Covering the Powers of the Complete Graph with a Bounded Number of Snakes

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ABSTRACT. Let K_n^d be the product of d copies of the complete graph K_n. Wojciechowski [4] proved that for any $d \geq 2$ the hypercube K_2^d can be vertex covered with at most 16 disjoint snakes. We show that for any odd integer $n \geq 3$, $d \geq 2$ the graph K_n^d, can be vertex covered with $2n^3$ snakes.

1 Introduction

Throughout this paper we consider only finite, undirected, simple graphs. We define a *path* in a graph G to be a sequence of distinct vertices of G with every pair of consecutive vertices being adjacent. A *closed* path is a path whose first vertex is adjacent to the last one. A *chord* of a path P in a graph G is an edge of G joining two nonconsecutive vertices of P. If e is a chord in a closed path P, then e is called *proper* if it is not the edge joining the first vertex of P to its last vertex. Note that a proper chord of a closed path corresponds to the standard notion of a chord in a cycle. A *snake* in a graph G is a closed path in G without proper chords, and an *open* snake is a path without chords.

The (cartesian) *product* of two graphs G and H is the graph $G \times H$ with the vertex set $V(G) \times V(H)$ and the edge set defined in the following way: (g_1, h_1) is adjacent to (g_2, h_2) if either $g_1g_2 \in E(G)$ and $h_1 = h_2$, or else $g_1 = g_2$ and $h_1h_2 \in E(H)$. Let K_n^d be the product of d copies of the complete graph K_n, $n \geq 2$, $d \geq 1$. It is convenient to think of the vertices of K_n^d as d-tuples of n-ary digits, *i.e.*, the elements of the set $\{0, 1, \cdots, n-1\}$, with edges between two d-tuples differing at exactly one coordinate.

Let $S(K_n^d)$ be the length of the longest snake in K_n^d. The problem of estimating the value of $S(K_n^d)$ was first met by Kautz [3] in the case $n = 2$ (known in the literature as the snake-in-the-box problem) in constructing a type of error-checking code for a certain analog-to-digital conversion systems. As a consequence several authors became interested in estimating the value of $S(K_2^d)$ and a large literature has evolved (see [2] for a list of references). Subsequently, the general case of the problem with an arbitrary value of n has been introduced by Abbott and Dierker [1].

During the XXIII Southeastern International Conference, Boca Raton 1992, Erdös posed the problem of deciding whether there is a number k such that for every $d \geq 2$ the vertices of K_2^d can be covered using at most k snakes, and if the answer to the problem is positive, then whether it can be done in such a way that the snakes are pairwise vertex-disjoint. Wojciechowski [4] proved the following stronger result.

Theorem 1. *For every $d \geq 2$, there is a subgroup $\mathcal{H}_d \subset K_2^d$ and a snake $C_d \subset K_2^d$ such that $|\mathcal{H}_d| \leq 16$ and C_d uses exacty one element of every coset of $\mathcal{H}_d$, where the group structure of K_2^d is of the product $(\mathcal{Z}_2)^d$.* □

Theorem 1 implies that for any $d \geq 2$ the vertices of K_2^d can be covered with at most 16 vertex disjoint snakes.

In this paper we prove that for any fixed odd integer $n \geq 3$ there is a constant r_n such that the graph K_n^d can be vertex covered with r_n snakes.

Theorem 2. *Let $n \geq 3$ be an odd integer and $r_n = 2n^3$. For any $d \geq 2$ the vertices of K_n^d can be covered with r_n snakes.* □

2 Basic definitions

We define an *m-path* in a graph G to be a path containing m vertices, *i.e.*, a path of length $m - 1$. If P is an m-path, then we will write $m = |P|$. A *chain* $\mathcal{P}$ of paths in a graph G is a sequence $(P_1, P_2, \cdots, P_m)$ of paths in G such that each path in $\mathcal{P}$ has at least two vertices, and the last vertex of P_i is equal to the first vertex of P_{i+1}, where $1 \leq i \leq m - 1$. When we need to specify the number m of paths in a chain, we refer to it as an *m-chain* of paths. An m-chain $\mathcal{P} = (P_i)_{i=1}^m$ of paths will be called *closed* if the first vertex of P_1 is equal to the last vertex of P_m.

Given an m-path $P = (a_i)_{i=1}^m$ in a graph G and an m-chain of paths $\mathcal{L} = (P_i)_{i=1}^m$ in a graph H, let $P \otimes \mathcal{L}$ be the $(\sum_{i=1}^m |P_i|)$-path in the graph $G \times H$ constructed in the following way. For any path $P_i = (b_{i1}, b_{i2}, \cdots, b_{ik_i})$ in $\mathcal{L}$, let P_i' be the path $((a_i, b_{i1}), (a_i, b_{i2}), \cdots, (a_i, b_{ik_i}))$ in $G \times H$. Note that for any $1 \leq i \leq m - 1$, the last vertex of the path P_i' is adjacent to the first vertex of the path P_{i+1}'. Let $P \otimes \mathcal{L}$ be the path obtained by joining together (juxtaposing) the paths $P_1', P_2', \cdots, P_m'$. We will say that $P \otimes \mathcal{L}$ is the path *generated* by P and $\mathcal{L}$. Note that the path generated by a closed path and a closed chain of paths is a closed path.

If $\mathcal{R}$ is an sm-chain of paths in a graph H, then the *m-splitting* of $\mathcal{R}$ is the sequence $(\mathcal{R}_1, \mathcal{R}_2, \cdots, \mathcal{R}_m)$ of s-chains of paths in H which joined together (juxtaposed) give $\mathcal{R}$. The above definition of the operation $\otimes$ can be generalized in the following way. Let $\mathcal{L} = (P_i)_{i=1}^m$ be an m-chain of s-paths in a graph G, let $\mathcal{R}$ be an sm-chain of paths in H, and let $(\mathcal{R}_1, \mathcal{R}_2, \cdots, \mathcal{R}_m)$ be the m-splitting of $\mathcal{R}$. Note that for any $1 \le i \le m-1$, the last vertex of the path $P_i \otimes \mathcal{R}_i$ in the graph $G \times H$ is equal to the first vertex of the path $P_{i+1} \otimes \mathcal{R}_{i+1}$. Set

$$\mathcal{L} \otimes \mathcal{R} = (P_1 \otimes \mathcal{R}_1, P_2 \otimes \mathcal{R}_2, \cdots, P_m \otimes \mathcal{R}_m).$$

We will say that $\mathcal{L} \otimes \mathcal{R}$ is the chain of paths *generated* by $\mathcal{L}$ and $\mathcal{R}$. Note that the chain of paths generated by two closed chains of paths is also a closed chain of paths.

Let $\mathcal{L} = (P_i)_{i=1}^m$ be a chain of paths in a graph G. We say that $\mathcal{L}$ is *openly separated* if for $i \le m-1$ and $j = i+1$, P_i and P_j have exactly one vertex in common, and otherwise P_i and P_j are vertex disjoint. We say that $\mathcal{L}$ is *closely separated* if $\mathcal{L}$ is closed, P_i and P_j have exactly one vertex in common when either $i \le m-1$ and $j = i+1$, or $i = 1$ and $j = m$ and otherwise P_i and P_j are vertex disjoint.

If P is a path, then let $-P$ be the path obtained from P by reversing the order of vertices, and if $\mathcal{L} = (P_i)_{i=1}^m$ is a chain of paths, then let $-\mathcal{L} = (-P_m, -P_{m-1}, \cdots, -P_1)$ be the chain of paths obtained from $\mathcal{L}$ by reversing the order of paths and reversing every path. The expression $(-1)^i X$, where X is a path or a chain of paths, will mean X for i even and $-X$ for i odd.

Let $\mathcal{L}$ be an sm-chain of paths, and let $\mathcal{R} = (\mathcal{L}_1, \mathcal{L}_2, \cdots, \mathcal{L}_m)$ be the m-splitting of $\mathcal{L}$. The *alternate matrix* of the splitting $\mathcal{R}$ is the following $(m \times s)$-matrix $\mathcal{A}$ of paths :

$$\mathcal{A} = \begin{pmatrix} \mathcal{L}_1 \\ -\mathcal{L}_2 \\ \vdots \\ (-1)^{m-1}\mathcal{L}_m \end{pmatrix} = \begin{pmatrix} Q_1^1 & Q_1^2 & \cdots & Q_1^s \\ Q_2^1 & Q_2^2 & \cdots & Q_2^s \\ \vdots & \vdots & & \vdots \\ Q_m^1 & Q_m^2 & \cdots & Q_m^s \end{pmatrix}$$

where $(Q_i^1, Q_i^2, \cdots, Q_i^s)$ is the sequence of paths forming the s-chain $(-1)^{i-1}$ $\mathcal{L}_i$. The splitting $\mathcal{R}$ will be called *openly alternating* if for every odd j, $1 \le j \le m-1$, the paths Q_j^s and Q_{j+1}^s have exactly one vertex in common, for every even j, $2 \le j \le m-1$, the paths Q_j^1 and Q_{j+1}^1 have exactly one vertex in common, and otherwise the paths Q_j^i and Q_l^i are vertex disjoint, $1 \le i \le s$, $1 \le j, l \le m$, $j \ne l$. Note that the splitting $\mathcal{R}$ is openly alternating if for every column of its alternate matrix $\mathcal{A}$ the paths in the column are mutually vertex disjoint except for the shared vertices which are necessary for $\mathcal{L}$ to be a chain of paths, *i.e.* Q_1^s and Q_2^s have exactly one vertex in common, Q_2^1 and Q_3^1 have exactly one vertex in common, and so on.

Assume that the sm-chain $\mathcal{L}$ is a closed chain of paths and m is even. Then, the splitting $\mathcal{R}$ is *closely alternating* if for each odd j, $1 \le j \le m-1$, the paths Q_j^s and Q_{j+1}^s have exactly one vertex in common, for each even j, $2 \le j \le m-1$, the paths Q_j^1 and Q_{j+1}^1 have exactly one vertex in common, the paths Q_1^1 and Q_m^1 have exactly one vertex in common, and otherwise the paths Q_j^i and Q_l^i are vertex disjoint, $1 \le i \le s$, $1 \le j, l \le m$, $j \ne l$. Note that the splitting $\mathcal{R}$ is closely alternating if for every column of its alternate matrix $\mathcal{A}$ the paths in the column are mutually vertex disjoint except for the shared vertices which are necesary for $\mathcal{L}$ to be a closed chain of paths.

Assume that $n \ge 3$ is a fixed odd integer. For any integer $d \ge 1$, we define the n^d-path π_n^d in K_n^d, and the closed $(n-1)n^d$-paths γ_n^{d+1} and $\hat{\gamma}_n^{d+1}$ in K_n^{d+1}.

Let π_n^1 be the n-path $(0, 1, \cdots, n-1)$ in K_n, and let $\gamma_n, \hat{\gamma}_n$ be the closed $(n-1)$-paths $(0, 1, \cdots, n-2)$ and $(1, 2, \cdots, n-1)$ in K_n, respectively. If $d \ge 1$ and the path π_n^d in K_n^d is defined, then let

$$\pi_n^{d+1} = \pi_n^1 \otimes (\pi_n^d, -\pi_n^d, \pi_n^d, -\pi_n^d, \cdots, \pi_n^d),$$

$$\gamma_n^{d+1} = \gamma_n \otimes (\pi_n^d, -\pi_n^d, \pi_n^d, -\pi_n^d, \cdots, -\pi_n),$$

and

$$\hat{\gamma}_n^{d+1} = \hat{\gamma}_n \otimes (\pi_n^d, -\pi_n^d, \pi_n^d, -\pi_n^d, \cdots, -\pi_n^d).$$

Let H be a graph, $d \ge 1$ be an integer, $\mathcal{L}$ be an n^d-chain of paths in H, and $\mathcal{D}$ be an $(n-1)n^d$-chain of paths in H. We define that $\mathcal{L}$ is *openly well distributed* if either $d = 1$ and $\mathcal{L}$ is an openly separated chain of open snakes, or $d \ge 2$, every chain $\mathcal{L}_i$ in the n-splitting $S = (\mathcal{L}_1, \mathcal{L}_2, \cdots, \mathcal{L}_n)$ of $\mathcal{L}$ is openly well distributed and S is openly alternating. We also say that $\mathcal{D}$ is *closely well distributed* if every chain $\mathcal{D}_i$ in the $(n-1)$-splitting $S' = (\mathcal{D}_1, \mathcal{D}_2, \cdots, \mathcal{D}_{n-1})$ of $\mathcal{D}$ is openly well distributed and S' is closely alternating.

3 Proof of Theorem 2

The construction establishing Theorem 2 is complicated. It will be convenient if we describe it in a sequence of lemmas. Two of these, Lemmas 1 and 3, were proved in [5].

Lemma 1. *If $d \ge 1$ and $\mathcal{L}$ is a closely well distributed $(n-1)n^d$-chains of paths in a graph H, then the path $\gamma_n^{d+1} \otimes \mathcal{L}$ is a snake in the graph $K_n^{d+1} \times H$.* □

Since any permutation of the digits at the first coordinate is an isomorphism of K_n^{d+1}, the following lemma immediately follows from Lemma 1.

Lemma 2. *If $d \geq 1$ and $\mathcal{C}$ is a closely well distributed $(n-1)n^d$-chain of paths in a graph H, then the path $\hat{\gamma}_n^{d+1} \otimes \mathcal{C}$ is a snake in the graph $K_n^{d+1} \times H$.* □

The following lemma was proved by Wojciechowski [5] (Lemma 4).

Lemma 3. *For each $d \geq 1$, there exists a closely well distributed $(n-1)n^d$-chain of paths in K_n^{d+1}.* □

Let

$$\alpha : \{0, 1, \cdots, n-1\} \rightarrow \{0, 1, \cdots, n-1\}$$

be a function defined by $\alpha(i) = i+1$ if $0 \leq i < n-1$ and $\alpha(n-1) = 0$. Let $x = (a_1, a_2, a_3) \in V(K_n^3)$, where $a_1, a_2, a_3 \in \{0, 1, \cdots, n-1\}$. Let

$$\sigma, \tau, \delta : V(K_n^3) \rightarrow V(K_n^3)$$

be permutations such that

$$\sigma(a_1, a_2, a_3) = (\alpha(a_1), a_2, a_3),$$
$$\tau(a_1, a_2, a_3) = (a_1, \alpha(a_2), a_3),$$

and

$$\delta(a_1, a_2, a_3) = (a_1, a_2, \alpha(a_3)).$$

Let Σ be the set of all permutations

$$f : V(K_n^3) \rightarrow V(K_n^3)$$

such that $f = \sigma^i \tau^j \delta^k$ with $i, j, k \in \{0, 1, \cdots, n-1\}$.

Lemma 4. *For any $x, y \in V(K_n^3)$, there is $f \in \Sigma$ with $y = f(x)$.*

Proof: Assume that $x = (x_1, y_1, z_1)$, $y = (x_2, y_2, z_2)$ be two vertices of K_n^3. One can easily verify that $f(x_1, y_1, z_1) = (x_2, y_2, z_2)$ if

$$f = \sigma^{(x_2 - x_1) \mod (n-1)} \tau^{(y_2 - y_1) \mod (n-1)} \delta^{(z_2 - z_1) \mod (n-1)}.$$

□

Let $f \in \Sigma$. Given a path $P = (u_1, u_2, \cdots, u_r)$ in K_n^3, let $f(P)$ be the path $(f(u_1), f(u_2), \cdots, f(u_r))$. Given a chain of paths $C = (P_1, P_2, \cdots, P_s)$, let $f(C)$ be the chain of paths $(f(P_1), f(P_2), \cdots, f(P_s))$.

Lemma 5. *Let $f \in \Sigma$ and $u, v \in V(K_n^3)$. Then u and v are adjacent in K_n^3 if and only if $f(u)$ and $f(v)$ are adjacent in K_n^3.*

Proof: Let $f \in \Sigma$ and $u, v \in V(K_n^3)$. Assume that $u = (a_1, a_2, a_3)$ and $v = (b_1, b_2, b_3)$ are adjacent in K_n^3, then u and v differ at exactly one position. Since α is a bijective function it follows that

$$f(u) = (f(a_1), f(a_2), f(a_3)),$$

and

$$f(v) = (f(b_1), f(b_2), f(b_3)),$$

are differing in exactly one position. Hence $f(u)$ and $f(v)$ are adjacent. Conversely, if $f(u)$ and $f(v)$ are adjacent in K_n^3, then similarly as above we show that u and v are adjacent. □

Lemma 6. *If P is an open snake in K_n^3 and $f \in \Sigma$, then $f(P)$ is also an open snake in K_n^3.*

Proof: Let $P = (u_1, u_2, \cdots, u_r)$ be an open snake in K_n^3 and let $f \in \Sigma$ be a given permutation. Since P does not have a chord, it follows from Lemma 5, that $f(P)$ does not have chords either. Hence $f(P)$ is also an open snake. □

Lemma 7. *If C is an openly separated chain of paths in K_n^3 and $f \in \Sigma$, then the chain $f(C)$ is also openly separated.*

Proof: Let $C = (P_1, P_2, \cdots, P_s)$ be an openly separated chain of paths in K_n^3 and let $f \in \Sigma$. Since C is an openly separated chain of paths, then for $i \leq s-1$ and $j = i+1$, P_i and P_j have exactly one vertex in common, and otherwise P_i and P_j are vertex disjoint. Since f is a bijection, it follows that $i \leq s-1$ and $j = i+1$, $f(P_i)$ and $f(P_j)$ have exactly one vertex in common, and otherwise $f(P_i)$ and $f(P_j)$ are vertex disjoint. Hence

$$f(C) = (f(P_1), f(P_2), \cdots, f(P_s)),$$

is also openly separated. □

Lemma 8. *If $f \in \Sigma$ and P is a path in K_n^3, then $f(-P) = -f(P)$.*

Proof: Let $f \in \Sigma$ and $P = (u_1, u_2, \cdots, u_r)$ be a path in K_n^3. Since $-P$ is the path obtained from P by reversing the order of the vertices, we have

$$\begin{aligned} f(-P) &= f(u_r, u_{r-1}, \cdots, u_1) \\ &= (f(u_r), f(u_{r-1}), \cdots, f(u_1)) \\ &= -(f(u_1), f(u_2), \cdots, f(u_r)) \\ &= -f(P). \end{aligned}$$

□

Lemma 9. *If $f \in \Sigma$ and C is a chain of paths in K_n^3, then $f(-C) = -f(C)$.*

Proof: Let $f \in \Sigma$ and $\mathcal{C} = (P_1, P_2, \cdots, P_s)$ be a chain of path in K_n^3. Then

$$\begin{aligned} f(-\mathcal{C}) &= f(-P_s, -P_{s-1}, \cdots, -P_1) \\ &= (f(-P_s), f(-P_{s-1}), \cdots, f(-P_1)) \\ &= (-f(P_s), -f(P_{s-1}), \cdots, -f(P_1)) \\ &= -(f(P_1), f(P_2), \cdots, f(P_s)) \\ &= -f(\mathcal{C}). \end{aligned}$$

□

Lemma 10. *Let $\mathcal{L}$ be an sm-chain of paths in K_n^3, and let $\mathcal{R} = (\mathcal{L}_1, \mathcal{L}_2, \cdots, \mathcal{L}_m)$ be the m-splitting of $\mathcal{L}$. If $\mathcal{R}$ is openly alternating and $f \in \Sigma$, then $f(\mathcal{R})$ is also openly alternating.*

Proof: Let

$$\mathcal{A} = \begin{pmatrix} \mathcal{L}_1 \\ -\mathcal{L}_2 \\ \vdots \\ (-1)^{m-1}\mathcal{L}_m \end{pmatrix} = \begin{pmatrix} Q_1^1 & Q_1^2 & \cdots & Q_1^s \\ Q_2^1 & Q_2^2 & \cdots & Q_2^s \\ \vdots & \vdots & & \vdots \\ Q_m^1 & Q_m^2 & \cdots & Q_m^s \end{pmatrix}$$

be the alternate matrix of $\mathcal{R}$. Then

$$\mathcal{A}' = \begin{pmatrix} f(\mathcal{L}_1) \\ -f(\mathcal{L}_2) \\ \vdots \\ (-1)^{m-1}f(\mathcal{L}_m) \end{pmatrix} = \begin{pmatrix} f(Q_1^1) & f(Q_1^2) & \cdots & f(Q_1^s) \\ f(Q_2^1) & f(Q_2^2) & \cdots & f(Q_2^s) \\ \vdots & \vdots & & \vdots \\ f(Q_m^1) & f(Q_m^2) & \cdots & f(Q_m^s) \end{pmatrix}$$

is the alternate matrix of $f(\mathcal{R})$. If $\mathcal{R}$ is openly alternating, then for every odd j, $1 \le j \le m-1$, the paths Q_j^s and Q_{j+1}^s have exactly one vertex in common, for every even j, $2 \le j \le m-1$, the paths Q_j^1 and Q_{j+1}^1 have exactly one vertex in common, and otherwise the paths Q_j^i and Q_l^i are vererx disjoint, $1 \le i \le s$, $1 \le j, l \le m$, $j \ne l$. Since f is a bijection, then for every odd j, $1 \le j \le m-1$, the paths $f(Q_j^s)$ and $f(Q_{j+1}^s)$ have exactly one vertex in common, for every even j, $2 \le j \le m-1$, the paths $f(Q_j^1)$ and $f(Q_{j+1}^1)$ have exactly one vertex in common, and otherwise the paths $f(Q_j^i)$ and $f(Q_l^i)$ are vertex disjoint, $1 \le i \le s$, $1 \le j, l \le m$, $j \ne l$. Hence $f(\mathcal{R})$ is also openly alternating. □

Similarly, we can prove the following lemma.

Lemma 11. *Let $\mathcal{L}$ be a closed sm-chain of paths, and let $\mathcal{R} = (\mathcal{L}_1, \mathcal{L}_2, \cdots, \mathcal{L}_m)$ be the m-splitting of $\mathcal{L}$. If $\mathcal{R}$ is closely alternating and $f \in \Sigma$, then $f(\mathcal{R})$ is also closely alternating.* □

Lemma 12. *If $\mathcal{C}$ is an openly well distributed chain of paths in K_n^3 and $f \in \Sigma$, then $f(\mathcal{C})$ is also openly well distributed.*

Proof: We are going to use induction with respect to d. For $d = 1$, the lemma follows from Lemma 7. Assume that the lemma is true for d, we show that it is true for $d+1$. Let $\mathcal{S} = (\mathcal{L}_1, \mathcal{L}_2, \cdots, \mathcal{L}_n)$ be the n-splitting of $\mathcal{C}$. Since $\mathcal{C}$ is openly well distributed, it follows that every chain $\mathcal{L}_i$, $i = 1, 2, \cdots n$, is openly well distributed and $\mathcal{S}$ is openly alternating. Then

$$f(\mathcal{S}) = (f(\mathcal{L}_1), f(\mathcal{L}_2), \cdots, f(\mathcal{L}_n)),$$

is the n-splitting of $f(\mathcal{C})$. By Lemma 10, $f(\mathcal{S})$ is openly alternating and by the induction hypothesis $f(\mathcal{L}_i)$ is openly well distributed, for $i = 1, 2, \cdots, n$. Hence $f(\mathcal{C})$ is openly well distributed. □

Lemma 13. *If v is a vertex of K_n^d, then v is a vertex of the path π_n^d.*

Proof: We are going to use induction with respect to d. For $d = 1$, the lemma is true since π_n^1 is the n-path $(0, 1, \cdots, n-1)$ in K_n. Assume that $d \geq 1$ and that the lemma is true for d, we show that it is true for $d+1$. Let $v = (a_1, a_2, \cdots, a_d, a_{d+1})$ be a vertex of K_n^{d+1}. By the inductive hypothesis, $(a_2, a_3, \cdots, a_{d+1})$ is a vertex of π_n^d. Since

$$\pi_n^{d+1} = \pi_n^1 \otimes (\pi_n^d, -\pi_n^d, \pi_n^d, -\pi_n^d, \cdots, \pi_n^d),$$

is a path in $K_n \times K_n^d = K_n^{d+1}$ and since a_1 is a vertex of π_n^1, it follows that $v = (a_1, a_2, \cdots, a_d, a_{d+1})$ is a vertex of π_n^{d+1}. □

Lemma 14. *If v is a vertex of K_n^d, then v is a vertex of γ_n^d or a vertex of $\hat{\gamma}_n^d$.*

Proof: Assume that $v = (a_1, a_2, \cdots, a_d)$ is a vertex of K_n^d. We have

$$\gamma_n^d = \gamma_n \otimes (\pi_n^{d-1}, -\pi_n^{d-1}, \pi_n^{d-1}, -\pi_n^{d-1}, \cdots, -\pi_n^{d-1}),$$

and

$$\hat{\gamma}_n^d = \hat{\gamma}_n \otimes (\pi_n^{d-1}, -\pi_n^{d-1}, \pi_n^{d-1}, -\pi_n^{d-1}, \cdots, -\pi_n^{d-1}),$$

which are paths in $K_n \times K_n^{d-1} = K_n^d$, and by Lemma 13, $(a_2, a_3, \cdots, a_d)$ is a vertex of π_n^{d-1}. If $a_1 \in \{0, 1, \cdots, n-2\}$, then a_1 is a vertex of γ_n, and if $a_1 = n-1$, then a_1 is a vertex of $\hat{\gamma}_n$ so it follows that $v = (a_1, a_2, \cdots, a_d)$ is a vertex of γ_n^d or a vertex of $\hat{\gamma}_n^d$. □

By Lemma 3, there is a closely well distributed $(n-1)n^d$-chain of paths $\mathcal{D}$ in K_n^3. Given $f \in \Sigma$, let $\mathcal{D}_f = f(\mathcal{D})$ and let

$$\mathcal{P}_f = \gamma_n^{d-3} \otimes \mathcal{D}_f,$$

and

$$\hat{\mathcal{P}}_f = \hat{\gamma}_n^{d-3} \otimes \mathcal{D}_f.$$

Lemma 15. *The chain of paths $\mathcal{D}_f$ is closely well distributed for every $f \in \Sigma$.*

Proof: Let $\mathcal{D}$ be a closely well distributed chain of paths and let $f \in \Sigma$. Since $\mathcal{D}$ is a closely well distributed chain of paths, every chain $\mathcal{D}_i$ in the $(n-1)$-splitting $S = (\mathcal{D}_1, \mathcal{D}_2, \cdots, \mathcal{D}_{n-1})$ of $\mathcal{D}$ is openly well distributed and S is closely alternating. By Lemma 12, every chain $f(\mathcal{D}_i)$ in the $(n-1)$-splitting

$$f(S) = (f(\mathcal{D}_1), f(\mathcal{D}_2), \cdots, f(\mathcal{D}_{n-1})),$$

of $\mathcal{D}_f$ is openly well distributed. By Lemma 11, $f(S)$ is closely alternating. Hence $\mathcal{D}_f$ is closely well distributed. □

The following lemma follows immediately from Lemmas 1, 2 and 15.

Lemma 16. *$\mathcal{P}_f$, $\hat{\mathcal{P}}_f$ are snakes in K_n^d for every $f \in \Sigma$.* □

Lemma 17. *For every vertex v of K_n^d, there exist $f \in \Sigma$ such that v is a vertex of $\mathcal{P}_f$ or v is a vertex of $\hat{\mathcal{P}}_f$.*

Proof: Suppose that $v = (a_1, a_2, \cdots, a_d)$ is any vertex of K_n^d. By Lemma 14, $(a_1, a_2, \cdots, a_{d-3})$ is a vertex of γ_n^{d-3} or of $\hat{\gamma}_n^{d-3}$. Assume first that $(a_1, a_2, \cdots, a_{d-3})$ is a vertex of γ_n^{d-3} and $\gamma_n^{d-3} = (v_1, v_2, \cdots, v_s)$, where $s = (n-1)n^{d-4}$. Then there is $i \in \{1, 2, \cdots, s\}$ with $v_i = (a_1, a_2, \cdots, a_{d-3})$. Assume that $\mathcal{D} = (P_1, P_2, \cdots, P_s)$ and let (b_{d-2}, b_{d-1}, b_d) be a vertex of P_i. By Lemma 4, there is $f \in \Sigma$ with

$$(a_{d-2}, a_{d-1}, a_d) = f(b_{d-2}, b_{d-1}, b_d).$$

Then (a_{d-2}, a_{d-1}, a_d) is a vertex of $f(P_i)$. Since

$$\mathcal{D}_f = (f(P_1), f(P_2), \cdots, f(P_s)),$$

it follows that $(a_1, a_2, \cdots, a_d)$ is a vertex of

$$\mathcal{P}_f = \gamma_n^{d-3} \otimes \mathcal{D}_f.$$

Similarly, if $(a_1, a_2, \cdots, a_{d-3})$ is a vertex of $\hat{\gamma}_n^{d-3}$, it follows that $(a_1, a_2, \cdots, a_d)$ is a vertex of

$$\hat{\mathcal{P}}_f = \hat{\gamma}_n^{d-3} \otimes \mathcal{D}_f.$$

□

Now we are ready to prove Theorem 2.

Proof of Theorem 2: Let $\mathcal{S} = \{\mathcal{P}_f : f \in \Sigma\} \cup \{\hat{\mathcal{P}} : f \in \Sigma\}$. By Lemma 16, the elements of $\mathcal{S}$ are snakes and by Lemma 17, they vertex-cover K_n^d. Since $|\Sigma| = n^3$, it follows that $|\mathcal{S}| = 2n^3$ and the proof is complete. □

4 Conclusion

The above construction relies heavily on the fact that n is odd. For the case where n is even, $n \geq 4$, we proved that the vertices of K_n^d can be covered with n^3 snakes. By including the construction for the even case the paper becomes too long, so we introduce it in a separate paper.

It still remains open problem whether the snakes in Theorem 2 can be made vertex-disjoint.

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