Large Neighbourhood Unions and Edge-disjoint Perfect Matchings

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Abstract

We show that if, for any fixed r, the neighbourhood unions of all r-sets of vertices are large enough, then G will have many edge-disjoint perfect matchings. In particular, we show that given fixed positive integers r and c and a graph G of even order n, if the minimum degree is at least $r + c - 1$ and if the neighbourhood union of each r-set of vertices is at least $n/2 + (2\lfloor(c+1)/2\rfloor - 1)r$, then G has c edge-disjoint perfect matchings, for n large enough. This extends earlier work by Faudree, Gould and Lesniak on neighbourhood unions of pairs of vertices.

1 Introduction

A graph is assumed to be finite with no multiple edges or loops. We use $\delta(G)$ to denote the minimum degree over all vertices in the graph G, and $\kappa'(G)$ to denote the edge-connectivity of G. The neighbourhood of a vertex v, denoted $N_G(v)$, is the set of edges adjacent to v in G. If there is no possibility of ambiguity, the subscript G will be omitted. Other notation not mentioned here can be found in [1].

This paper addresses the problem of determining whether or not a graph has c edge-disjoint perfect matchings, for any given c.

For $c \geq 2$, this problem is NP-complete. Ian Holyer [3] has shown that determining the edge-colourability of a 3-regular graph is NP-complete. Vizing's Theorem (proven in [1]) tells us that a 3-regular graph is either 3-edge-colourable

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or 4-edge-colourable. Since a 3-regular graph is 3-edge-colourable if and only if it has 3 edge-disjoint perfect matchings, it follows that the problem of determining whether or not a 3-regular graph has 3 (or equivalently 2) edge-disjoint perfect matchings is NP-complete. Thus, in general, determining whether or not a graph has $c \geq 2$ edge-disjoint perfect matchings is NP-complete.

In this paper, we obtain a sufficient condition for a graph to have c edge-disjoint perfect matchings, based on the size of neighbourhood unions of r-sets of vertices in the graph. This extends earlier work by Faudree, Gould and Lesniak on neighbourhood unions of pairs of vertices.

2 Background Results

Tutte [1] characterized when a graph has a perfect matching:

Theorem 1 *A graph G has a perfect matching if and only if*

$$odd(G - S) \leq |S| \qquad \forall S \subseteq V,$$

where $odd(G - S)$ is the number of odd-order components in $G - S$.

A path or cycle is hamiltonian if it passes through every vertex of G. A graph is hamiltonian if it has a hamiltonian cycle. The following results on hamiltonicity, which are due to Ore [1], will be useful:

Theorem 2 *Let G be a graph of order n. If $d(u) + d(v) \geq n$ for all non-adjacent pairs of vertices u, v, then G is hamiltonian.*

Theorem 3 *Let G be a graph of order n. If $d(u) + d(v) \geq n + 1$ for all non-adjacent pairs of vertices u, v, then G is hamiltonian connected, i.e. there is a hamiltonian path joining any 2 vertices of G.*

Faudree, Gould and Lesniak [2] have shown:

Theorem 4 *If a graph $G = (V, E)$ with even order n satisfies:*

1. $\delta(G) \geq c + 1$
2. $\kappa'(G) \geq c$
3. $|N(u_1) \cup N(u_2)| \geq n/2 \qquad \forall u_1,\ u_2 \in V$

then G has c edge-disjoint perfect matchings, for n large enough.

They have also observed in their paper that each of these conditions is required and is also sharp.

3 Main Results

In Theorem 4, it is shown that an even order graph with sufficiently high minimum degree and edge-connectivity will have a large number of edge-disjoint perfect matchings provided the neighbourhood union of all pairs of vertices is at least one half the order of the graph. In the following result, we show that by demanding neighbourhood unions which contain just one more vertex, we can avoid checking the edge-connectedness condition altogether.

Corollary 5 *Let c be a positive integer. Let $G = (V, E)$ be a graph with even order n satisfying:*

1. $\delta(G) \geq c + 1$
2. $|N(u_1) \cup N(u_2)| \geq n/2 + 1$

then G has c edge-disjoint perfect matchings, for n large enough.

Proof Let G satisfy the conditions of the corollary. Note that G is connected since each component having at least 2 vertices must have more than half the vertices in the graph by the neighbourhood condition. Furthermore, the minimum degree condition does not allow for an isolated vertex.

Suppose that $\kappa'(G) < c$. Then there is a minimal set S of $\leq c$ edges for which $G - S$ is disconnected. Let C be a component of $G - S$. Let $V(S)$ be the set of endpoints of the edges in S. If $C \cap V(S)$ were empty, then G would be disconnected, so we can take $u \in C \cap V(S)$. Then, $d_S(u) = |N(u) \cap V(S)| \leq c$. To see this, note that u cannot be adjacent to vertices in any other component of $G - S$ except through edges of S, so this gives at least $|N(u) \cap V(S) \cap V(G - C)|$ edges in S, all of which are incident with u. Also, there is at least one edge in the cut set for each vertex in $C \cap V(S)$ adjacent to u, so this gives $|N(u) \cap V(S) \cap C|$ edges in S, none of which is incident with u. These edges are all distinct since if an edge of S has both endpoints in C, we can delete it from S and have a smaller set leaving $G - S$ disconnected. Thus,

$$\begin{aligned} c &\geq |S| \\ &\geq |N(u) \cap V(S) \cap V(G-C)| + |N(u) \cap V(S) \cap C| \\ &= d_S(u) \end{aligned}$$

But $d_G(u) \geq c+1$, so there exists a vertex $x \in C$ such that u is adjacent to x, and $x \notin C \cap V(S)$. Since $|C \cap V(S)| \leq |S| \leq c$ and the vertex x also has $d(x) \geq c+1$, there exists at least one other vertex $y \in C$ besides x and those in $C \cap V(S)$. But then $|C| \geq |N(x) \cup N(y)| \geq n/2+1$. Thus, every component has more than half the vertices, so there is only one component, a contradiction.

Thus, $\kappa'(G) \geq c$ and Theorem 4 applies to show that G has c edge-disjoint perfect matchings. □

Thus we do away with the edge-connectivity requirement and instead add a suitable constant to the neighbourhood union requirement to obtain many edge-disjoint perfect matchings. It is worth mentioning that we cannot simply omit the edge-connectivity condition in the above corollary without increasing the necessary neighbourhood union size to $n/2+1$. Faudree, Gould and Lesniak mentioned in [2] that the graph obtained by adding $c-1$ edges between two disjoint copies of $K_{n/2}$ with $n \equiv 2 (\text{mod } 4)$, satisfies (for n sufficiently large) conditions (1) and (3) of Theorem 4, but does not have c edge-disjoint perfect matchings.

In the remainder of this section, we consider neighbourhood unions of r vertices for any fixed positive integer r instead of just pairs of vertices, and attempt to generalize Corollary 5. The penalty we pay for using r-sets of vertices (rather than pairs) is that while Corollary 5 only requires neighbourhood unions of size $n/2+1$, the proof of our corresponding result for r-sets will require adding to $n/2$ a constant depending both on c and r, rather than just the fixed value 1.

We will need the following lemma:

Lemma 6 *Let r be a positive integer. Let G be a graph of large enough even order n satisfying:*

1. $\delta(G) \geq r$
2. $|N(u_1) \cup N(u_2) \cup \cdots \cup N(u_r)| \geq n/2 \qquad \forall u_1, u_2, \cdots, u_r \in V.$

If there exists S such that $odd(G-S) > |S|$, then $S = \emptyset$.

Proof Suppose there exists an $S \neq \emptyset$ such that $odd(G-S) > |S|$. Then, $G-S$ has at least $s+2$ components, where $s = |S|$, since the parities of $odd(G-S)$ and $|S|$ are the same.

Let k be the number of components of $G - S$ of order $< r$. Then $s + 2 - k \leq$ the number of components of $G - S$ of order at least r. Thus, n is at least $k + s + (n/2 - s)(s + 2 - k)$ since the neighbourhood condition implies that any component of order at least r also has order at least $n/2 - s$. This quadratic equation in s reduces to

$$(s - (n/2 - 1))(s - k) \geq 0.$$

Thus, $s \leq k$ or $s \geq n/2 - 1$ (or both).

If $s \geq n/2 - 1$, then $s + 2 \geq n/2 + 1$ and all components are singletons. If $n/2 + 1 \geq r$ (i.e. for n large enough), then the neighbourhood condition implies that $s \geq n/2$. It follows that $G - S$ has at least $n/2 + 2$ vertices, so the total number of vertices in G is at least $n + 2 > n$, a contradiction. Thus we can conclude that $s \leq k$.

Since $S \neq \emptyset$, we have $1 \leq s \leq k$.

Suppose first that $s \geq n/2 - r^2$. Let k_1 be the number of singleton components. Note that $k_1 \leq r - 1$ since $\geq r$ singleton components would imply that $s \geq n/2$ by the neighbourhood condition. Thus, the number of vertices in G is

$$\begin{aligned}
n &\geq s + (\text{ the number of vertices in components of size } < r) \\
&\geq n/2 - r^2 + k_1 + 2(k - k_1) \\
&= n/2 - r^2 + 2k - k_1 \\
&\geq n/2 - r^2 + 2k - (r - 1) \\
&\geq n/2 - r^2 - r + 1 + 2s \qquad \text{because } k \geq s \\
&\geq n/2 - r^2 - r + 1 + 2(n/2 - r^2) \\
&= n + n/2 - 3r^2 - r + 1
\end{aligned}$$

Thus, $3r^2 + r \geq n/2 + 1$, a contradiction for n large enough.

Thus we assume that $s < n/2 - r^2$.

Let $C_1, C_2, \cdots, C_k$ be the components of $G - S$ which have size $< r$.

Case 1: $|C_1| + |C_2| + \cdots + |C_k| \geq r$.

Let $p = |C_1| + |C_2| + \cdots + |C_t|$ where t is the first integer with this sum $\geq r$.

Then, $|C_1| + |C_2| + \cdots + |C_{t-1}| \leq r - 1$, $|C_t| \leq r - 1$, so that $p \leq 2(r - 1)$. Also, by the neighbourhood condition applied to r vertices from these t components, $s \geq n/2 - p \geq n/2 - 2r + 2$.

Thus, $n/2 - r^2 > s \geq n/2 - 2r + 2$, so $(r - 1)^2 + 1 < 0$, a contradiction.

Case 2: $|C_1| + |C_2| + \cdots + |C_k| < r$.

Since $k \neq 0$, we have $1 \leq |C_1| + |C_2| + \cdots + |C_k| \leq r - 1$. Let i be the size of the smallest component, $1 \leq i \leq r - 1$. Then,

$$\begin{aligned} r - 1 &\geq |C_1| + |C_2| + \cdots + |C_k| \\ &\geq ik \\ &\geq is \end{aligned}$$

But $s \geq r - i + 1$ since $\delta(G) \geq r$. Thus,

$$\begin{aligned} r - 1 &\geq i(r - i + 1) \\ r - 1 &\geq ir - i^2 + i \\ i^2 - i(r + 1) + r - 1 &\geq 0. \end{aligned}$$

Thus, $i^2 - i(r + 1) + r > 0$, i.e. $(i - r)(i - 1) > 0$.

Thus, either $i > r$, or $i < 1$, a contradiction in either case. Thus, this case cannot occur either, and so the result holds.

This completes the proof of the lemma. □

We are now in a position to prove the main result in this paper:

Theorem 7 *Let r, c be fixed positive integers. Let G be a graph of even order n satisfying:*

1. $\delta(G) \geq r + c - 1$
2. $|N(u_1) \cup N(u_2) \cup \cdots \cup N(u_r)| \geq n/2 + (2\lfloor(c+1)/2\rfloor - 1)r$ $\forall\, u_1, u_2, \cdots, u_r \in V$.

Then, G has c edge-dijoint perfect matchings, for n large enough.

Proof

Case 1: r=1. We use induction on $\lfloor(c+1)/2\rfloor$. If $\lfloor(c+1)/2\rfloor = 1$, i.e. if $c = 1$ or 2, then $d(u) \geq n/2 + 1$ for all u. It follows that G is hamiltonian, by Theorem 2, and thus that G has 2 edge-disjoint perfect matchings, since G has even order.

If $\lfloor(c + 1)/2\rfloor = m \geq 2$, then $d(u) \geq n/2 + 2m - 1$ for all $u \in V$. G is hamiltonian, again by Theorem 2, so G has 2 edge-disjoint perfect matchings. Delete a hamiltonian cycle and call the resulting graph H.

Then, $d_H(u) \geq n/2 + 2(m - 1) - 1$ for all $u \in V$. By the induction hypothesis, H has $c - 2$ edge-disjoint perfect matchings since $m - 1 = \lfloor((c-2)+1)/2\rfloor$. But then H together with the hamiltonian cycle deleted from G forms c edge-disjoint perfect matchings of G.

This finishes the proof of case 1.

Case 2: $r \geq 2$

We use induction on c.

If $c = 1$, the hypothesis of the theorem tells us that $\delta(G) \geq r$, and $|N(u_1) \cup N(u_2) \cup \cdots \cup N(u_r)| \geq n/2 + r$ for all $u_1, u_2, \cdots, u_r \in V$. If G does not have a perfect matching, then there exists $S \subseteq V$ such that $odd(G - S) > |S|$, by Theorem 1. By the lemma, $S = \emptyset$, and G has at least 2 odd components C_1, C_2. But, since $\delta(G) \geq r$, this implies that $|C_i| \geq r$ and thus $|C_i| \geq n/2 + r$ for all i, by the neighbourhood condition. This gives a contradiction because C_1 and C_2 cannot both have more than half the vertices.

If $c = 2$, we are given that $\delta(G) \geq r + 1$, and $|N(u_1) \cup N(u_2) \cup \cdots \cup N(u_r)| \geq n/2 + r$ for all $u_1, u_2, \cdots, u_r \in V$. Now, G has a perfect matching M, by the case $c = 1$. Let $H = G - M$. Then $\delta(H) \geq r$, and $|N_H(u_1) \cup N_H(u_2) \cup \cdots \cup N_H(u_r)| \geq n/2$ for all $u_1, u_2, \cdots, u_r \in V(H)$. Thus, if there exists an $S \subseteq V(H)$ such that $odd(H - S) > |S|$, then $S = \emptyset$, by the lemma.

Now, $\delta(H) \geq r \Rightarrow |C_i| \geq r + 1$ for C_i a component of H. But this implies $|C_i| \geq n/2$, by the neighbourhood condition. Thus, H must consist of 2 odd components, C_1 and C_2, each of order $n/2$. Note that since $r \geq 2$, each C_i has at least 3 vertices, being of odd order.

Suppose that u, v are non-adjacent vertices in C_1. If there are r vertices in C_1 which are not adjacent to v, then v is not in the neighbourhood union of these r vertices, so that these r vertices has a neighbourhood union of size at most $n/2 - 1$, a contradiction.

Thus, $d_H(v) \geq n/2 - (r - 1)$ and similarly, $d_H(u) \geq n/2 - (r - 1)$. Thus,

$$d_H(u) + d_H(v) \geq n - 2(r - 1) \geq n/2 + 1$$

for n large enough. Thus, C_1 is hamiltonian connected, by Theorem 3. Similarly, C_2 is hamiltonian connected. Also, for n large enough, $n/2 > r$, and for any r vertices in C_1, their neighbourhood union consists of at least $n/2 + r$ vertices and therefore at least r vertices from C_2. Thus, in G, there are at least r edges of M joining C_1 and C_2. In particular, since $r \geq 2$, there are at least 2 edges of M joining C_1 and C_2, say u_1u_2, v_1v_2 with $u_1, v_1 \in C_1, u_2, v_2 \in C_2$.

Note that $u_1 \neq v_1$, and $u_2 \neq v_2$ since M is a matching. Letting P_1 be a hamiltonian path in C_1 with u_1 and v_1 as endpoints and P_2 a hamiltonian path in C_2 having u_2 and v_2 as endpoints, then $P_1 \cup u_1u_2 \cup P_2 \cup v_2v_1$ is a hamiltonian cycle in G, so G has 2 edge-disjoint perfect matchings.

Thus the base cases $c = 1,\ 2$ are shown.

Now let $c = 2q+1,\ q \geq 1$. Then $\delta(G) \geq r + 2q$, and $|N(u_1) \cup N(u_2) \cup \cdots \cup N(u_r)| \geq n/2 + (2q+1)r$ for all $u_1,\ u_2, \cdots, u_r \in V$.

Then G has 2q edge-disjoint perfect matchings for n large enough, by the induction hypothesis. Then let H be the graph resulting from deleting these $2q$ edge-disjoint perfect matchings. Then H satisfies $\delta(H) \geq r$, and $|N_H(u_1) \cup N_H(u_2) \cup \cdots \cup N_H(u_r)| \geq n/2 + r$ for all $u_1,\ u_2, \cdots, u_r \in V(H)$. So if H doesn't have a perfect matching, we know from Theorem 1 that there exists a set S such that $odd(H-S) > |S|$ and from the lemma, that $S = \emptyset$. Thus, $odd(H) \geq 2$. But, since each component of H has at least $\delta(H) + 1 = r + 1$ vertices, each component of H will also have at least $n/2 + r$ vertices, a contradiction. Thus, H has a perfect matching and G has $2q + 1$ edge-disjoint perfect matchings.

Now let $c = 2q+2,\ q \geq 1$. Then $\delta(G) \geq r + 2q + 1,\ |N(u_1) \cup N(u_2) \cup \cdots \cup N(u_r)| \geq n/2 + (2q+1)r$ for all $u_1,\ u_2, \cdots, u_r \in V$.

Then G has $2q+1$ edge-disjoint perfect matchings by the case $c = 2q+1$. Then let H be the graph obtained from deleting these 2q+1 edge-disjoint perfect matchings. Then H satisfies $\delta(H) \geq r,\ |N(u_1) \cup N(u_2) \cup \cdots \cup N(u_r)| \geq n/2$ for all $u_1,\ u_2, \cdots, u_r \in V$.

If H does not have a perfect matching, then H consists of 2 odd components, by the lemma. Again $|C_i| \geq n/2$, by the neighbourhood condition. Also, as before, C_1 and C_2 are hamiltonian connected.

In G, there exist at least $(2q+1)r$ edges between C_1 and C_2, since for any r vertices in C_1, there are at least $(2q+1)r$ vertices from C_2 in their neighbourhood union. So at least one of the $2q+1$ perfect matchings, say M, has at least $r \geq 2$ edges between C_1 and C_2. Then $H \cup M$ has 2 edge-disjoint perfect matchings since C_1 and C_2 are hamiltonian connected.

Thus, the $2q$ edge-disjoint perfect matchings other than M together with $H \cup M$ has $2q+2$ edge-disjoint perfect matchings. Thus, G has $2q+2$ edge-disjoint perfect matchings, and this completes the induction argument.

□

Note: for n to be large enough, it is enough for n to be at least $6r^2 + 2r$.

4 Conclusion

We have obtained a corollary of a result of Faudree, Gould, and Lesniak which guarantees c edge-disjoint perfect matchings in an even-order graph if the minimun degree and the neighbourhood union of all pairs of vertices are large enough. We show that this corollary can be extended so that large neighbourhood unions are required for r-sets rather than pairs of vertices.

The advantage of this result is that by considering r-sets, our result may be applied to graphs where the average degree is smaller. In particular, the neighbourhood union requirement in Theorem 4 requires that at most 1 vertex can have degree less than $n/4$. Our result may still apply as long as at most $r - 1$ vertices have degree less than $n/(2r) + k(c)$, for any fixed constant r, where $k(c)$ is a constant depending only on c, the number of edge-disjoint perfect matching wanted.

The natural question then arises whether or not Theorem 4 can be extended to a similar result using neighbourhood unions of r-sets of vertices, by keeping the edge-connectivity and neighbourhood union conditions the same and only adding to the minimum degree condition. Also, since Theorem 7 is not sharp for $r = 2$, can it be improved for general r?

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