Induced Graph Theorem on Magic Valuations

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ABSTRACT. Let G be a graph. A bijection f from $V(G) \cup E(G)$ to $\{1, 2, \ldots, |V(G)| + |E(G)|\}$ is called a magic valuation if $f(u)+f(v)+f(uv)$ is constant for any edge uv in G. A magic valuation f of G is called a supermagic valuation if $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$. The following theorem is proved.

Theorem. For any graph H, there exists a connected graph G so that G contains H as an induced subgraph and G has a supermagic valuation.

1 Introduction

Throughout this paper, we only consider finite, undirected graphs without loops or multiple edges. For a graph G, let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. A *labeling* f of G is a bijection from $V(G) \cup E(G)$ to $\{1, 2, \ldots, |V(G)| + |E(G)|\}$.

In [6], Kotzig and Rosa introduced the notion of magic valuations. A labeling f of G is called a *magic valuation* (in short, *M-valuation*) if there exists a constant s such that $f(u) + f(v) + f(uv) = s$ for any edge uv in G. We call this constant s the *magic number* of f. A magic valuation

f is called a *supermagic valuation* (in short, *SM-valuation*) if $f(V(G)) = \{1, 2, \ldots, |V(G)|\}$.

Kotzig and Rosa [6, 7] proved many results on M-valuations including the ones in the following list.

- All cycles, complete bipartite graphs and caterpillars have M-valuations.
- A complete graph K_n has an M-valuation if and only if $n \in \{1, 2, 3, 5, 6\}$.
- Let nK_2 denote a graph consisting of n independent edges. Then nK_2 has an M-valuation if and only if n is odd.

They actually showed that every caterpillar and nK_2 with n odd has an SM-valuation in their proof. It is proved in [1] that the following statements hold on SM-valuations.

- A complete graph K_n has an SM-valuation if and only if $n \in \{1, 2, 3\}$.
- A complete bipartite graph $K_{m,n}$ has an SM-valuation if and only if $m = 1$ or $n = 1$.
- A cycle C_n has an SM-valuation if and only if n is odd.

2 Main Results

It appears more difficult to find an M-valuation for a dense graph than to do so for a sparse one. Especially, from the fact that any complete graph with at least 7 vertices has no M-valuation, one might think that any graph containing a large complete graph has no M-valuation. The next theorem asserts that this intuitive feeling is not true.

Theorem 1. *Let H be a graph with n vertices and m edges. Then there exists a connected graph G with $|V(G)| \leq 2m + 2n^2 + o(n^2)$ such that G contains H as an induced subgraph and G has an SM-valuation.*

In Theorem 1, we can not take G such that $|V(G)|$ is close to $|V(H)|$ with no restriction. More precisely, the next theorem was proved in [7].

Theorem A. *Let G be a graph containing a complete graph K_n for $n \geq 9$. If G has an M-valuation, then $|V(G)| + |E(G)| \geq n^2 - 5n + 14$.* □

We give an asymptotically better bound by using a result on Sidon set in [2].

Theorem 2. *Let G be a graph containing a complete graph K_n. Then the following statements* (i) *and* (ii) *hold.*

(i) *If G has an M-valuation, then $|V(G)| + |E(G)| \geq 2n^2 - O(n^{3/2})$.*

(ii) *If G has an SM-valuation, then* $|V(G)| \geq n^2 - O(n^{3/2})$.

Before closing this section, we fix a few notations. If a and b are two integers with $a < b$, we denote the set of all integers i with $a \leq i \leq b$ by $[a, b]$. Let G be a graph. Let X, Y be subsets of $V(G)$. We denote the set of edges $\{xy \in E(G) \colon x \in X \text{ and } y \in Y\}$ by $E(X, Y)$. We simply write $E(X)$ instead of $E(X, X)$. Moreover, if H_1 and H_2 are induced subgraphs of G, we denote $E(V(H_1), V(H_2))$ by $E(H_1, H_2)$.

3 Proof of Theorem 1

We employ Singer's difference set [8] for the labeling of G.

Lemma 3. *For any positive integer n, there exists a function ξ from $[0, n]$ to the set of non-negative integers such that*

(i) $0 = \xi(0) < \xi(1) < \xi(2) < \ldots < \xi(n)$,

(ii) $\xi(j) - \xi(i) \neq \xi(l) - \xi(k)$ *for any* $\{i, j\} \neq \{k, l\}$,

(iii) $\xi(n) \leq n^2(1 + o(1))$.

Proof: For a given n, let q be the minimum prime power not less than n. The existence of Singer's difference set implies that there exist $\{0 = x_0 < x_1 < \cdots < x_q\}$ in $[0, q^2 + q]$ so that any element of $\mathbf{Z}_{q^2+q+1} \setminus \{0\}$ is uniquely represented by $x_i - x_j$ for some $i, j \in [0, q]$. Define $\xi(i) = x_i$ for $0 \leq i \leq n$. Then ξ satisfies the conditions (i) and (ii). It is left to show that (iii) holds. Let p_m be the mth prime. It is known in number theory that $p_{m+1} - p_m \leq O(m^{\frac{11}{20}+\varepsilon})$ for any $\varepsilon > 0$ [4]. Hence if $p_m \leq n < p_{m+1}$ then it follows that $q - n \leq p_{m+1} - p_m \leq O(m^{\frac{11}{20}+\varepsilon}) \leq o(n)$. Therefore, $\xi(n) = x_n \leq q^2 + q \leq n^2(1 + o(1))$, as claimed. □

Proof of Theorem 1: For a given graph H, set $n = |V(H)|$, $m = |E(H)|$ and $V(H) = \{u_1, u_2, \ldots, u_n\}$. Let ξ be a function guaranteed by Lemma 3 for n.

Step 1. Construct a connected graph G' containing H and an odd cycle C.

Set

$$t = 2m + 2n + 1.$$

Let C be a cycle of order t with

$$V(C) = \{v_1, v_2, \ldots, v_t\}$$

so that the vertices are arranged as $v_1, v_{\frac{t+3}{2}}, v_2, v_{\frac{t+5}{2}}, v_3, \ldots, v_t, v_{\frac{t+1}{2}}$ along the cycle. We define a graph G' such that

$$V(G') = V(H) \cup V(C),$$
$$E(G') = E(H) \cup E(C) \cup \{uv_t : u \in V(H)\}.$$

Step 2. Define a function f' on $V(G') \cup E(G')$.

First we define two integers s and M as

$$s = 3t + 2(\xi(n) + \xi(n-1)) + n + 1,$$
$$M = s - \frac{t+3}{2}.$$

Later in step 4, we shall show that s and M are the magic number and the maximum label of an SM-valuation of a required graph G, respectively.

We define a function f' from $V(G') \cup E(G')$ to the set of positive integers such that

$$\begin{aligned} f'(v_i) &= i && \text{if } v_i \in V(C),\\ f'(u_i) &= t + \xi(i) && \text{if } u_i \in V(H),\\ f'(u_iu_j) &= s - 2t - \xi(i) - \xi(j) && \text{if } u_iu_j \in E(H),\\ f'(u_iv_t) &= s - 2t - \xi(i) && \text{if } u_i \in V(H),\\ f'(v_iv_j) &= s - i - j && \text{if } v_iv_j \in E(C). \end{aligned}$$

Claim 2.1. $f'(V(C)) = [1, t]$ and $f'(E(C)) = [M - t + 1, M]$.

We only need to show the second equality. From the definition of C, the sums $i + j$ for $v_iv_j \in E(C)$ cover the interval from $1 + \frac{t+1}{2} = \frac{t+3}{2}$ to $\frac{t+1}{2} + t = \frac{t+3}{2} + t - 1$, as claimed.

Claim 2.2. f' is an injection from $V(G') \cup E(G')$ to $[1, M]$.

Claim 2.1 implies that $f'|_{V(C)}$ and $f'|_{E(C)}$ are injections. It is also easy to see that $\max f'(V(C)) < \min f'(V(H))$, and $\max f'(E(H) \cup E(H, \{v_t\})) < \min f'(E(C))$. Moreover, $f'|_{V(H)}$ and $f'|_{E(H) \cup E(H, \{v_t\})}$ are injections by the property of ξ. It only suffices to show that $\max f'(V(H)) < \min f'(E(H))$. Indeed,

$$\begin{aligned} \max f'(V(H)) &= t + \xi(n)\\ &= s - 2t - \xi(n) - 2\xi(n-1) - n - 1\\ &< s - 2t - \xi(n) - \xi(n-1)\\ &\le \min f'(E(H)). \end{aligned}$$

Hence, Claim 2.2 holds.

The proof of Claim 2.2 also implies the following claim.

Claim 2.3. $\max f'(V(G')) < \min f'(E(G'))$.

Claim 2.4. $f'(x) + f'(y) + f'(xy) = s$ for any edge xy in G'.

If $xy = u_iu_j \in E(H)$, then $f'(u_i) + f'(u_j) + f'(u_iu_j) = t + \xi(i) + t + \xi(j) + s - 2t - \xi(i) - \xi(j) = s$. If $xy = u_iv_t \in E(H, \{v_t\})$, then $f'(u_i) + f'(v_t) + f'(u_iv_t) = t + \xi(i) + t + s - 2t - \xi(i) = s$. If $xy = v_iv_j \in E(C)$, then $f'(v_i) + f'(v_j) + f'(v_iv_j) = i + j + s - i - j = s$. Hence, Claim 2.4 holds.

Step 3. Attach new pairs of a vertex and an edge to G' to complete a required graph G.

We denote $[1, M]\setminus f'(V(G')\cup E(G'))$ by P, which is the set of remaining labels. Then, we have

$$\begin{aligned}|P| &= M - |f'(V(G') \cup E(G'))| \\ &= s - \frac{t+3}{2} - (|V(H)| + |V(C)| + |E(H)| + |E(C)| + |E(H, \{v_t\})|) \\ &= 3t + 2(\xi(n) + \xi(n-1)) + n + 1 - \frac{t+3}{2} - (n + t + m + t + n) \\ &= 2(\xi(n) + \xi(n-1)).\end{aligned}$$

Let

$$P = \{a_1 < a_2 < \ldots < a_k < b_k < b_{k-1} < \ldots < b_2 < b_1\},$$

where

$$k = \xi(n) + \xi(n-1).$$

We build a graph G by adding k new pairs of a vertex and an edge to G' in order to dispose of $2k$ labels in P.

Claim 3.1. $\max f'(V(G')) < a_k < \min f'(E(G'))$. In particular, $a_k = n + t + k$.

Since $\max f'(V(G')) - |f'(V(G'))| = t + \xi(n) - (n + t) < k$, the first inequality holds. Similarly, since $\min f'(E(G')) - |f'(V(G'))| \geq s - 2t - \xi(n) - \xi(n-1) - (n + t) = k + 1$, the second inequality holds. Thus $a_k = |f'(V(G'))| + k = n + t + k$.

Claim 3.2. $a_i \in [t + i, t + i + n]$ for $1 \leq i \leq k$ and $b_i \in [M - t + 1 - i - (m + n), M - t + 1 - i]$ for $1 \leq i \leq k$.

Note that Claim 2.1 implies $t + 1 \leq a_1$ and $b_1 \leq M - t$. From Claim 3.1, it follows that $a_i \geq a_1 + (i - 1) \geq t + i$ and $a_i \leq a_k - (k - i) = t + i + n$. Similarly, $b_i \leq b_1 - (i - 1) \leq M - t + 1 - i$ and $b_i \geq a_k + (k - i + 1) = n + t + 2k - i + 1 = M - t + 1 - i - (m + n)$, as claimed.

Now, we are ready to define a graph G ;

$$\begin{aligned}V(G) &= V(G') \cup \{w_1, w_2, \ldots, w_k\}, \\ E(G) &= E(G') \cup \{w_iv_{s-a_i-b_i} : 1 \leq i \leq k\}.\end{aligned}$$

This definition is consistent since $1 \leq s - a_i - b_i \leq t$ holds for any $1 \leq i \leq k$. Indeed, from Claim 3.2, we have

$$\begin{aligned}s - a_i - b_i &\geq s - (t + i + n) - (M - t + 1 - i) \\ &= m + 1 \\ &\geq 1,\end{aligned}$$

and

$$\begin{aligned} s - a_i - b_i &\le s - (t+i) - (M - t + 1 - i - m - n) \\ &= t. \end{aligned}$$

Note that G is connected and contains H as an induced subgraph.

Step 4. Define an SM-valuation f on G.

We define a labeling f on G so that f is an extension of f';

$$\begin{aligned} f(x) &= f'(x) && \text{if } x \in V(G'), \\ f(xy) &= f'(xy) && \text{if } xy \in E(G'), \\ f(w_i) &= a_i && \text{if } 1 \le i \le k, \\ f(w_i v_{s-a_i-b_i}) &= b_i && \text{if } 1 \le i \le k. \end{aligned}$$

Since f' is an injection from $V(G') \cup E(G')$ to $[1, M] \setminus P$, it follows that f is a bijection from $V(G) \cup E(G)$ to $[1, M]$. From Claim 2.4, we have $f(x) + f(y) + f(xy) = s$ for any edge $xy \in E(G')$. Moreover, we have $f(w_i) + f(v_{s-a_i-b_i}) + f(w_i v_{s-a_i-b_i}) = a_i + s - a_i - b_i + b_i = s$ for $1 \le i \le k$. Hence, f is an M-valuation. Furthermore, Claim 2.3 and Claim 3.1 with the fact $a_i < b_j$ for any i and j imply that f is an SM-valuation. It is only left to estimate the number of vertices of G. Since $\xi(n-1) < \xi(n) \le n^2(1 + o(1))$ holds from Lemma 3, we have

$$\begin{aligned} |V(G)| &= |V(G')| + k \\ &= n + t + \xi(n) + \xi(n-1) \\ &= 2m + 3n + 1 + \xi(n) + \xi(n-1) \\ &\le 2m + 2n^2 + o(n^2). \end{aligned}$$

This completes the proof. □

4 Proof of Theorem 2

First we introduce some terminology according to [5]. Let $X = \{x_1 < \ldots < x_n\}$ be a set of positive integers. X is called a sequence for a *well spread set of integers* (in short, *WS-sequence*) if the sums $x_i + x_j$ for $i < j$ are all different. We define *the smallest span* $\sigma^*(n)$ and *the smallest span of pairwise sums* $\rho^*(n)$ as follows.

$$\sigma^*(n) = \min\{x_n - x_1 + 1 \,:\, X \text{ is a WS-sequence of order } n\}$$
$$\rho^*(n) = \min\{x_n + x_{n-1} - x_2 - x_1 + 1 \,:\, X \text{ is a WS-sequence of order } n\}.$$

The following four lemmas were proved in [5] and [7].

Lemma 4. $\sigma^*(2) = 2$, $\sigma^*(3) = 3$, $\sigma^*(4) = 5$, $\sigma^*(5) = 8$, $\sigma^*(6) = 13$, $\sigma^*(7) = 19$, $\sigma^*(8) = 25$, *and* $\sigma^*(n) \geq \frac{1}{2}n^2 - \frac{3}{2}n + 5$ *for* $n \geq 9$. □

Lemma 5. $\rho^*(n) \geq 2\sigma^*(n-1)$ *for* $n \geq 4$. □

Lemma 6. $\rho^*(2) = 1$, $\rho^*(3) = 3$, $\rho^*(4) = 6$, $\rho^*(5) = 11$, $\rho^*(6) = 19$, $\rho^*(7) = 30$, $\rho^*(8) = 43$, *and* $\rho^*(n) \geq n^2 - 5n + 14$ *for* $n \geq 9$. □

Lemma 7. *Let* G *be a graph containing a complete graph* K_n. *If* G *has an* M*-valuation, then* $|V(G)| + |E(G)| \geq \rho^*(n)$. □

Theorem A in Section 2 follows immediately from Lemma 6 and Lemma 7. A similar result to Lemma 7 holds on SM-valuations.

Lemma 8. *Let* G *be a graph containing a complete graph* K_n. *If* G *has an* SM*-valuation, then* $|V(G)| \geq \sigma^*(n)$.

Proof: Let f be an SM-valuation of G. Then $\{f(v) : v \in V(K_n)\}$ is a well spread of integers, because for any two distinct edges xy and zw in $E(K_n)$, there holds $f(x) + f(y) = s - f(xy) \neq s - f(zw) = f(z) + f(w)$. Since any label at most $\max f(V(K_n))$ is assigned to some vertex of G, it follows that $|V(G)| \geq \max f(V(K_n)) \geq \sigma^*(n)$. □

We owe the lower bound of σ^* to the following result of Erdös and Turán [2].

Theorem B. *Let* $x_1 < x_2 < \ldots < x_n \leq s$ *be positive integers such that the sums* $x_i + x_j$ *for* $i \leq j$ *are all different. Then* $s \geq n^2 - O(n^{\frac{3}{2}})$. □

We need to replace the condition "$i \leq j$" in Theorem B to "$i < j$" for our purpose. Is the bound affected by this change? In fact, as shown in the proof of Lemma 6 in [3], a modification of the proof of Theorem B assures the same asymptotic bound for σ^*.

Theorem B′. $\sigma^*(n) \geq n^2 - O(n^{\frac{3}{2}})$.

Now we are ready to complete the proof of Theorem 2.

Proof of Theorem 2: (i) follows from Lemma 5, Lemma 7 and Theorem B'. (ii) follows from Lemma 8 and Theorem B', as required. □

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