The fine structure of balanced ternary designs with block size three, index three and $\rho_2 = 1, 2$

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ABSTRACT: In this paper necessary and sufficient conditions for a vector to be the fine structure of a balanced ternary design with block size 3, index 3 and $\rho_2 = 1$ and 2 are determined with one unresolved case.

1 Introduction and definitions

A *balanced ternary design* is a collection of multi-sets of size k, chosen from a v-set in such a way that each element occurs 0, 1 or 2 times in any one block, each pair of non-distinct elements, $\{x, x\}$, occurs in ρ_2 blocks of the design and each pair of distinct elements, $\{x, y\}$, occurs λ times throughout the design. We denote these parameters by $(v; \rho_2; k, \lambda)$BTD. A BTD on the element set V is denoted by $(V, \mathcal{B})$, where $\mathcal{B}$ is the collection of multi-subsets of V. It is easy to see that each element must occur singly in a constant number of blocks, say ρ_1 blocks, and so each element occurs altogether $r = \rho_1 + 2\rho_2$ times. Also if b is the number of blocks in the design, then

$$vr = bk \quad \text{and} \quad \lambda(v-1) = r(k-1) - 2\rho_2.$$

(For further information [3] should be consulted.) In this note we concentrate on the case $k = \lambda = 3$ and $\rho_2 = 1$ and 2. A necessary and sufficient

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condition for the existence of such a balanced ternary design is that $v \equiv 3 \pmod 6$ and $v \geq 9$ when $\rho_2 = 2$. (See for example [3].)

A BTD *with a hole*, or a *frame*-BTD, is a collection of multi-sets (blocks) of size k chosen from a v-set V so that the following conditions hold:

(i) $\{\infty_i \mid i = 1, 2, \ldots, h\} = H$ is a subset of V called a *hole*;

(ii) any element in $V \setminus H$ occurs 0, 1 or 2 times per block, and precisely twice in ρ_2 blocks;

(iii) at most one (counting repetitions) element of each block is in H;

(iv) any pair xy, where x and y are distinct elements, not both in H, occurs λ times throughout the design.

We write the parameters of a frame-BTD as $(v[h]; \rho_2; k, \lambda)$. Of course a BTD is a frame with $h = 0$.

Given a $(v; \rho_2; k, \lambda)$BTD, the fine structure of the system is the vector $(c_1, c_2, \ldots, c_\lambda)$, where c_i is the number of blocks repeated precisely i times. There are some works on the fine structure for designs (see for example, [2,6,7,8,9]).

In this paper we shall determine necessary and sufficient conditions for a vector to be the fine structure of a balanced ternary design with block size 3, index 3 and $\rho_2 = 1$ and 2. The case $\rho_2 = 0$ was settled in [6] and [7] and the case $\rho_2 = 3$ appeared in [8].

Since any two of $\{c_1, c_2, c_3\}$ determine the third, we use a more convenient notation for the fine structure: (t, s) is said to be the fine structure of a $(v; \rho_2; 3, 3)$BTD, if $c_2 = t$ and $c_3 = v(v - 2\rho_2 - 1)/6 - s$, where $\rho_2 = 1, 2$. We first need to know the pairs (t, s) which can possibly arise as fine structures. We define

$$\begin{aligned} \mathrm{Adm}_1(v) &= \{(t,s) \mid 0 \leqslant t \leqslant s \leqslant v(v-3)/6\} \setminus \{(0,1),(0,2),(0,3)\}, \\ \mathrm{Adm}_2(v) &= \{(t,s) \mid 0 \leqslant t \leqslant s \leqslant v(v-5)/6\} \end{aligned}$$

and use the notations $\mathrm{Fine}_1(v)$ and $\mathrm{Fine}_2(v)$ for the set of fine structures which actually arise in $(v; \rho_2; 3, 3)$BTDs, where $\rho_2 = 1$ and 2 respectively. Our result is as follows.

MAIN THEOREM

(i) $\mathrm{Fine}_1(v) = \mathrm{Adm}_1(v)$ for all $v \equiv 3 \pmod 6$, $v \notin \{9, 15\}$,
$\mathrm{Fine}_1(9) = \mathrm{Adm}_1(9) \setminus \{(0,4),(1,1)\}$,
$\mathrm{Adm}_1(15) \setminus \{(0,4)\} \subseteq \mathrm{Fine}_1(15)$ and $\mathrm{Fine}_1(15) \subseteq \mathrm{Adm}_1(15)$.

(ii) $\mathrm{Fine}_2(v) = \mathrm{Adm}_2(v)$ for all $v \equiv 3 \pmod 6$, $v \geq 9$.

2 Necessary conditions

It is straightforward (see Lemma 1.1 of [8]) to prove:

Lemma 2.1

(i) If $(t,s) \in \mathrm{Fine}_1(v)$ then $0 \leqslant t \leqslant s \leqslant v(v-3)/6$;

(ii) If $(t,s) \in \mathrm{Fine}_2(v)$ then $0 \leqslant t \leqslant s \leqslant v(v-5)/6$.

Before proceeding, we require some notation and preliminary results. Let $(V, \mathcal{B})$ be a $(v;1;3,3)$BTD with the fine structure (t,s), and let $T \subseteq \mathcal{B}$ be those blocks which are not triply repeated. We define

$$T_1 = \{\{a,a,b\} \mid \{a,a,b\} \in \mathcal{B}\}, \text{ and } T_2 = T \setminus T_1.$$

It is straightforward to verify that $|T| = 4v/3 + 3s$, $|T_1| = v$ and $|T_2| = v/3 + 3s$.

If $x, y \in V$ and $S \subseteq \mathcal{B}$, then we denote by $r_x(S)$ the number of occurrences of x in S, and by $\lambda_{xy}(S)$ the number of occurrences of the pair xy in the blocks of S. Also, let $\alpha = |A|$, where $A = \{a_1, a_2, \ldots, a_\alpha\} = \{x \mid r_x(T) = 7\}$, let $\beta = |B|$, where $B = \{b_1, b_2, \ldots, b_\beta\} = \{x \mid r_x(T) \geq 10\}$, and let $\gamma = |C|$, where $C = \{xy \mid \lambda_{xy}(T_2) = 3\}$.

The following lemma contains several results which we will require later.

Lemma 2.2

1. For all $x \in V$, $r_x(T_1) \geq 2$, $r_x(T) \geq 4$ and $r_x(T) \equiv 1 \pmod 3$;
2. for all distinct $x, y \in V$, $\lambda_{xy}(T) = 0$ or 3;
3. $\sum_{x \in V} r_x(T) = 4v + 9s$;
4. $3\alpha + 6\beta \leq 9s$;
5. for all distinct $x, y \in V$, $\lambda_{xy}(T_1) \equiv 0 \pmod 2$;
6. $\gamma = 3s$;
7. for all distinct $x, y \in V$, $\lambda_{xy}(T_2) \neq 2$; and
8. for all $x \in V$, $r_x(T_2) \geq 1$.

Proof: Note that $\mathcal{B} \setminus T$ contains exactly the triply repeated blocks.
(1) It follows from the definition that $r_x(T_1) \geq 2$. A simple counting argument verifies that $r_x(\mathcal{B}) \equiv 1 \pmod 3$ for all $x \in V$, and it is clear that

$r_x(\mathcal{B} \setminus T) \equiv 0 \pmod 3$; hence $r_x(T) \equiv 1 \pmod 3$ for all $x \in V$. From $r_x(T_1) \geq 2$ and $r_x(T) \equiv 1 \pmod 3$, it follows that $r_x(T) \geq 4$.
(2) Since $\lambda_{xy}(\mathcal{B} \setminus T)$ must be either 0 or 3, it follows that $\lambda_{xy}(T) = 0$ or 3 for all distinct $x, y \in V$.
(3) This is easily verified by counting the number of blocks which are not triply repeated.
(4) Let $\delta = |\{x : r_x(T) = 4\}|$. Then from (3), we have

$$4\delta + 7\alpha + 10\beta \leq 4v + 9s.$$

But $\delta + \alpha + \beta = v$, and so we have $3\alpha + 6\beta \leq 9s$.
(5) This is immediate since the triples in T_1 are all of the form xxy.
(6) By (5), for distinct $x, y \in V$, we have $\lambda_{xy}(T_1) = 0$ or 2, and so there are exactly v pairs xy $(x \neq y)$ with $\lambda_{xy}(T_1) = 2$. Hence there are exactly v pairs xy with $\lambda_{xy}(T_2) = 1$ (using (2)). Since $|T_2| = v/3 + 3s$, we must have $\sum \lambda_{xy}(T_2) = v + 9s$, and so it follows that $\gamma = 3s$.
(7) This is a consequence of (2) and (5).
(8) We have $xxy \in T_1$, for some y, and so the pair xy must occur in a triple in T_2. □

Lemma 2.3 If $\lambda_{xy}(T_2) = 3$ then $r_x(T) \geq 7$.

Proof: Clearly, $r_x(T_2) \geq 3$. Hence, since $r_x(T_1) \geq 2$, we must have $r_x(T) \geq 7$ (see (1) in Lemma 2.2). □

Lemma 2.4 If $(V, \mathcal{B})$ is a $(v; 1; 3, 3)$BTD with the fine structure $(0, s)$ and there exist distinct x, y, z with $\lambda_{xy}(T_2) = \lambda_{xz}(T_2) = 3$ then $r_x(T) \geq 10$.

Proof: We make use of the fact that there are no repeated blocks in T. The blocks of T_2 which contain the pairs xy and xz must be either:
(1) xya, xyb, xyc, xzd, xze, xzf, where $\{x, y, z\} \cap \{a, b, c, d, e, f\} = \emptyset$ and a, b, c, d, e, $f \in V$; or
(2) xyz, xya, xyb, xzc, xzd, where $a, b, c, d \in V$, $\{x, y, z\} \cap \{a, b, c, d\} = \emptyset$, $a \neq b$ and $c \neq d$.

In case (1), $r_x(T_2) \geq 6$ and $r_x(T_1) \geq 2$, so $r_x(T) \geq 10$ (using Lemma 2.2 (1)). For case (2), suppose that xyz, xya, xyb, xzc, xzd and xxe are the only blocks in T which contain x. Then we must have $e = a$, since otherwise $\lambda_{xa}(T) \neq 3$, but similarly we must have $e = b$, which is not possible. Hence $r_x(T) > 7$, and so $r_x(T) \geq 10$. □

Lemma 2.5 For a $(v; 1; 3, 3)$BTD with the fine structure $(0, s)$, $\gamma \leq \binom{\beta}{2} + \alpha$, and if $\beta = 0$ then $\gamma \leq \lfloor \alpha/2 \rfloor$.

Proof: By Lemma 2.3, if $\lambda_{xy}(T_2) = 3$ then $x, y \in A \cup B$. Moreover, by Lemma 2.4, for $i = 1, 2, \ldots, \alpha$ there is at most one pair xa_i containing a_i and having $\lambda_{xa_i}(T_2) = 3$. There are at most $\lfloor \alpha/2 \rfloor$ pairs a_ia_j, with no a_i occurring in more than one such pair, and so if $\beta = 0$, $\gamma \le \lfloor \alpha/2 \rfloor$. Also, there are at most $\binom{\beta}{2}$ pairs b_ib_j with $\lambda_{b_ib_j}(T_2) = 3$, and so the result follows.
□

We are now ready to prove the following.

Lemma 2.6 $(0,1), (0,2), (0,3) \notin \text{Fine}_1(v)$.

Proof: Let $(V, \mathcal{B})$ be a $(v; 1; 3, 3)$BTD with the fine structure $(0, s)$, and let α, A, β, B, γ and C be as defined above. If $\beta = 0$, then $3s \le \lfloor \alpha/2 \rfloor \le \alpha/2$ (from Lemma 2.2 (5) and Lemma 2.5), and $3\alpha \le 9s$ (Lemma 2.2 (4)), which tells us that $\alpha \le \alpha/2$, and so $\alpha = 0$ and $s = 0$. If $\beta \ne 0$, then by Lemma 2.2 (5) and Lemma 2.5 we have $3s \le \binom{\beta}{2} + \alpha$, and so using Lemma 2.2 (4) we see that $3\alpha + 6\beta \le 3(\binom{\beta}{2} + \alpha)$, and so $\beta \ge 5$. Then Lemma 2.2 (4) tells us that $9s \ge 30$, and so $s \ge 4$. Hence $(0,1), (0,2), (0,3) \notin \text{Fine}_1(v)$.
□

Lemma 2.7 $(1,1) \notin \text{Fine}_1(9)$.

Proof: Suppose $(V, \mathcal{B})$ is a (9;1;3,3) BTD of type $(1,1)$ with $V = \{1, 2, \ldots, 9\}$. Without loss of generality we can assume $\{123, 123, 124, 135, 236, 789\} \subseteq T_2$ (see Lemma 2.2 (5),(7) and (8)). Now consider the pairs 16, 17, 18 and 19. These pairs must be in triply repeated triples; either 167 and 189, 168 and 179, or 169 and 178. This is impossible since each of the pairs 78, 79 and 89 have already occurred in the triple 789. □

Lemma 2.8 If $(V, \mathcal{B})$ is a BTD of type $(0, 4)$ then $(\alpha, \beta) = (0, 6)$ or $(2, 5)$.

Proof: The result follows immediately from Lemma 2.2 (4) and Lemma 2.5. □

Lemma 2.9 $(0, 4) \notin \textit{Fine}_1(9)$.

Proof: Suppose $(V, \mathcal{B})$ is a $(9; 1; 3, 2)$BTD of type $(0, 4)$ with $V = \{1, 2, \ldots, 9\}$ and let T, T_1, T_2, α and β be as defined earlier. By Lemma 2.8 we must have $(\alpha, \beta) = (0, 6)$ or $(2, 5)$.

If $(\alpha, \beta) = (0, 6)$ then by Lemma 2.2 (3) we can assume without loss of generality that $r_1(T) = r_2(T) = \ldots = r_6(T) = 10$ and $r_7(T) = r_8(T) = r_9(T) = 4$.

Now, there are exactly five triply repeated triples in $\mathcal{B}\backslash T$ and $r_7(\mathcal{B}\backslash T) = r_8(\mathcal{B} \setminus T) = r_9(\mathcal{B} \setminus T) = 9$. Hence 7, 8 and 9 must each occur three times in these five triples but with no pair 78, 79 or 89 in more than one. This is impossible and so $(\alpha, \beta) \ne (0, 6)$.

If $(\alpha, \beta) = (2, 5)$ then by Lemma 2.2 (3) we can assume without loss of generality that $r_1(T) = r_2(T) = \ldots = r_5(T) = 10$, $r_6(T) = r_7(T) = 7$ and $r_8(T) = r_9(T) = 4$. Since $\binom{\beta}{2} + \alpha = 12$ (see Lemma 2.5) all of the pairs $\{i, j\}$ with $i, j \in \{1, 2, 3, 4, 5\}$ must occur exactly three times in T_2. Hence 1, 2, 3, 4, 5 are in distinct triples in $\mathcal{B} \backslash T$ and we can assume without loss of generality that the five triply repeated triples (in $\mathcal{B} \setminus T$) are 189, 268, 378, 469, 579. Also (since $\binom{\beta}{2} + \alpha = 12$) the other two pairs which occur three times in T_2 must be $\{i, 6\}$ and $\{j, 7\}$ for some $i, j \in \{1, 2, 3, 4, 5\}$. Moreover, looking at the five triply repeated triples we see that $i \in \{1, 3, 5\}$ and $j \in \{1, 2, 4\}$.

If $i = 3$ then $36x, 36y, 36z \in T_2$ for distinct x, y and z. But looking at the five triply repeated triples we see that $x, y, z \notin \{2, 4, 7, 8, 9\}$ which leaves only two possibilities (1 and 5) for x, y and z and so $i \neq 3$. It is straight forward to check, in a similar manner, that $i \neq 5, j \neq 2$ and $j \neq 4$. Hence we must have $i = j = 1$, but this is also impossible as the triple $11x \in T_1$ forces $\lambda_{1x} > 3$ for any $x \in \{2, 3, \ldots, 9\}$. □

Remark. Whether or not $(0, 4) \in \text{Fine}_1(15)$ is unresolved.

We make use of *group divisible designs* in the next sections. A group divisible design, $(K, \lambda, M; v)$ GDD, is a collection of subsets of size $k \in K$, called blocks, chosen from a v-set, where the v-set is partitioned into disjoint subsets (called groups) of size $m \in M$ such that each block contains at most one element from each group, and any two elements from distinct groups occur together in λ blocks. If $M = \{m\}$ and $K = \{k\}$ we write $(k, \lambda, m; v)$ GDD.

3 The fine structure of small orders

In this section we deal with small cases which are needed for the recursive construction in Section 4 and those which are not covered by these constructions.

Lemma 3.1 (i) $\text{Fine}_1(9) = \text{Adm}_1(9) \setminus \{(0, 4), (1, 1)\}$;
(ii) $\text{Fine}_2(9) = \text{Adm}_2(9)$.

Proof. (i) By Lemmas 2.6, 2.7 and 2.9 we have that $(0, 1)$, $(0, 2)$, $(0, 3)$, $(0, 4)$, and $(1, 1)$ are not in $\text{Fine}_1(9)$. Now apply Lemmas 2.1 and 2.6 and see [1] for a $(9; 1; 3, 3)$BTD of type $(t, s) \in \text{Adm}_1(9) \setminus \{(0, 4), (1, 1)\}$.
(ii) Apply Lemma 2.1 and see [1] for a $(9; 2; 3, 3)$BTD of type $(t, s) \in \text{Adm}_2(9)$. □

Lemma 3.2 (i) There exists a $(9[3];1;3,3)$ frame-BTD with c_2 doubly and c_3 triply repeated blocks where $(c_2,c_3) \in \{(a,b)|\ 0 \leqslant a+b \leqslant 9\} \setminus R$ and

$$R = \{(0,4),(1,4),(0,5),(1,5),(0,6),(3,6),(0,7),(0,8),(1,8)\};$$

(ii) There exists a $(9[3];2;3,3)$ frame-BTD with c_2 doubly and c_3 triply repeated blocks where $(c_2,c_3) \in \{(a,b)|\ 0 \leqslant a+b \leqslant 7\}$.

Proof. See [1] for these designs. □

Lemma 3.3 (i) $\mathrm{Adm}_1(15) \setminus \{(0,4)\} \subseteq \mathrm{Fine}_1(15)$;
(ii) $\mathrm{Fine}_2(15) = \mathrm{Adm}_2(15)$.

Proof. (i) First apply Lemmas 2.1 and 2.6. Secondly, let $(V,\mathcal{B}_1,\mathcal{G})$ and $(V,\mathcal{B}_2,\mathcal{G})$ be two $(3,1,3;15)$ GDD with m blocks in common, where $m \in \{0,1,2,\dots,30\}\setminus\{1,2,3,5\}$ (see [4]). Form a $(3;1;3,3)$BTD on the elements of each group $g \in \mathcal{G}$ and let $\mathcal{B}_3$ be the collection of these blocks. Then $\mathcal{B}_1 \cup \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$ yields a $(15;1;3,3)$BTD of type (m,m). Finally, see [1] for the remaining types.
(ii) Apply Lemma 2.1 and see [1] for a $(15;2;3,3)$BTD of type $(t,s) \in \mathrm{Adm}_2(15)$. □

Lemma 3.4 (i) $(0,10),(1,1) \in \mathrm{Fine}_1(v)$ for $v = 21$ and 27;
(ii) $(0,4) \in \mathrm{Fine}_1(v)$ for $v = 21$, 27, 33 and 39.

Proof. See [1] for these designs. □

We also need the following well-known result.

Lemma 3.5 There exists a $(3,3,3;9)$ GDD with c_2 doubly and c_3 triply repeated blocks where $(c_2,c_3) \in \{(0,0),(9,0),(0,9)\}$.

Proof. For the type $(c_2,c_3) = (0,9)$ we take three copies of a $(3,1,3;9)$ GDD which exists (see [5]). For the type $(c_2,c_3) = (9,0)$ we proceed as follows. Let $(V,\mathcal{B}_1,\mathcal{G})$ and $(V,\mathcal{B}_2,\mathcal{G})$ be two $(3,1,3;9)$ GDD with zero blocks in common (see [4]). Then $(V,\mathcal{B}_1 \cup \mathcal{B}_1 \cup \mathcal{B}_2,\mathcal{G})$ is a $(3,3,3;9)$ GDD with the desired structure. Finally, the following blocks yield a $(3,3,3;9)$ GDD of type $(c_2,c_3) = (0,0)$. Here the groups are $\{1,2,3\}$, $\{4,5,6\}$ and $\{7,8,9\}$. (The block $\{a,b,c\}$ is denoted by abc.)

159	357	147	267	168	258	348	249	369
359	257	347	167	368	158	248	149	269
259	157	247	367	268	358	148	349	169

□

4 Constructions

We start this section with the following two similar constructions.

Construction A *Let* $w \equiv 0$ *or* 1 (*mod* 3), $w \geq 3$, *and* $\rho_2 = 1$ *or* 2. *Then there exists a* $(6w+3;\rho_2;3,3)BTD$.

Proof. Let $(V, \mathcal{B}, \mathcal{G})$ be a $(3,1,2;2w)$ GDD (see for example [5]). We form the desired design on the set $(V \times \{1,2,3\}) \cup \{\infty_1,\infty_2,\infty_3\}$. For each block $b \in \mathcal{B}$ we take the blocks of a $(3,3,3;9)$ GDD on the set $b \times \{1,2,3\}$ with groups $b \times \{i\}$, $i = 1,2,3$. For each group $g \in \mathcal{G}$ except one group, say g_w, we take the blocks of a $(9[3];\rho_2;3,3)$ frame-BTD on the set $(g \times \{1,2,3\}) \cup \{\infty_1,\infty_2,\infty_3\}$ such that $\{\infty_1,\infty_2,\infty_3\}$ are the hole elements. Finally we take the blocks of a $(9;\rho_2;3,3)$BTD on the set $(g_w \times \{1,2,3\}) \cup \{\infty_1,\infty_2,\infty_3\}$. The collection of these blocks yields a $(6w+3;\rho_2;3,3)$BTD on the set $(V \times \{1,2,3\}) \cup \{\infty_1,\infty_2,\infty_3\}$. □

Construction B *Let* $w \equiv 2$ (*mod* 3), $w \geq 5$, *and* $\rho_2 = 1$ *or* 2. *Then there exists a* $(6w+3;\rho_2;3,3)BTD$.

Proof. Let $(V, \mathcal{B}, \mathcal{G})$ be a $(3,1,\{2,4^*\};2w)$ GDD (see for example [5]). We form the desired design on the set $(V \times \{1,2,3\}) \cup \{\infty_1,\infty_2,\infty_3\}$. For each block $b \in \mathcal{B}$ we take the blocks of a $(3,3,3;9)$ GDD on the set $b \times \{1,2,3\}$ with groups $b \times \{i\}$, $i = 1,2,3$. For each group $g \in \mathcal{G}$ with $|g| = 2$, we take the blocks of a $(9[3];\rho_2;3,3)$ frame-BTD on the set $(g \times \{1,2,3\}) \cup \{\infty_1,\infty_2,\infty_3\}$ such that $\{\infty_1,\infty_2,\infty_3\}$ are the hole elements. Finally for the group of size four, say g_{w-1}, we take the blocks of a $(15;\rho_2;3,3)$BTD on the set $(g_{w-1} \times \{1,2,3\}) \cup \{\infty_1,\infty_2,\infty_3\}$. The collection of these blocks yields a $(6w+3;\rho_2;3,3)$BTD on the set $(V \times \{1,2,3\}) \cup \{\infty_1,\infty_2,\infty_3\}$. □

Lemma 4.1 Let $v \equiv 3 \pmod 6$. (i) If $v \geq 33$ then $(0,10),(1,1) \in Fine_1(v)$;
(ii) If $v \geq 45$ then $(0,4) \in Fine_1(v)$.

Proof. (i) First note that there exists a $(3,1,\{6,12^*\};v-3)$ GDD for all $v \equiv 3 \pmod 6$, $v \geq 33$, (see [5]). Triplicate the blocks of this GDD and use a $(9[3];1;3,3)$BTD for groups of size 6 and a $(15;1;3,3)$BTD for the group of size 12. Since $(0,10),(1,1) \in Fine_1(15)$ it follows that $(0,10),(1,1) \in Fine_1(v)$.
(ii) Triplicate the blocks of a $(3,1,\{6,18^*\};v-3)$ GDD which exists for all $v \equiv 3 \pmod 6$, $v \geq 42$, (see [4]). Then use a $(9[3];1;3,3)$BTD for groups of size 6 and a $(21;1;3,3)$BTD for the group of size 18. Since $(0,4) \in Fine_1(21)$ it follows that $(0,4) \in Fine_1(v)$. □

Lemma 4.2 Let $w \equiv 0$ or $1 \pmod 3$, $w \geq 3$. Then $\text{Fine}_1(6w+3) = \text{Adm}_1(6w+3)$.

Proof. Applying Construction A and using designs of different types for the ingredients we can find all the types $(t,s) \in \text{Adm}_1(6w+3)$ except $(t,s) \in \{(0,10),(0,4),(1,1)\}$. These types are covered by Lemmas 3.4 and 4.1. So $\text{Adm}_1(6w+3) \subseteq \text{Fine}_1(6w+3)$ and using Lemmas 2.1 and 2.6 we have the equality. □

Lemma 4.3 Let $w \equiv 2 \pmod 3$, $w \geq 5$. Then $\text{Fine}_1(6w+3) = \text{Adm}_1(6w+3)$.

Proof. If we apply Construction B and use designs of different types for the ingredients we can find all the types $(t,s) \in \text{Adm}_1(6w+3)$ except the type $(0,4)$. This type is covered by Lemmas 3.4 and 4.1. So $\text{Adm}_1(6w+3) \subseteq \text{Fine}_1(6w+3)$ and by Lemmas 2.1 and 2.6 we have the equality. □

So far we have proved the following result which is part (i) of the main theorem.

Theorem 4.4 Let $v \equiv 3 \pmod 6$, $v \geq 21$. Then $\text{Fine}_1(v) = \text{Adm}_1(v)$.

The second part of the Main theorem is proved in the following theorem.

Theorem 4.5 Let $v \equiv 3, \pmod 6$, $v \geq 9$. Then $\text{Fine}_2(v) = \text{Adm}_2(v)$.

Proof. Let $v = 6w+3$. If $w = 1$ or 2 apply Lemmas 3.1 and 3.3 part (ii). So let $w \geq 3$ and proceed as follows. If $w \equiv 0$ or $1 \pmod 3$ apply Construction A and if $w \equiv 2 \pmod 3$ apply Construction B to construct a $(v;2;3,3)$BTD. Now using designs of different types for the ingredients we find that $\mathit{Fine}_2(v) \subseteq \text{Adm}_2(v)$. So the result follows by Lemma 2.1. □

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