Ramsey Sets for Matchings

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Abstract

In this note we characterize the members of the Ramsey set $\mathcal{R}(2K_2, tK_2)$ of all $(2K_2, tK_2)$-minimal graphs using factor-critical graphs. Moreover, the sets $\mathcal{R}(2K_2, tK_2)$ are determined for $t \leq 5$.

1 Introduction

For (simple) graphs F, G and H we write $F \rightarrow (G, H)$ to mean that in any 2-coloring of the edges of F with green and red there is a green subgraph isomorphic to G or a red subgraph isomorphic to H. F is said to be a (G, H)-*minimal* graph if $F \rightarrow (G, H)$ and $F' \not\rightarrow (G, H)$ for every proper subgraph F' of F. The most general problem in graph Ramsey theory is that of characterizing those F satisfying $F \rightarrow (G, H)$ for a given pair of graphs (G, H). This problem is solved if the *Ramsey set* $\mathcal{R}(G, H)$ of all (G, H)-minimal graphs (up to isomorphism) is determined. Various results have been obtained concerning the question whether, for given (G, H), $\mathcal{R}(G, H)$ is finite or infinite, but the complete determination of $\mathcal{R}(G, H)$ is an extremely difficult problem which has been solved only for some very special pairs (G, H).

In [1] Burr, Erdös, Faudree and Schelp proved that $\mathcal{R}(G, H)$ is finite if G is a matching mK_2 and H an arbitrary graph. In [2] Burr, Erdös, Faudree, Rousseau and Schelp studied the special case $G = 2K_2$ and $H = tK_2$. They showed how the members of $\mathcal{R}(2K_2, tK_2)$ with connectivity at most one can be constructed using the sets $\mathcal{R}(2K_2, t'K_2)$ with $t' < t$. Moreover, they described a large family of members of $R(2K_2, tK_2)$ and determined the sets for small t.

In this paper we will extend the results from [2] and characterize the members of $\mathcal{R}(2K_2, tK_2)$. This characterization essentially uses factor-critical graphs. Moreover, a well-known method for constructing factor-critical graphs will be used to determine the sets $\mathcal{R}(2K_2, tK_2)$ for $t \leq$

5. (The sets $\mathcal{R}(2K_2, tK_2)$ for $t \leq 4$ were already given in [2], but one member of $\mathcal{R}(2K_2, 4K_2)$ is missing there.) Additionally, we will answer some questions raised in [2] concerning the maximum order and size of the members of $\mathcal{R}(2K_2, tK_2)$.

All notation and terminology not specifically mentioned will follow that in [3].

2 Factor-critical graphs

Here we will present some properties of factor-critical graphs which will be used later in connection with $\mathcal{R}(2K_2, tK_2)$.

As usual, $V(G)$ and $E(G)$ denote the vertex-set and the edge-set of a graph G. A graph G is said to be a *factor-critical* graph if $G-v$ contains a perfect matching for every $v \in V(G)$. Thus, we obtain

Property 1. $|V(G)| = 2\beta_1(G) + 1$ for any factor-critical graph G, where $\beta_1(G)$ denotes the edge independence number of G. Moreover, $\beta_1(G-v) = \beta_1(G)$ for every $v \in V(G)$.

The following properties 2 - 6 can be found in [4], pp. 196 - 204.

Property 2. A factor-critical graph G of order at least three can be represented as $G = P^{(0)} + P^{(1)} + \ldots + P^{(l)}$ where $P^{(0)}$ is an odd cycle and, for $j = 0, \ldots, l-1$, $P^{(j+1)}$ is either a path of odd length with both end-vertices but no internal vertex in $G_j = P^{(0)} + P^{(1)} + \ldots + P^{(j)}$ or an odd cycle having exactly one vertex in common with G_j. This representation is called an ***ear decomposition*** of G with ***ears*** $P^{(1)}, \ldots, P^{(l)}$. All ear decompositions of G must have the same number of ears, namely $l = |E(G)| - |V(G)|$. Moreover, it is easy to see that any graph permitting an ear decomposition is factor-critical.

Property 3. A factor-critical graph is connected and bridgeless.

Property 4. A 2-connected factor-critical graph G of order at least three has an ear decomposition $G = P^{(0)} + P^{(1)} + \ldots + P^{(l)}$, where $P^{(1)}, \ldots, P^{(l)}$ are paths of odd lengths. $G_j = P^{(0)} + P^{(1)} + \ldots + P^{(j)}$ is a 2-connected factor-critical graph for $j = 0, \ldots, l-1$.

A factor-critical graph G is said to be *minimal factor-critical* if $G - e$ is not factor-critical for every $e \in E(G)$.

Property 5. A graph is minimal factor-critical if and only if it is connected and each of is blocks is minimal factor-critical.

Property 6. A minimal factor-critical graph contains no subgraph C_4. Every subgraph K_3 of a minimal factor-critical graph G has to be a block of G.

From properties 4 and 6 we can deduce

Property 7. A 2-connected minimal factor-critical graph G of order at least three has an ear decomposition $G = P^{(0)} + P^{(1)} + \ldots + P^{(l)}$, where $P^{(1)}, \ldots, P^{(l)}$ are paths of odd lengths at least three. For $j = 0, \ldots, l-1$, the graph $G_j = P^{(0)} + P^{(1)} + \ldots + P^{(j)}$ is 2-connected and minimal factor-critical, and the end-vertices of $P^{(j+1)}$ are non-adjacent in G_j if $P^{(j+1)}$ has length three.

3 General results on $\mathcal{R}(2K_2, tK_2)$

It is easy to see that $\mathcal{R}(2K_2, K_2) = \{2K_2\}$ and $(t+1)K_2 \in \mathcal{R}(2K_2, tK_2)$. We define $\mathcal{R}'(2K_2, tK_2) = \mathcal{R}(2K_2, tK_2) \setminus \{(t+1)K_2\}$. First we will derive a simple but useful characterization of the members of $\mathcal{R}'(2K_2, tK_2)$.

Lemma 1. Let F be a graph and let $S_1, \ldots, S_k$ be the components of F. Then $F \in \mathcal{R}'(2K_2, tK_2)$ if and only if the following conditions hold for $1 \leq i \leq k$.

(i) $S_i \neq K_1$.

(ii) $\sum_{i=1}^{k} \beta_1(S_i) = t$.

(iii) $\beta_1(S_i - v) = \beta_1(S_i)$ for every $v \in V(S_i)$.

(iv) $\beta_1(S_i - E(K_3)) = \beta_1(S_i)$ for every $K_3 \subset S_i$.

(v) For every $e \in E(S_i)$ there exists a $v \in V(S_i)$ such that $\beta_1((S_i - e) - v) < \beta_1(S_i)$ or a subgraph $K_3 \subset S_i$ such that $\beta_1((S_i - e) - E(K_3)) < \beta_1(S_i)$.

Proof. Suppose first that $F \in \mathcal{R}'(2K_2, tK_2)$. Then the minimality of F implies (i). Using that $(t+1)K_2 \in \mathcal{R}(2K_2, tK_2)$ and the minimality of F we obtain that $\sum_{i=1}^{k} \beta_1(S_i) = \beta_1(F) \leq t$. Equality must hold since otherwise a coloring of the edges of F only with red would imply that $F \not\rightarrow (2K_2, tK_2)$. This proves (ii). Moreover, in any 2-coloring of F where the green subgraph is either a K_3 or a star there must be t independent red edges. This yields (iii) and (iv). The minimality of F implies that $F - e \not\to (2K_2, tK_2)$ for every $e \in E(F)$. This means that $F - e$ can be colored with green and red such that the green edges form either a star or a K_3 and at most $t-1$ independent red edges occur. This implies (v) because of (ii) - (iv). Similarly it can be seen that F belongs to $\mathcal{R}'(2K_2, tK_2)$ if (i) - (v) are fulfilled. ■

Next we will derive an additional much more restrictive property of the components of the members of $\mathcal{R}'(2K_2, tK_2)$.

Lemma 2. Any component S of a graph $F \in \mathcal{R}'(2K_2, tK_2)$ must be a K_3-free minimal factor-critical graph with $\beta_1(S) \geq 2$.

Proof. Lemma 1(iii) and Gallai's Lemma (see [4], p. 89) imply that S has to be factor-critical.

Suppose first that S contains a subgraph K_3. Then one of the following two cases must occur.

Case I: S contains a block K_3. Let $V(K_3) = \{v_1, v_2, v_3\}$, and let S_i, $1 \leq i \leq 3$, be the component of $S - E(K_3)$ containing v_i. Since S is factor-critical, we can find a perfect matching in $S - v_i$, and this must contain a perfect matching of $S_i - v_i$ for $1 \leq i \leq 3$. Thus, the number of vertices in S_i has to be odd. But this implies that $\beta_1(S - E(K_3)) < \beta_1(S)$ in contradiction to Lemma 1(iv).

Case II: S contains a subgraph K_3 but no block of S is a K_3. In view of property 6 of factor-critical graphs, S cannot be minimal factor-critical. Thus we can find a spanning minimal factor-critical proper subgraph S' of S. Note that S' has to contain a subgraph K_3: Otherwise replace S by S'. This yields a proper subgraph of F belonging to $\mathcal{R}'(2K_2, tK_2)$ by Lemma 1, a contradiction to the minimality of F. Choose now a minimal factor-critical spanning subgraph S' with minimum number of edges and, in addition, with minimum number of subgraphs K_3 among all spanning minimal factor-critical subgraphs with $|E(S')|$ edges. Consider a subgraph K_3 of S'. As mentioned above, it has to be a block of S'. Again let $V(K_3) = \{v_1, v_2, v_3\}$, and let S'_i, $1 \leq i \leq 3$, be the component of $S' - E(K_3)$ containing v_i. As in case I, it can be proved that the number of vertices in S'_i is odd for $1 \leq i \leq 3$. This implies a perfect matching in $S'_i - w$ for every $w \in V(S'_i)$. Since the $K_3 = [v_1, v_2, v_3]$ is not a block in S, we can find an edge $uv \in E(S) \setminus E(S')$ with $u \in S'_i$ and $v \in S'_j$ where $i \neq j$, say $u \in S'_1$ and $v \in S'_2$. Delete the edge $v_1 v_2$ from S' and add the edge uv. Let S'' be the resulting spanning subgraph of S. Then $S'' - w$ has a perfect matching for every $w \in V(S'')$: If $w \in V(S'_3)$, take perfect matchings of $S'_3 - w$, $S'_1 - u$

and $S'_2 - v$ and add uv. If $w \in V(S'_1)$, take perfect matchings of $S'_1 - w$, $S'_2 - v_2$ and $S'_3 - v_3$ and add v_2v_3. The case that $w \in V(S'_2)$ is equivalent. Thus, S'' is factor-critical. Moreover, it has to be minimal factor-critical since $|E(S'')| = |E(S')|$. But S'' contains a smaller number of subgraphs K_3 than S', a contradiction to the choice of S'.

It remains that S is factor-critical and K_3-free. Then Lemma 1(v) and property 1 of factor-critical graphs imply that S is minimal factor-critical. Moreover, $\beta_1(S) \neq 0$ by Lemma 1(i), and $\beta_1(S) \neq 1$ since K_3 is the only factor-critical graph with edge independence number 1. This completes the proof of Lemma 2. ■

The following theorem characterizes the graphs in $\mathcal{R}'(2K_2, tK_2)$ using factor-critical graphs.

Theorem 1. Let $\mathcal{S}_n$ be the class of K_3-free minimal factor-critical graphs with edge independence number n. Then $F \in \mathcal{R}'(2K_2, tK_2)$ if and only if $F = \bigcup_{i=1}^{k} S_i$ with $k \geq 1$, $S_i \in \mathcal{S}_{t_i}$, $t_1, \ldots, t_k \geq 2$, $\sum_{i=1}^{k} t_i = t$ and $V(S_i) \cap V(S_j) = \emptyset$ if $i \neq j$.

Proof. Lemma 1 and Lemma 2 imply that every $F \in \mathcal{R}'(2K_2, tK_2)$ must have the structure given in Theorem 1. Furthermore, the connectivity of factor-critical graphs and Lemma 1 imply that every graph of this structure belongs to $\mathcal{R}'(2K_2, tK_2)$. ■

By Theorem 1, $\mathcal{R}'(2K_2, tK_2)$ is determined if $\mathcal{S}_n$ is known for $n = 2, \ldots, t$. The following lemma shows that the 2-connected graphs from $\mathcal{S}_2, \ldots, \mathcal{S}_n$ are essential for the construction of $\mathcal{S}_n$.

Lemma 3. Let S be a graph with blocks $B_1, \ldots, B_l$ and let $\mathcal{S}^*_m$ be the subclass of the 2-connected members of $\mathcal{S}_m$. Then S belongs to $\mathcal{S}_n$ if and only if it is connected and, for $i = 1, \ldots, l$, $B_i \in \mathcal{S}^*_{m_i}$ where $m_i \geq 2$ and $\sum_{i=1}^{l} m_i = n$.

Proof. Using properties 1 and 5 of factor-critical graphs and taking into account that $|V(S)| = 1 - l + \sum_{i=1}^{l} |V(B_i)|$ if S is connected, the assertion of the lemma is obtained. ■

In the proof of Theorem 3 we will describe a method to construct $\mathcal{S}^*_m$ from $\mathcal{S}^*_2, \ldots,$
$\mathcal{S}^*_{m-1}$. Thus, in view of Theorem 1 and Lemma 3, we will obtain a method to determine $\mathcal{R}'(2K_2, tK_2)$. Moreover, Lemma 3 produces a fairly large class of members of $\mathcal{R}'(2K_2, tK_2)$ if suitable odd cycles of lengths at least

five are taken as blocks (trivially, $C_{2m+1} \in S_m^*$). This class has been given already in [2]. It can be enlarged considerably by using the 2-connected minimal factor-critical graphs given in the following lemma instead of odd cycles.

Lemma 4. Let $m \geq 2$ and $\Pi_{2m-1} = \{(i_1, \ldots, i_k) : k \geq 2, i_1 \equiv 1 \pmod 2, i_1 \geq 1, i_2, \ldots, i_k \equiv 0 \pmod 2, 2 \leq i_2 \leq \ldots \leq i_k \text{ and } i_1 + \ldots + i_k = 2m-1\}$. Let $P_{i_1,\ldots,i_k}$ be the graph consisting of two vertices a and b joined by k internal-vertex-disjoint paths $P_{i_1+2}, \ldots, P_{i_k+2}$ and $\mathcal{P}_m = \{P_{i_1,\ldots,i_k} : (i_1, \ldots, i_k) \in \Pi_{2m-1}\}$. Then $\mathcal{P}_m \subset S_m^*$.

Proof. It is easy to see that every $G \in \mathcal{P}_m$ is 2-connected. Trivially, G permits an ear decomposition and property 2 from Section 2 implies G to be factor-critical. Moreover, $G - e$ contains a bridge for every $e \in E(G)$. Thus, property 3 of factor-critical graphs yields the minimality. ■

Obviously, $|\mathcal{P}_m| = 2 - m + \sum_{j=1}^{m-1} p(j)$, where $p(j)$ denotes the number of unordered partitions of j into natural numbers (note that $P_{i_1,i_2} = C_{2m+1}$ for every $(i_1, i_2) \in \Pi_{2m-1}$). Because of $p(j) \sim e^{\pi\sqrt{2j/3}}/(4j\sqrt{3})$ Lemma 4 describes a large class of 2-connected minimal factor-critical graphs for m large.

Next we will answer the questions concerning the number of vertices and edges of a graph $F \in \mathcal{R}(2K_2, tK_2)$.

Lemma 5. Let $t \geq 2$, $F \in \mathcal{R}'(2K_2, tK_2)$ and let $\omega(F)$ denote the number of components of F. Then

$$|V(F)| = 2t + \omega(F),\ 1 \leq \omega(F) \leq \lfloor t/2 \rfloor,\ 2t + \omega(F) \leq |E(F)| \leq 3t - \omega(F).$$

Proof. Let $S_1, \ldots, S_k$, $k = \omega(F)$, be the components of F. Theorem 1 and property 1 of factor-critical graphs yield that $|V(S_i)| = 2\beta_1(S_i) + 1$, $\beta_1(S_i) \geq 2$ and $\sum_{i=1}^k \beta_1(S_i) = t$. This implies $|V(F)| = \sum_{i=1}^k |V(S_i)| = 2t + k$ and $k \leq \lfloor t/2 \rfloor$.

To prove the bounds on $|E(F)|$ we will make use of property 2 of factor-critical graphs. Thus, for $i = 1, \ldots, k$, $|E(S_i)| = |V(S_i)| + l_i$ where l_i denotes the number of ears in an ear decomposition of S_i. The lower bound on $|E(F)|$ follows immediately. To obtain the upper bound, note that the minimality of S_i implies that every ear has to contain at least two internal vertices. Moreover, no subgraph K_3 in S_i forces the odd cycle $P^{(0)}$ to have length at least five. This gives $l_i \leq (|V(S_i)| - 5)/2$ implying the desired upper bound on $|E(F)|$. ■

Theorem 2. Let $p_{\max}(t)$ be the maximum order and let $q_{\max}(t)$ be the maximum size of a graph $F \in \mathcal{R}(2K_2, tK_2)$. Let the sets of graphs $F \in \mathcal{R}(2K_2, tK_2)$ of order $p_{\max}(t)$ and size $q_{\max}(t)$ respectively be denoted by $\mathcal{F}_{p_{\max}}(t)$ and $\mathcal{F}_{q_{\max}}(t)$. For $t \geq 4$ put $\mathcal{F}'(t) = \{\frac{t}{2}C_5\}$ for t even and $\mathcal{F}'(t) = \{\frac{t-3}{2}C_5 \cup C_7, \frac{t-3}{2}C_5 \cup P_{1,2,2}\}$ for t odd. Furthermore, let $P_{1,(n-1)\times 2}$ denote the graph $P_{i_1,\ldots,i_n}$ where $i_1 = 1$ and $i_2 = \ldots = i_n = 2$. Then

$$p_{\max}(t) = \begin{cases} 2t+2 & \text{if } 1 \leq t \leq 3, \\ \lfloor 5t/2 \rfloor & \text{if } t \geq 4, \end{cases}$$

$$\mathcal{F}_{p_{\max}}(t) = \begin{cases} \{(t+1)K_2\} & \text{if } 1 \leq t \leq 3, \\ \{(t+1)K_2\} \cup \mathcal{F}'(t) & \text{if } 4 \leq t \leq 5, \\ \mathcal{F}'(t) & \text{if } t \geq 6, \end{cases}$$

$$q_{\max}(t) = 3t - 1 \text{ for } t \geq 1,$$

$$\mathcal{F}_{q_{\max}}(t) = \begin{cases} \{2K_2\} & \text{if } t = 1, \\ \{P_{1,(t-1)\times 2}\} & \text{if } t \geq 2. \end{cases}$$

Proof. The case $t = 1$ is trivial since $\mathcal{R}(2K_2, K_2) = \{2K_2\}$. In the following let $t \geq 2$.

Lemma 5 yields that $|V(F)| \leq \lfloor 5t/2 \rfloor$ for every $F \in \mathcal{R}'(2K_2, tK_2)$, and $\lfloor 5t/2 \rfloor$ vertices occur if and only if $\omega(F) = \lfloor t/2 \rfloor$. Theorem 1 implies that in case of t even $\omega(F) = \lfloor t/2 \rfloor$ is attained if and only if every component of F belongs to $\mathcal{S}_2$. Using the ear decomposition we see that $\mathcal{S}_2 = \{C_5\}$ yielding that $F = \frac{t}{2}C_5$. In case of t odd and $\omega(F) = \lfloor t/2 \rfloor$ one component of F must belong to $\mathcal{S}_3$ and all others to $\mathcal{S}_2$. Again using the ear decomposition we see that $\mathcal{S}_3 = \{C_7, P_{1,2,2}\}$ yielding that $F = \frac{t-3}{2}C_5 \cup C_7$ or $F = \frac{t-3}{2}C_5 \cup P_{1,2,2}$. Taking into account that $\mathcal{R}(2K_2, tK_2) = \mathcal{R}'(2K_2, tK_2) \cup \{(t+1)K_2\}$ we obtain the desired results on $p_{\max}(t)$ and $\mathcal{F}_{p_{\max}}(t)$.

Using Lemma 5, we see that $q_{\max}(t) \leq 3t - 1$ for $t \geq 2$. The upper bound is attained by the graph $P_{1,(t-1)\times 2}$ which belongs to $\mathcal{S}_t^* \subset \mathcal{R}(2K_2, tK_2)$ by Lemma 4. It remains to show that for $t \geq 2$ no further graph of size $3t - 1$ occurs in $\mathcal{R}(2K_2, tK_2)$.

Let $F \in \mathcal{R}(2K_2, tK_2)$ of size $3t - 1$ and $t \geq 2$. Then $F \in \mathcal{R}'(2K_2, tK_2)$ and Lemma 5 yields $\omega(F) = 1$. This implies that F is a minimal factor-critical K_3-free graph. We obtain that $|V(F)| = 2t + 1$ and $|E(F)| - |V(F)| = t - 2$. Thus, F must have an ear-decomposition $F = P^{(0)} + P^{(1)} + \ldots + P^{(t-2)}$ where $P^{(0)}$ is an odd cycle of length at least five. Moreover, the minimality of F implies that for $1 \leq j \leq t - 2$ the ear $P^{(j)}$ has length at least three. In view of $|V(F)| = 2t + 1$ we see that $P^{(0)} = C_5$ and that every ear has length exactly three.

To show that $F = P_{1,(t-1)\times 2}$ we apply induction on t. The assertion holds for $t = 2$ since $C_5 = P_{1,1\times 2}$. Now let $t \geq 3$ and suppose that $\mathcal{F}_{q_{max}}(t-1) = \{P_{1,(t-2)\times 2}\}$. Let $F = P^{(0)} + P^{(1)} + \ldots + P^{(t-2)}$ be an ear decomposition of F and $G = P^{(0)} + P^{(1)} + \ldots + P^{(t-3)}$. G has to be a factor-critical K_3-free graph of order $2(t-1)+1$ and size $3(t-1)-1$. In addition, the minimality of F implies the minimality of G. Thus, $G \in \mathcal{F}_{q_{max}}(t-1)$ by Theorem 1 and $G = P_{1,(t-2)\times 2}$ by the induction hypothesis. Moreover, F can be obtained from G by adding a path of length 3 as an ear. It can be checked that (up to isomorphism) $P_{1,(t-1)\times 2}$ is the only minimal factor-critical graph obtained in this way. This completes the proof of Theorem 2. ∎

4 The sets $\mathcal{R}(2K_2, tK_2)$ for $t \leq 5$

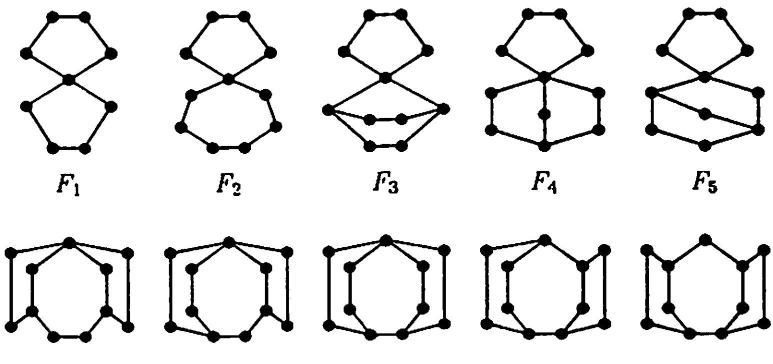
Here we will use the results from Section 3 to determine $\mathcal{R}(2K_2, tK_2)$ explicitly for some small t.

Theorem 3. Let $F_1, \ldots, F_{10}$ be the graphs given in Figure 1 and let $\mathcal{P}_m$ be defined as in Lemma 4, i.e., $\mathcal{P}_2 = \{C_5\}$, $\mathcal{P}_3 = \{C_7, P_{1,2,2}\}$, $\mathcal{P}_4 = \{C_9, P_{1,2,4}, P_{1,2,2,2}, P_{3,2,2}\}$, and $\mathcal{P}_5 = \{C_{11}, P_{1,2,6}, P_{1,2,2,4}, P_{1,2,2,2,2}, P_{1,4,4}, P_{3,2,4}, P_{3,2,2,2}, P_{5,2,2}\}$. Then

$$\mathcal{R}(2K_2, tK_2) = \begin{cases} \{2K_2\} & \text{if } t = 1, \\ \mathcal{P}_2 \cup \{3K_2\} & \text{if } t = 2, \\ \mathcal{P}_3 \cup \{4K_2\} & \text{if } t = 3, \\ \mathcal{P}_4 \cup \{5K_2, 2C_5, F_1\} & \text{if } t = 4, \\ \mathcal{P}_5 \cup \{6K_2, C_5 \cup C_7, C_5 \cup P_{1,2,2}, F_2, \ldots, F_{10}\} & \text{if } t = 5. \end{cases}$$

Proof. We know already that $(t+1)K_2 \in \mathcal{R}(2K_2, tK_2)$ and $\mathcal{R}(2K_2, K_2) = \{2K_2\}$. It remains to determine $\mathcal{R}'(2K_2, tK_2) = \mathcal{R}(2K_2, tK_2) \setminus \{(t+1)K_2\}$ for $t \geq 2$. This can be done as follows: First construct the sets $\mathcal{S}_2^*, \ldots, \mathcal{S}_t^*$, then $\mathcal{S}_2, \ldots, \mathcal{S}_t$ with Lemma 3, and then $\mathcal{R}'(2K_2, tK_2)$ with Theorem 1.

To construct $\mathcal{S}_2^*, \ldots, \mathcal{S}_t^*$ we can make use of property 7 of factor-critical graphs. It implies that any 2-connected minimal factor-critical graph G is either an odd cycle or can be obtained by taking a suitable 2-connected minimal factor-critical graph G' of order at least five and adding a path P of odd length at least three such that $V(P) \cap V(G') = \{a, b\}$, where a and b are the end-vertices of P. Moreover, the vertices a and b must be non-adjacent in G' if P has length three.



F_1 F_2 F_3 F_4 F_5

F_6 F_7 F_8 F_9 F_{10}

FIGURE 1

Thus, $\mathcal{S}_2^* = \{C_5\}$, and if $\mathcal{S}_2^*, \ldots, \mathcal{S}_{m-1}^*$ are constructed, then $\mathcal{S}_m^*$ with $m \geq 3$ can be obtained as follows: Take, for $k = 2, \ldots, m-1$, the members of $\mathcal{S}_k^*$ and add, as described above, a path of length $2m - 2k + 1$ in all possible ways. Then the resulting graphs are all 2-connected, factor-critical (since they permit an ear decomposition) and K_3–free. Hence, $\mathcal{S}_m^*$ consists of C_{2m+1} and those (nonisomorphic) of the obtained graphs which are, in addition, minimal factor-critical.

This procedure yields $\mathcal{S}_m^* = \mathcal{P}_m$ for $2 \leq m \leq 4$ and $\mathcal{S}_5^* = \mathcal{P}_5 \cup \{F_6, \ldots, F_{10}\}$. (For these m, the minimality or non-minimality of the graphs constructed to determine $\mathcal{S}_m^*$ is easy to check: Those graphs yielding a graph containing a bridge after deletion of any edge are minimal in view of property 3 of factor-critical graphs. Additionally, the graph F_8 is minimal. All remaining graphs contain one of these graphs or one of the (factor-critical) graphs F_1, C_9 in case of $m = 4$ and F_2, F_4, C_{11} in case of $m = 5$ as a proper spanning subgraph and are non-minimal.) With these $\mathcal{S}_m^*$, the desired sets $\mathcal{R}'(2K_2, tK_2)$ can be obtained for $2 \leq t \leq 5$ using Lemma 3 and Theorem 1. ■

Using the same method, $\mathcal{R}(2K_2, tK_2)$ could be determined explicitly for other small t. Of course it would be more interesting to solve the problem of characterizing the graphs in $\mathcal{R}(sK_2, tK_2)$ for $s, t \geq 3$. But this seems to be very difficult.

References

[1] S.A. Burr, P. Erdös, R. J. Faudree, and R. H. Schelp, A class of Ramsey-finite graphs, *Congr. Numer.* **21** (1978), 171 - 180.

[2] S.A. Burr, P. Erdös, R. J. Faudree, C. C. Rousseau and R. H. Schelp, Ramsey minimal graphs for matchings, in: *Theory and Applications of Graphs*, (ed. G. Chartrand), Wiley, New York, 1981, 159 - 168.

[3] F. Harary, *Graph Theory*, Addison-Wesley, Reading, Mass., 1969.

[4] L. Lovász and M. D. Plummer, *Matching Theory*, North Holland, Amsterdam, 1986.