω is the sum $\Omega_1 + \Omega_2$ of perfect cwatsets Ω_1 and Ω_2 if and only if up to a reordering of the columns of Ω:

(*i*) Every element $\mathbf{a} \in \Omega$ is of the form $\mathbf{bc}$ with $\mathbf{b} \in \Omega_1$ and $\mathbf{c} \in \Omega_2$, and

(*ii*) For every element $\mathbf{b} \in \Omega_1$, there is an element $\mathbf{c} \in \Omega_2$ with $\mathbf{bc} \in \Omega$ (and, similarly, for all $\mathbf{c} \in \Omega_2$, there exists $\mathbf{b} \in \Omega_1$ with $\mathbf{bc} \in \Omega$).

Of course this definition makes sense even if Ω, Ω_1, and Ω_2 are not perfect, but then there is no analogous hypergraph statement. Note that there are obvious projection maps $\pi_1 : \Omega_1+\Omega_2 \to \Omega_1$ and $\pi_2 : \Omega_1+\Omega_2 \to \Omega_2$, but these do not necessarily make Ω into a product; indeed, if Ω_0 is another cwatset with $\varphi_1 : \Omega_0 \to \Omega_1$ and $\varphi_2 : \Omega_0 \to \Omega_2$, then there is a map $\varphi_1 + \varphi_2 : \Omega_0 \to \Omega$ making the diagram below commute if and only if for all $\mathbf{d} \in \Omega_0$, $\varphi_1(\mathbf{d})\varphi_2(\mathbf{d}) \in \Omega$.



Definition 12 *A hypergraph H is a reduced sum $H_1 + H_2$ if it satisfies the conditions of Definition 11 and if furthermore, in H, $v_i \cong v_j$ if and only if $1 \leq i \leq l$ and $1 \leq j \leq l$ or $l+1 \leq i \leq d$ and $l+1 \leq j \leq d$. A reduced sum of more than 2 cwatsets is defined analogously.*

Lemma 2 *Let H be a reduced sum of $H_1 + H_2$, where H is the hypergraph of a perfect cwatset. For an edge e in H_1, let $n_e = \#\{e'$ is an edge in $H | e' \cap \{v_1, v_2, \ldots, v_l\} = e\}$. Then, for any two edges e and f in H_1, $n_e = n_f := n_1$. (And, similarly, for any two edges e and f in H_2, $n_e = n_f := n_2$.)*

PROOF: Let e and f be edges in H_1. Then let $\{e_1, e_2, \ldots, e_{n_e}\}$ be edges in H such that $e_i \cap \{v_1, v_2, \ldots, v_l\} = e$, and $\{f_1, f_2, \ldots, f_{n_f}\}$ be edges in H

such that $f_i \cap \{v_1, v_2, \ldots, v_l\} = f$. Since H is edge-symmetric, there is a symmetry ϕ of H such that $\phi(e_1) = f_1$. Since by the definition of a reduced sum $\phi(\{v_1, v_2, \ldots, v_l\}) = \{v_1, v_2, \ldots, v_l\}$, then $\phi(e_1 \cap \{v_1, v_2, \ldots, v_l\}) = \phi(e_1) \cap \{v_1, v_2, \ldots, v_l\} = f_1 \cap \{v_1, v_2, \ldots, v_l\} = f$, so there exists a j such that $\phi(e_1) = f_j$. In fact, for every i, there exists a j_i such that $\phi(e_i) = f_{j_i}$, so $\phi : \{e_1, e_2, \ldots, e_{n_e}\} \mapsto \{f_1, f_2, \ldots, f_{n_f}\}$. Since ϕ is bijective it induces a bijection on edges, so $n_e = n_f$. □

Lemma 3 *If H is the hypergraph of a perfect cwatset and H is the reduced sum $H_1 + H_2$, then H_1 and H_2 are both hypergraphs of perfect cwatsets.*

PROOF: We will prove the lemma, without loss of generality, for H_1 only. We must show that H_1 is regular and uniform, and that its automorphism group acts transitively on its edges. H_1 is regular, because there is some integer d such that for all i, v_i is incident to d edges in H. By Lemma 2, if $1 \leq i \leq l$, then each edge of H_1 containing v_i corresponds to exactly n_1 edges of H, so there are $\frac{d}{n_1}$ edges of H_1 incident to each vertex v_i of H_1, and hence H_1 is regular. We now show that all of the edges of H_1 are isomorphic and therefore that H_1 is uniform. Fix e_i^1 and e_j^1 in H_1. We know that there exists $\sigma \in S(H)$ such that $\sigma(e_i) = e_j$. Since $\sigma(e_i^1) \subset \{v_1, v_2, \ldots, v_l\}$ and $\sigma(e_i^1) \subset e_j$, $\sigma(e_i^1) \subset e_j^1$. But $\sigma^{-1}(e_j^1) \subset e_i^1$ (by the same reasoning), so e_i^1 and e_j^1 have the same number of vertices so $\sigma(e_i^1) = e_j^1$. This obviously means that H_1 is uniform. □

Theorem 4 *The hypergraph $H(\Omega)$ of a perfect cwatset Ω can be expressed as a reduced sum of hypergraphs of perfect cwatsets each of which has the property that its symmetry group acts transitively on its vertices.*

PROOF: Let G denote the structure group of Ω. We say that two vertices v_1 and v_2 are equivalent if there exists $\sigma \in G$ such that $\sigma(v_1) = v_2$ (i.e., in earlier notation, if $v_1 \cong v_2$), and in this case, we write $v_1 \sim v_2$. This is easily seen to be an equivalence relation.

For every equivalence class $[v]$, let $H_{[v]}$ be the hypergraph whose edges are
$\{e_i \cap [v] \mid e_i \text{ is an edge of } H\}$. It is evident that

$$H = \sum_{v \in H} H_{[v]}.$$

Also, from Lemma 3, $H_{[v]}$ is the hypergraph of a perfect cwatset for every v, and the fact that the sum is reduced is merely the definition of the reduced sum. □

We call a hypergraph *irreducible* if it cannot be expressed as a reduced sum of two hypergraphs, and we call a perfect cwatset irreducible if its hypergraph is irreducible. Notice that there are perfect cwatsets which are not irreducible [5], and therefore Theorem 4 (which yields a decomposition of a cwatset to a collection of irreducible cwatsets) is meaningful.

However, the sum is not a uniquely defined operation. There exist hypergraphs $H \not\cong H'$, and hypergraphs H_1 and H_2 such that $H = H_1 + H_2$ and $H' = H_1 + H_2$. Since Theorem 4 provides a decomposition into reduced sums, it is only necessary for our purposes to understand the reduced sum.

Theorem 5 *Let H be the reduced sum $H_1 + H_2$ where H_1 and H_2 are hypergraphs of perfect cwatsets. Then H is uniform. Furthermore, let d_1 (resp. d_2) be the degree of each vertex in H_1 (resp. H_2) and e_1 (resp. e_2) the number of edges in H_1 (resp. H_2). Then H is regular if and only if $\frac{d_1}{e_1} = \frac{d_2}{e_2}$.*

PROOF: (*i*) Since H_1 and H_2 are both hypergraphs of perfect cwatsets, H_1 and H_2 are both uniform; suppose that each edge of H_1 has u_1 vertices and each edge of H_2 has u_2 vertices. Then, clearly, each edge of H has $u_1 + u_2$ vertices, so H is uniform.

(*ii*) Let v_1 be a vertex of H_1. The degree of v_1 in H_1 is d_1, so there are d_1 edges of H_1 incident to v_1. However, each edge in H_1 corresponds to n_1 edges in H, and therefore v_1 is incident to $n_1 d_1$ edges in H. Let e be the number of edges in H. Then $e = e_1 n_1$, so $n_1 = \frac{e}{e_1}$, so v_1 is incident to $n_1 d_1 = \frac{e}{e_1} d_1$ edges in H. Similarly, if v_2 is a vertex of H_2, v_2 is incident to $n_2 d_2 = \frac{e}{e_2} d_2$ edges in H. Thus, H is regular if and only if $\frac{e}{e_2} d_2 = \frac{e}{e_1} d_1$, or $\frac{d_1}{e_1} = \frac{d_2}{e_2}$. □

Lemma 4 *For any regular, edge-symmetric hypergraph with v vertices, e edges, d edges incident to each vertex and u vertices incident to each edge, $vd = eu$.*

PROOF: We count the edges of the hypergraph: by multiplying the number of vertices by the number of edges incident to each vertex, we count all of the edges as many times as there are vertices in each edge, so we have counted each edge u times. Then by dividing by u we obtain $e = \frac{vd}{u}$. Hence, $vd = eu$. □

Corollary 2 *Suppose H is the reduced sum $H_1 + H_2$, and suppose furthermore that H, H_1, and H_2 are all hypergraphs of perfect cwatsets. Then we have the following table, where the first column is the number of vertices, the second column is the number of edges, the third column is the number* *of edges incident to each vertex, the fourth column is the number of vertices incident to each edge, and k is an integer with $1 \le k \le \gcd(e_1, e_2)$:*

	v	e	d	u
H_1	v_1	e_1	d_1	$\frac{d_1}{e_1} v_1$
H_2	v_2	e_2	$\frac{d_1}{e_1} e_2$	$\frac{d_1}{e_1} v_2$
H	$v_1 + v_2$	$k \operatorname{lcm}(e_1, e_2)$	$\frac{d_1}{e_1} k \operatorname{lcm}(e_1, e_2)$	$\frac{d_1}{e_1}(v_1 + v_2)$

PROOF: We will denote by v_*, e_*, d_*, and u_* the number of vertices, edges, degree of each vertex, and number of vertices incident to each edge of H. We obtain $u_1 = \frac{d_1}{e_1} v_1$ as a direct result of Lemma 4; $d_2 = \frac{d_1}{e_1} e_2$ is due to Theorem 5. Hence, by Lemma 4, $u_2 = \frac{d_1}{e_1} v_2$. It is clear that $v_* = v_1 + v_2$ from the definition of the sum, and $u_* = \frac{d_1}{e_1}(v_1 + v_2)$ from Theorem 5.

Since we know from Lemma 2 that e_1 and e_2 both divide e_*, we know that $\operatorname{lcm}(e_1, e_2)$ also divides e_*, so $e_* = k \operatorname{lcm}(e_1, e_2)$. Since the set of edges E of H is a subset of $E_1 \times E_2$, where E_1 is the set of edges of H_1 and E_2 is the set of edges of H_2, $\#(E) \le \#(E_1 \times E_2) = \#(E_1) \cdot \#(E_2) = e_1 e_2 = \operatorname{lcm}(e_1, e_2) \gcd(e_1, e_2)$, so $k \le \gcd(e_1, e_2)$.

Finally, we have $d_* = \frac{d_1}{e_1} k \operatorname{lcm}(e_1, e_2)$ because from Theorem 5, $d_* = u_1 d_1 = \frac{d_1}{e_1} e_* = \frac{d_1}{e_1} k \operatorname{lcm}(e_1, e_2)$. □

6 Perfect $(d, 2)$-Cwatsets of Prime Degree

Now, armed with the previous results involving the structure group and decomposition of a perfect cwatset, we continue the combinatorial study of the hypergraphs introduced in [7]. We obtain combinatorial properties of the hypergraphs which are useful in determining specific classes of cwatsets, as we do in the next two sections.

Since $(d, 2)$-cwatsets can be represented by graphs, we begin our study of perfect cwatsets with $(d, 2)$-cwatsets. Initially, we classify an even smaller class of perfect cwatsets: the irreducible $(d, 2)$-cwatsets that have prime dimension.

In a graph, we will denote by uv the edge whose vertices are u and v, and we call the vertices u and v adjacent.

Definition 13 *A graph is* circulant *if there is an ordering $v_1, v_2, \ldots, v_d$ of its vertices such that if $v_i v_j$ is an edge, then so is $v_{i+k} v_{j+k}$ for every k, where the indeces are considered modulo d. If the graph associated with a $(d, 2)$-cwatset is circulant, we call that cwatset circulant as well.*

It is known [4] that a point-symmetric graph with p vertices where p is prime is necessarily circulant (in fact, we know that the group of symmetries of such a graph contains D_{2p}, the dihedral group of $2p$ elements). Hence, in this section, we will consider only circulant graphs.

We are now ready to completely classify the $(p,2)$-cwatsets where p is prime. Recall that $\mathbb{F}_p$ is the field of integers modulo p and that $\mathbb{F}_p^* = \mathbb{F}_p - \{0\}$ is the multiplicative group of $\mathbb{F}_p$.

We know ([2] and [4]) that there exists a point-symmetric and edge-symmetric (and hence circulant) graph Γ with p vertices, each of which are incident to h edges, if and only if h is even and divides $p-1$. It is further shown ([2] and [4]) that there is only one isomorphism class of such graphs for each such p and h. This gives us the following result.

Theorem 6 *The irreducible $(p,2)$-cwatsets with p prime have order $\frac{hp}{2}$ and splitting number $\frac{hp}{2} - h$ where h is an even divisor of $p-1$. Furthermore, the graph of such a cwatset is isomorphic to the graph whose vertices are elements of $\mathbb{F}_p$ and whose edges are $\{a, a+l\}$ where $a \in \mathbb{F}_p$ and $l \in L$, where L is the subgroup of $\mathbb{F}_p^*$ of order h.*

For example, since 4 and 6 divide 12, there are $(13,2)$-cwatsets of order 26 and 39 (in addition to the cyclic one of order 13 and the complete one of order 78).

Apart from completely classifying the $(p,2)$-cwatsets with p prime, this theorem is of importance because it is the first example of the already-existing literature about graphs or hypergraphs readily giving results about cwatsets: up to now, the hypergraph representation had only conceptual value. Furthermore, this theorem indicates the usefulness of circulant graphs. While not all irreducible $(d,2)$-cwatsets are circulant, it does seem likely that a classification of all $(d,2)$-cwatsets will be preceded by a classification of circulant $(d,2)$-cwatsets.

7 Perfect $(d,2)$-Cwatsets of Arbitrary Degree

Theorem 6 suggests that a complete listing of irreducible $(d,2)$-cwatsets based simply on order and derived from only the combinatorics of the graphs might be possible. At the very least, it seems that it should be possible to solve this problem for the circulant case. We will, for the remainder of this paper, study circulant $(d,2)$-cwatsets only. By definition, if the incidence matrix is of the form $[a_{ij}]$, then we have the following three properties:

(*i*) $a_{ij} = a_{ji}$

(*ii*) $a_{ii} = 0$

(*iii*) $a_{ij} = a_{kl}$, where $k \equiv i+1 \pmod{d}$ and $l \equiv j+1 \pmod{d}$.

The first two are true of any incidence matrix. The third results from the isomorphism of all vertices under rotation (i.e. circulance). An edge between vertex a and vertex b implies that there is an edge between vertex $a + j$ and vertex $b + j$, for every integer j. Notice that since each row has the same number of 1's, the graph is regular.

However, not all matrices of this form correspond to cwatsets. A matrix of this form does in fact correspond to a cwatset if the associated graph is edge-symmetric, that is, given any two 1's in the places a_{ij} and a_{kl} in the matrix, we can apply one permutation to both the rows and the columns that maps 1's onto 1's and a_{ij} onto a_{kl}. Clearly, any $(d, 2)$-cwatset must have order of the form $\frac{dl}{2}$ where l is the degree of each vertex.

Theorem 7 *If l divides d and d is even, then there is a circulant $(d, 2)$-cwatset of degree d and order $\frac{dl}{2}$.*

PROOF: Let $d = \mu l$. We form an incidence matrix A as follows: $a_{1\,j} = 0$ when μ does not divide $j - 1$ and $a_{1\,j} = 1$ otherwise. The rest of the rows of the matrix are determined by property (*iii*) above. Since a rotation of the vertices can map any 1 in the matrix onto a 1 in the first row, we must only show that we can map any 1 in the first row onto any other. Choose two arbitrary 1's, $a_{1\,\alpha\mu}$ and $a_{1\,\beta\mu}$. We apply the permutation $\sigma = (\alpha\mu\ \beta\mu)$ to the rows and columns of A. Obviously, σ maps $a_{1\,\alpha\mu}$ onto $a_{1\,\beta\mu}$. Also, for every natural number γ, $a_{k\ \gamma\mu} = 1$ if and only if $k \equiv 1 \pmod{\mu}$, so the columns $\alpha\mu$ and $\beta\mu$ are identical. Similarly, $a_{\gamma\mu\ k} = 1$ if and only if $k \equiv 1 \pmod{\mu}$, so the rows $\alpha\mu$ and $\beta\mu$ are identical as well. Therefore, σ preserves A, so A is the incidence matrix of a graph which represents a cwatset. Also, that cwatset must have order $\frac{dl}{2}$, because the first row of the matrix has l 1's, so, since each row is a permutation of the others, each row has l 1's, and therefore the matrix has dl 1's. Each edge is represented by two 1's in the matrix, so the order of the cwatset is $\frac{dl}{2}$. □

Each vertex of such a cwatset clearly has degree l, since the number of edges, $\frac{dl}{2}$, is equal to the number of vertices, d, times the number of edges incident to each vertex divided by the number of vertices incident to each edge. Thus the splitting number of the $(d, 2)$-cwatset given by Theorem 7 is $\frac{dl}{2} - l$.

Note that the set of elements generated by the edge from vertex d to vertex a is a cycle if and only if $\gcd(d, a) = 1$.

Theorem 8 *If d is not a prime power and $d \not\equiv 2 \pmod{4}$, there is a $(d, 2)$-cwatset of order $2d$.*

PROOF: We claim that there exist $a < \frac{d}{2}$, and $b < \frac{d}{2}$ such that $\gcd(a, d) = \gcd(b, d) = 1$ and $a \equiv cb \pmod{d}$, where $c^2 \equiv 1 \pmod{d}$.

If $d \not\equiv 2 \pmod 4$), then $d = 4s$, or d is odd.

If $d = 4s$, then let $a = 2s - 1 = c$ and $b = 1$. Then $c = ab$ and $c^2 = (2s-1)^2 = 4s^2 - 4s + 1$, which is congruent to $1 \pmod d$.

If d is odd and not a prime power, then we may write $d = jk$ where $\gcd(j, k) = 1$ and $j, k \neq 1$. Let $a = \frac{j+k}{2}$ and $b = \frac{j-k}{2}$, both of which are integers because j and k are both odd. Since $\gcd(d, b) = 1$, there is an integer c such that $a \equiv cb \pmod d$. Then

$$a^2 \equiv c^2 b^2 \pmod d,$$

so

$$(j+k)^2 \equiv c^2 (j-k)^2 \pmod d,$$

so

$$j^2 + 2kj + k^2 \equiv c^2 (j^2 - 2kj + k^2) \pmod d,$$

or, eliminating the kj's since $d = kj$,

$$(j^2 + k^2)(c^2 - 1) \equiv 0 \pmod d.$$

Since $\gcd(j^2 + k^2, d) = 1$, it follows that $c^2 \equiv 1 \pmod d$.

Hence, the claim is true.

Now let Γ be the circulant graph on d ordered vertices such that $v_i v_l$ is an edge if and only if $i \equiv l \pm \lambda \pmod d$ where $\lambda = a$ or $\lambda = b$. Certainly it is uniform (since every graph is uniform) and regular (since each vertex is incident to exactly 4 edges), and it is edge-symmetric. Indeed, since the graph is circulant, $\langle (1\,2 \cdots d) \rangle < S(\Gamma)$, so all of the edges of the form $v_i v_{i+a}$ are in the same orbit under the action of the automorphism group, as are all edges of the form $v_i v_{i+b}$. Hence, to demonstrate edge-symmetry, we must only exhibit an element of $S(\Gamma)$ which send $v_0 v_b$ onto $v_0 v_a$. Define $\sigma \in S_d$ by $\sigma : i \mapsto ci$. Then $\sigma(\{0, b\}) = \{0, bc\} = \{0, a\}$. Also, $\sigma \in S(\Gamma)$ because $\sigma(\{i, i \pm b\}) = \{ci, ci \pm bc\} = \{ci, ci \pm a\}$ which is an edge, and $\sigma(\{i, i \pm a\}) = \sigma(\{i, i \pm bc\}) = \{ci, ci \pm bc^2\} = \{ci, ci \pm b\}$, which is an edge. □

Theorem 8 is representative of the sorts of combinatorial and number-theoretic methods which one can use to determine orders of cwatsets (see also [7]). Only one such theorem was included here because other similar theorems are proven using very similar arguments.

Acknowledgments

I would like to thank Professor John Rickert of the Rose-Hulman Institute of Technology and Professors Ciprian Foias and Kent Orr of Indiana Univeristy, who were of great help in discussing cwatsets.

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