Further results on almost Moore digraphs

Edy Tri BASKORO
Department of Mathematics,
Institut Teknologi Bandung, Ganesa 10 Bandung Indonesia
e-mail: p4mitb@bdg.centrin.net.id

Mirka MILLER
Department of Computer Science,
The University of Newcastle NSW 2308 Australia,
e-mail: mirka@cs.newcastle.edu.au

Ján PLESNÍK
Department of Numerical and Optimization Methods,
Faculty of Mathematics and Physics,
Comenius University, 842 15 Bratislava, Slovak Republic,
e-mail: plesnik@fmph.uniba.sk

Abstract

The nonexistence of digraphs with order equal to the Moore bound $M_{d,k} = 1+d+\ldots+d^k$ for $d, k > 1$ has lead to the study of the problem of the existence of 'almost' Moore digraphs, namely digraphs with order close to the Moore bound. In [1], it was shown that almost Moore digraphs of order $M_{d,k} - 1$, degree d, diameter k $(d, k \geq 3)$ contain either no cycle of length k or exactly one such cycle. In this paper we shall derive some further necessary conditions for the existence of almost Moore digraphs for degree and diameter greater than 1. As a consequence, for diameter $k = 2$ and degree d, $2 \leq d \leq 12$, we show that there are no almost Moore digraphs of order $M_{d,2} - 1$ with one vertex in a 2-cycle C_2 except the digraphs with every vertex in C_2.

ARS COMBINATORIA 56(2000), pp. 43-63

1 Introduction

By a *digraph* we mean a structure $G = (V, A)$ where $V(G)$ is a nonempty set of distinct elements called *vertices*; and $A(G)$ is a set of ordered pairs (u, v) of distinct vertices $u, v \in V$ called *arcs*.

A digraph H is a *subdigraph* of G if $V(H) \subset V(G)$ and $A(H) \subset A(G)$. Let $V' \subset V(G)$. The subdigraph of G whose vertex set is V' and whose arc set is the set of all those arcs of G which have ends in V' is called the *induced subdigraph* of G by V' and is denoted by $G[V']$.

The *order* of a digraph G is the number of vertices in G, i.e., $|V(G)|$. An *in-neighbour* of a vertex v in a digraph G is a vertex u such that $(u, v) \in G$. Similarly, an *out-neighbour* of a vertex v is a vertex w such that $(v, w) \in A(G)$. For $S \subset V(G)$ denote by $N^-(S)$ (respectively $N^+(S)$) the set of all in-neighbours (respectively out-neighbours) of elements of S. The *in-degree* (respectively *out-degree*) of a vertex $v \in V(G)$ is the number of its in-neighbours (respectively out-neighbours) in G. If in a digraph G, the in-degree equals the out-degree ($= d$) for every vertex, then G is called a *diregular* digraph of degree d.

A *walk* W of length k in G is an alternating sequence $(v_0 a_1 v_1 a_2 ... a_k v_k)$ of vertices and arcs in G such that $a_i = (v_{i-1}, v_i)$ for each i. If only the endpoints v_0, v_k of a walk are known we use the shorter notation *walk* v_0–v_k to denote a walk from v_0 to v_k. Vertices other than v_0 and v_k are called *internal vertices*. A *closed* walk has $v_0 = v_k$. If the arcs $a_1, a_2, ..., a_k$ of a walk W are distinct, W is called a *trail*. If, in addition, the vertices $v_0, v_1, ..., v_k$ are also distinct, W is called a *path*. A *cycle* C_k of length k is a closed trail of length $k > 0$ with all vertices distinct (except the first and the last).

The *distance* from vertex u to vertex v in G, denoted by $\delta(u, v)$, is defined as the length of the shortest path from vertex u to vertex v. Note that in general $\delta(u,v)$ is not necessarily equal to $\delta(v,u)$. The *diameter* k of a digraph G is the maximum distance between any two vertices in G.

Let one vertex be distinguished in a digraph of maximum out-degree d and diameter k, having n vertices. Let n_i, $i = 0, 1, ..., k$ be the number of vertices at distance i from the distinguished vertex. Then,

$$n_i \leq d^i \qquad \text{for } i = 0, 1, ..., k. \tag{1}$$

Hence,

$$n = \sum_{i=0}^{k} n_i \leq 1 + d + d^2 + ... + d^k. \tag{2}$$

If the equality sign holds in (2) then such a digraph is called a *Moore digraph*. The right-hand side of (2) is called the *Moore bound* $M_{d,k}$.

Digraphs with the maximum possible number of vertices are required in the construction of optimal networks [6],[9],[12],[17]. It is well known that except for trivial cases (for $d = 1$ or $k = 1$) Moore digraphs do not exist (see [16] or [5]). The trivial cases are the cycles C_{k+1} of length $k+1$ and the complete (symmetric) digraphs K_{d+1} on $d+1$ vertices.

Since Moore digraphs do not exist for $k \neq 1$ and $d \neq 1$, the problem of the existence of almost Moore digraphs, i.e., digraphs of diameter $k \geq 2$, maximum out-degree $d \geq 2$ and the number of vertices $M_{d,k} - \Delta_{d,k}$ ($\Delta_{d,k}$ is called the *defect*) becomes an interesting problem. If $\Delta_{d,k} < M_{d,k-1}$ then such an almost Moore digraph G is always regular of out-degree d [10]. This can be easily seen since if there exists a vertex $v \in G$ with out-degree $d_1 < d$, then the order of G must satisfy:

$$n \leq 1 + d_1 + d_1 d + ... + d_1 d^{k-1} = 1 + d_1(1 + d + ... + d^{k-1})$$

$$= M_{d,k} - (d - d_1)(1 + d + ... + d^{k-1}) \leq M_{d,k} - M_{d,k-1} \tag{3}$$

$$< M_{d,k} - \Delta_{d,k}$$

which is impossible.

Next, let d be the maximum in-degree in the digraph and suppose there exists a vertex $v \in G$ with in-degree $d_1 < d$. Then, once again, the order of G must satisfy (3). Thus, if $\Delta_{d,k} < M_{d,k-1}$ then G is diregular of degree d. Consequently, we need consider only diregular digraphs of degree d. The first result in this problem was due to [8] which showed that almost Moore digraphs of diameter 2, degree d and $\Delta_{d,k} = 1$ do exist; interestingly, one such digraph is the line digraph of K_{d+1}. In particular, for degree 2 there are exactly three non-isomorphic digraphs of order $M_{d,k} - 1$ [13] (see Figure 1) while for degree 3 there is exactly one such digraph, i.e., the line digraph of K_4 [3]. The problem of finding all non-isomorphic almost Moore digraphs of order $M_{d,k} - 1$ is rather difficult for an arbitrary degree greater than 3. Nevertheless, a number of necessary conditions for the existence of such digraphs was given in [1],[2],[3].

In [14], Miller and Fris proved that almost Moore digraphs of diameter $k \geq 3$ and $\Delta_{d,k} = 1$ do not exist for degree 2. Subsequently we proved that for degree 3 such digraphs do not exist if k is odd or $k+1$ does not divide $\frac{9}{2}(3^k - 1)$ [2]. For $\Delta_{d,k} \geq 1$, the only result known to us is that almost Moore digraphs of degree 2, diameter $k \geq 3$ and $\Delta_{d,k} = 2$ do not exist for most values of k [15]. For other recent related results see also [7], [9] and [18].

From now on we shall consider only almost Moore digraphs of degree $d \geq 2$, diameter $k \geq 2$ and $\Delta_{d,k} = 1$. Such digraphs will be called (d,k)-*digraphs*. Every (d,k)-digraph G has the characteristic property that for every vertex $v \in G$ there is a unique vertex $y \in G$ such that there are two walks of lengths $\leq k$ from v to y in G [2]. Such a vertex y is called the *repeat* of x, denoted by $r(x)$. If $r(x) = y$ then $r^{-1}(y) = x$. In the case of $r(x) = x$, x is called a *selfrepeat* of G. For $S \subset V(G)$ we define $r(S) = \bigcup_{v \in S} r(v)$ and similarly, $r^{-1}(S) = \bigcup_{v \in S} r^{-1}(v)$. The function r can be considered as a permutation on the vertex set of G. Figure 1 illustrates the notion of repeat for the three existing $(2,2)$-digraphs [3]. Each permutation is expressed as a set of permutation cycles. The cycle $(v_1 v_2 v_3 \cdots v_t)$ is a *permutation cycle of length t* in a (d,k)-digraph if $r(v_1) = v_2, r(v_2) = v_3, \cdots, r(v_t) = v_1$. Note that the digraph of Figure 1(b) was already considered in [10].

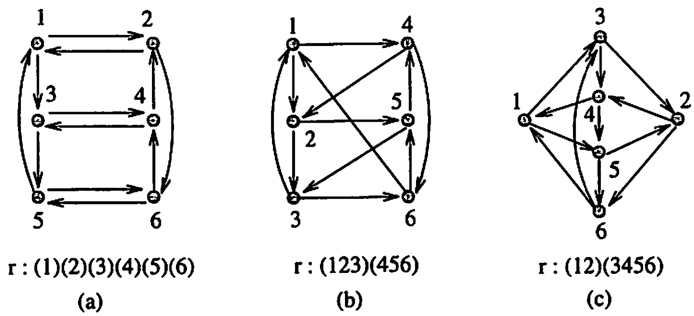

Figure 1: The permutation cycles of the three $(2,2)$-digraphs

We study the existence of (d,k)-digraphs and in the case that they do exist, we would like to know something about their structures. However, in general to decide whether such a digraph exists or not is very difficult. We may divide (d,k)-digraphs into two classes according to whether or not they contain selfrepeat vertices. For example, in Figure 1 the first $(2,2)$-digraph has every vertex a selfrepeat while the other two $(2,2)$-digraphs are without selfrepeats.

In [1], we have shown a necessary condition for the existence of a (d,k)-digraph for $k \geq 3$: such digraphs contain either no selfrepeat or exactly k selfrepeats. In this paper we derive some further necessary conditions for the existence of (d,k)-digraphs of general degree and diameter greater than 1. In Section 2, we show that the existence of a (d,k)-digraphs with a selfrepeat requires the existence of one or more (d',k)-digraphs with $1 \leq d' \leq d$. The interesting cases are those with one or more d' such that

$d' \neq 1, d$. In such a case we are often able to make use of known results and further necessary conditions for the (smaller) (d', k)-subdigraph. To illustrate this approach, we show that a $(d, 2)$-digraph contains either all selfrepeats or none for small degrees up to 12 (Section 3).

2 Further necessary conditions

In [1] we gave a necessary condition for the existence of a (d, k)-digraph G for $k \geq 3$, namely, that G may contain at most one k-cycle. In this section a further necessary condition will be presented. To do this, let us begin with several definitions and some previous results which will be used to derive further necessary conditions.

As in [1], iterated repeats and neighbourhoods are defined in the following way.

For $S \subset V(G)$

$$\begin{aligned} r^p(S) &= r(r^{p-1}(S)) && \text{if } p \geq 0, \\ r^p(S) &= r^{-1}(r^{p+1}(S)) && \text{if } p < 0. \\ N^p(S) &= \textstyle\bigcup_{v \in S} N^p(v) \end{aligned}$$

where $r^0(S) = S$ and $r^1(S) = r(S)$; $N^1 = N^+$ and $N^{-1} = N^-$. The following theorem was proved in [1].

Theorem A *Given a (d, k)-digraph G and any integers p, q:*
(a) (commutability) $S \subset V(G) : r^p(N^q(S)) = N^q(r^p(S))$;
(b) (automorphism) $u, v \in V(G) : (u, v) \in A(G) \iff (r^p(u), r^p(v)) \in A(G)$.

For every vertex v of a (d, k)-digraph there exists a smallest natural number $\omega(v)$ called the *order* of v, such that $r^{\omega(v)}(v) = v$ (that is, $\omega(v)$ is the length of the permutation cycle containing v). For instance, in the third $(2, 2)$-digraph of Figure 1, the order of vertex 1 is 2 and the order of vertex 5 is 4, i.e., $\omega(1) = 2$ and $\omega(5) = 4$.

We say that a sequence of positive integers $a_1, a_2, ..., a_q$ is *monotonically divisible* if it is monotonic and for any two elements a_i, a_j of the sequence a_i divides a_j or a_j divides a_i. The following three results have been proved in [1].

Lemma A (monotonic divisibility). *Let $(v_0, v_1, ..., v_p)$ be a walk of length $p \leq k$ in a (d,k)-digraph G. If v_0 or v_p is a selfrepeat then the sequence of orders $\omega(v_0), \omega(v_1), ..., \omega(v_p)$ is monotonically divisible.*

Lemma B *The permutation $r(N^+(v))$ has the same cycle structure for every selfrepeat v of a (d,k)-digraph.*

Lemma C *For any selfrepeat v of a (d,k)-digraph G, the permutation $r(N^+(v))$ has the same cycle structure as the permutation $r(N^-(v))$.*

Lemma 1 *Let G be a (d,k)-digraph with a selfrepeat, $d,k \geq 2$. Let α be an order of some vertex in G. Let $V_\alpha = \{ x \in V(G) \mid r^\alpha(x) = x \}$. Then each vertex of V_α has the same number of out-neigbours which are in V_α.*

Proof For $d = 2$, the only $(2,k)$-digraph with a selfrepeat is the one with $k = 2$, i.e., the line digraph of K_3. Such a digraph has selfrepeat vertices only and so $\alpha = 1$ and $V_1 = V(G)$. From now on we assume that $d \geq 3$. It is obvious that all selfrepeat vertices of G are in V_α.
If $\alpha = 1$, by [1, Theorem 3] the induced subdigraph $G[V_1]$ is either a k-cycle or a (d_1,k)-digraph with $2 \leq d_1 \leq d$. Then each vertex of V_1 has the same number of out-neigbours which are in V_1.
Assume $\alpha \neq 1$. By Lemma B, each selfrepeat of G has the same number of out-neighbours which are in V_α. Let $v \in V_\alpha$ and $r(v) \neq v$. Let y and z be two selfrepeats which lie in the same cycle of length k in G. Then we can not have both $v \in N^-(y)$ and $v \in N^-(z)$, since otherwise v has more than one repeat, namely y and z. We can assume $v \notin N^-(z)$. Since

the permutation cycle structures of $r(N^+(z))$ and $r(N^-(z))$ are the same (Lemma C), then to finish the proof we shall show that both $N^+(v)$ and $N^-(z)$ contain the same number of vertices of V_α.

Let $N^-(z) = \{x, x_1, ..., x_{d-1}\}$. Since $v \notin N^-(z)$ then $r(v) \notin N^-(z)$. Thus to reach all vertices of $N^-(z)$ from v there exists a system of d internally disjoint walks from v to $N^-(z)$ (one walk $(v, s, ..., x)$ of length at most $k-1$ and $(d-1)$ walks of form $(v, s_i, ..., x_i)$ of length k). By Lemma A, x together with s must be in V_α. Now, consider one of the walks $(v, s_i, ..., x_i, z)$ of length $k+1$. If $x_i \notin V_\alpha$ then we claim that $s_i \notin V_\alpha$. Assume $s_i \in V_\alpha$. This means $r^\alpha(s_i) = s_i$. Thus, we have two walks $(s_i, ..., x_i, z)$ and $(s_i = r^\alpha(s_i), ..., r^\alpha(x_i), r^\alpha(z) = z)$ of lengths less than or equal k. Therefore, $r(s_i) = z$ which contradicts z being a selfrepeat. If $x_i \in V_\alpha$ then we shall show that $s_i \in V_\alpha$. Assume $s_i \notin V_\alpha$. This means $r^\alpha(s_i) \neq s_i$. Thus, we have two walks $(v, s_i, ..., x_i)$ and $(v = r^\alpha(v), r^\alpha(s_i), ... r^\alpha(x_i) = x_i)$ of lengths less than or equals k. Therefore, $r(v) = x_i \in N^-(z)$ which contradicts the fact that $v \notin N^-(z)$ and z is a selfrepeat. Therefore both $N^+(v)$ and $N^-(z)$ contain the same number of vertices of V_α. □

Lemma 2 *Let G be a (d,k)-digraph with selfrepeats, $k \geq 2$. Let a be a selfrepeat and v be a vertex of G such that $\delta(a, v) \leq k-1$. If $x \in N^-(a)$ and $(v, v_1, ..., x)$ is a walk of length $\leq k$ from v to x, then* $\omega(v_1) = \mathrm{lcm}(\omega(v), \omega(x))$.

Proof Since $\delta(a, v) \leq k-1$ we have a walk $(a, ..., v, v_1)$ of length $\leq k$. By Lemma A, $\omega(v) | \omega(v_1)$. On the other hand, by using the same argument on walk $(v_1, ..., x, a)$ we get $\omega(x) | \omega(v_1)$. Let $p = \mathrm{lcm}(\omega(v), \omega(x))$. Assume that $\omega(v_1) \neq p$. Then $r(v) = x$ as there are two walks $(v, v_1, ..., x)$ and $(v = r^p(v), r^p(v_1), ..., r^p(x) = x)$ of lengths $\leq k$. As a is a selfrepeat and $x \in N^-(a)$, then by Theorem A there must be an arc (v, a) in G. Therefore,

$(a, \ldots, v, a)$ form a cycle of length $\leq k$. This implies that v is a selfrepeat which is a contradiction with $r(v) = x$. Therefore, $\omega(v_1) = \text{lcm}(\omega(v), \omega(x))$. □

Corollary 1 *Let G be a (d,k)-digraph with selfrepeats, $k \geq 2$. Let a be a selfrepeat and v be a vertex of G such that $\delta(a,v) \leq k-1$. Then the multiset of orders of $N^+(v)$ is equal to $\{\, \text{lcm}(\omega(v), \omega(x)) \mid \forall x \in N^-(a) \,\}$.*

Theorem 1 *Let G be a (d,k)-digraph with selfrepeats, $k \geq 2$. Let z be a selfrepeat of G. Let α be the order of some vertex in G. Then the induced subdigraph $G[V_\alpha]$ is either a cycle of length k or a (d',k)-digraph, where $V_\alpha = \{x \in V(G) \mid r^\alpha(x) = x\}$ and $d' = |\, \{v \in N^+(z) \mid r^\alpha(v) = v\} \,|$.*

Proof By [1, Theorem 3] we can assume that $\alpha \geq 2$. Consider the induced subdigraph $G[V_\alpha]$. $G[V_\alpha]$ has the same out-degree d' for every vertex guaranteed by Lemma 1. Let x be a vertex of $G[V_\alpha]$. Consider all the vertices of $G[V_\alpha]$ at distance $\leq k$ from x as depicted in Figure 2.

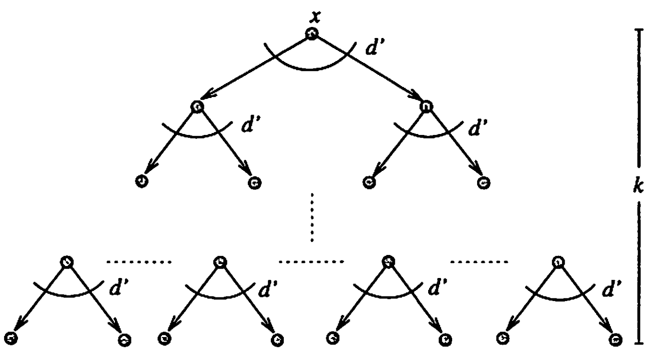

Figure 2: All the vertices at distance $\leq k$ from vertex x.

We shall show that $G[V_\alpha]$ has order $M_{d',k} - 1$, diameter k and the same in-degree d' for every vertex. Take $x = z$ which is a selfrepeat of $G[V_\alpha]$.

Then, by Lemma A, every vertex of $G[V_\alpha]$ can be reached from z by a path of length $\leq k$ involving vertices of $G[V_\alpha]$ only. Since z is a selfrepeat then one of the vertices in the bottom level of Figure 2 must be z and the remaining vertices must be all distinct. Therefore the order of $G[V_\alpha]$ is $M_{d',k} - 1$. It is obvious that the diameter of $G[V_\alpha]$ is at least k. From the above reasoning we note that all the vertices of $G[V_\alpha]$ are reachable from any selfrepeat by using paths of lengths $\leq k$ in $G[V_\alpha]$. Now, consider x which is not a selfrepeat of $G[V_\alpha]$. Since the order of $G[V_\alpha]$ is $M_{d',k} - 1$ then among the vertices of distance $\leq k$ from x there are exactly two vertices which are the same (otherwise x has more than one repeat). This means that all vertices of $G[V_\alpha]$ are reachable within k steps from x. Therefore the diameter of $G[V_\alpha]$ is exactly k. Moreover, this implies that the in-degree of every vertex of $G[V_\alpha]$ must be also d'. □

In this section, we have shown a necessary condition for the existence of (d,k)-digraphs with selfrepeats, $k \geq 2$. This condition is quite significant and useful for constructions since it gives us an insight to the structures of such digraphs (if they exist).

To conclude this section, we list some open problems:

Problem 1 *Characterize $(d,2)$-digraphs with all selfrepeat vertices for a given degree d.*

We note that for $k \geq 3$ there are no (d,k)-digraphs with every vertex a selfrepeat [4], [2].

Problem 2 *Give necessary (and sufficient) conditions for the existence of (d,k)-digraphs without selfrepeats, $k \geq 2$.*

3 Diameter 2

In this section we shall apply the results of Section 2 to $(d,2)$-digraphs for $d \geq 2$, i.e., digraphs of degree d, diameter 2 and $d+d^2$ vertices. The line digraph of the complete digraph K_{d+1} is one example of $(d,2)$-digraphs. (It is worthwhile to notice that each such line digraph consists of all selfrepeats.)

There are two significant results from [3] regarding the structure of the permutation cycles of $(d,2)$-digraphs. Since those results will be used frequently in this section, we state them in the following theorems. Let m_l denote the number of permutation cycles of length l $(l = 1,2,...,n)$ in a $(d,2)$-digraph.

Theorem B *For the numbers m_l $(l \geq 2)$ of the permutation cycles of even length of a $(d,2)$-digraph, $\sum_{l\ even} m_l$ is even.*

Theorem C *For the numbers m_l of the permutation cycles of length l, $l = 1,2,...,n$, of a $(d,2)$-digraph there are nonnegative integers u and v_{lq} fulfilling the following equalities for $q = 1,...,\lfloor\frac{l-1}{2}\rfloor$.*

$$d - u + \sum_{\substack{l\ odd\\ l\geq 3}} \sum_{q=1}^{(l-1)/2} [-2(m_l - v_{lq}) + 2(2v_{lq} - m_l)re\{x(l,q)\}]$$
$$+ \sum_{l\ even} \sum_{q=1}^{\frac{1}{2}l-1} [-2(m_l - v_{lq}) + 2(2v_{lq} - m_l)re\{x(l,q)\}] - \frac{1}{2}\sum_{l\ even} m_l = 0 \quad (4)$$

$$d^2 + u + \sum_{\substack{l\ odd\\ l\geq 3}} \left[-m_l + \sum_{q=1}^{(l-1)/2} (-2v_{lq} + 2(m_l - 2v_{lq})re\{x(l,q)\})\right]$$
$$+ \sum_{l\ even} \sum_{q=1}^{\frac{1}{2}l-1} [-2v_{lq} + 2(m_l - 2v_{lq})re\{x(l,q)\}] - \frac{3}{2}\sum_{l\ even} m_l = m_1 \quad (5)$$

where $c_{l_q} = \cos\frac{2\pi q}{l}$ *and* $re\{x(l,q)\} = -\frac{1}{2}+\frac{1}{2\sqrt{2}}\left[\sqrt{\sqrt{25-24c_{l_q}}+4c_{l_q}-3}\,\right]$.

Theorem 2 *For* $d \geq 2$ *there is no* $(d,2)$*-digraph with selfrepeats containing a vertex of order either 2 or 3.*

Proof Assume such a $(d,2)$-digraph G exists. Let $\alpha = 2$ or 3. By Theorem 1, the induced subdigraph $G[V_\alpha]$ is a $(d',2)$-digraph with $d' = d_\alpha + d_1$ and the number of vertices $n = (d')^2 + d'$, where d_α and d_1 are the number of vertices of $N^+(z)$ of orders α and 1 respectively for any selfrepeat z. Since the subdigraph $G[V_1]$ induced by the set of all selfrepeats has $d_1^2+d_1$ vertices, then the number of vertices of $G[V_\alpha]$ of order α is $(d')^2 + d' - (d_1^2 + d_1)$ $= (d' - d_1)(d' + d_1 + 1)$. Therefore, the number of permutation cycles of length α in $G[V_\alpha]$, $m_\alpha = \frac{1}{\alpha}(d' - d_1)(d' + d_1 + 1)$.

Let $\alpha = 2$. Since $G[V_2]$ exists, then by Theorem C there exists nonnegative integer u such that

$$d' - u - \frac{1}{2}m_2 = 0$$

$$\Longleftrightarrow \quad u = d' - \frac{1}{4}(d' - d_1)(d' + d_1 + 1)$$

$$\Longleftrightarrow \quad u = d' - \frac{1}{4}d_2(d_2 + 2d_1 + 1)$$

with $d_2 \geq 2$ and d_2 even. However, this is not possible since if $d_2 = 2$ then u is always a fraction and if $d_2 \geq 4$ then u is always negative. Therefore G can not contain a vertex of order 2.

Let $\alpha = 3$. By a similar argument, we have

$$d' - u - 2(m_3 - v_{3,1}) + 2(2v_{3,1} - m_3)\ re\{x(3,1)\} = 0.$$

But, $re\{x(3,1)\}$ is irrational, and so we must have $2v_{3,1} - m_3 = 0$. This implies

$$d' - u - m_3 = 0$$

$$\iff u = d' - \frac{1}{3}(d' - d_1)(d' + d_1 + 1)$$

$$\iff u = d' - \frac{1}{3}d_3(d' + d_1 + 1)$$

with $d_3 \geq 3$. Then u is always negative which is a contradiction. Therefore G can not contain a vertex of order 3. □

Lemma 3 *There is no $(d, 2)$-digraph with a selfrepeat v such that $r(N^+(v))$ contains odd number of permutation cycles of length α, where α is the smallest length of the cycles with order > 1, and α even.*

Proof Suppose such a digraph G exists. Consider the induced subdigraph $G[V_\alpha]$. In this subdigraph, the vertices are all of orders either 1 or α. From the proof of Theorem 2 we have the number of permutation cycles of length α, $m_\alpha = \frac{1}{\alpha}d_\alpha(d_\alpha + 2d_1 + 1)$. Since there is odd number of permutation cycles of length α in $N^+(v)$ then $\frac{1}{\alpha}d_\alpha$ is always odd. Therefore, m_α is odd too, which is not possible by Theorem B. □

Lemma 4 *Let G be a $(d, 2)$-digraph $(d \geq 2)$ which has only selfrepeat vertices. If x and y have the same $d-1$ out- (resp. in-) neighbours then $N^+(x) = N^+(y)$ (resp. $N^-(x) = N^-(y)$).*

Proof Suppose $N^+(x) = S \cup \{v_1\}$ and $N^+(y) = S \cup \{w_1\}$, $|S| = d - 1$. For a contradiction assume $v_1 \neq w_1$ (as depicted in Figure 3). Clearly, $v_1, w_1 \notin N^+(S)$. To reach v_1 and w_1 from y and x respectively we should have (v_1, w_1) and $(w_1, v_1) \in G$. Now it is impossible to reach other vertices of $N^+(w_1)$ from x. The proof for in-neighbours is similar. □

Lemma 5 *Let G be a $(d, 2)$-digraph $(d \geq 3)$ which has only selfrepeat vertices. If x and y have the same $d-2$ out- (resp. in-) neighbours then $N^+(x) = N^+(y)$ (resp. $N^-(x) = N^-(y)$).*

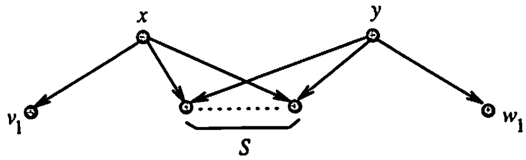

Figure 3: The out-neighbours of x and y differ in exactly one vertex.

Proof Suppose $N^+(x) = S \cup \{v_1, v_2\}$ and $N^+(y) = S \cup \{w_1, w_2\}$, $|S| = d-2$. For a contradiction assume $v_i \neq w_j$, $i, j \in \{1,2\}$ (as depicted in Figure 4). It is clear that v_1, v_2, w_1 and w_2 are not in $N^+(S)$. Without loss of generality we can assume $(v_1, w_1) \in G$ to have w_1 reachable from x. To have w_2 reachable from x we will show that (v_2, w_2) must be in G. Assume this is not the case. Then $(v_1, w_2) \in G$. Now, we will have a problem trying to reach all vertices of $N^+(w_1)$ and $N^+(w_2)$ from x. There are exactly $2(d-1)$ vertices of $N^+(w_1) \cup N^+(w_2)$ which differ from v_1 and v_2. We cannot reach them all through v_2, so there is at least one of them, say z, which is reachable from x through v_1. This leads to z being another repeat of v_1. Therefore (v_2, w_2) must be in G in order to reach w_2 from x (as in Figure 4). There is no arc from v_1 to $N^+(w_1)$, since otherwise there will be a second repeat of v_1. Hence all the remaining out-neighbours of v_1 must be from $N^+(w_2)$. This implies that v_1 and w_2 have only one different out-neighbour which is a contradiction by Lemma 4. The proof of the lemma for in-neighbourhouds is similar. □

Lemma 6 *Let G be a $(d,2)$-digraph ($d \geq 4$) which has only selfrepeat vertices. If x and y have the same $d-3$ out- (resp. in-) neighbours then $N^+(x) = N^+(y)$ (resp. $N^-(x) = N^-(y)$).*

Suppose $N^+(x) = S \cup \{v_1, v_2, v_3\}$ and $N^+(y) = S \cup \{w_1, w_2, w_3\}$, $|S| = d-3$. For a contradiction assume $v_i \neq w_j$, $i, j \in \{1,2,3\}$. If $(v_i, w_j) \in G$ for some fixed i and $j \in \{1,2,3\}$ then it is not possible to have $(v_i, w_k) \in G$ for $k \neq j$.

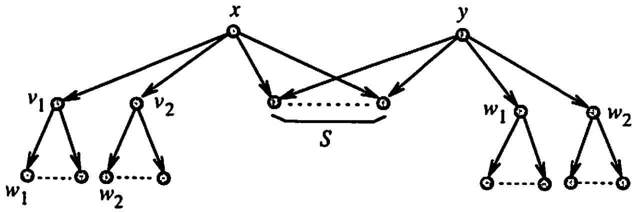

Figure 4: The out-neighbours of x and y differ in exactly two vertices.

This can be proved as follows. Since $d \geq 4$, let $p \in N^+(v_i)$, and $p \neq w_i$, $i = 1,2,3$. As (v_i, w_j) and (v_i, w_k) are in G then $p \notin N^+(w_j) \cup N^+(w_k)$. This means that $p \in N^+(w_l)$, where $l \neq j,k$. This implies that G contains two vertices v_i and w_l with exactly $d-2$ common out-neighbours, which is a contradiction by Lemma 5. Therefore, without loss of generality we assume: $(v_1, w_1), (v_2, w_2)$ and $(v_3, w_3) \in G$. Now, we shall show that arcs

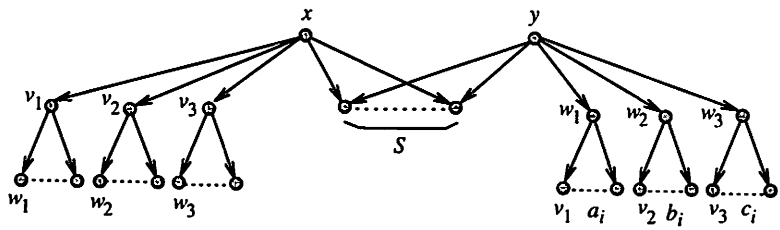

Figure 5: The out-neighbours of x and y differ in exactly three vertices.

(w_i, v_i), $i = 1,2,3$ are in G too. Since v_i, $i = 1,2,3$ must be reachable by y and each $v_i \notin N^+(S)$, it suffices to show that arc $(w_i, v_j) \notin G$ for $i \neq j$. Suppose $(w_i, v_j) \in G$ for $i \neq j$. Let $p \in N^+(v_j)$, $p \neq w_j$. It is obvious that $p \notin N^+(w_j)$. Moreover, $p \notin N^+(w_i)$ since otherwise p will be a second repeat of w_i. Therefore, $p \in N^+(w_k)$, $k \neq i,j$. This holds for each $p \in N^+(v_j)$, $p \neq w_j$. Hence we have two vertices v_j and w_k with exactly $d-1$ common out-neighbours, which is not possible by Lemma 4 (see Figure 5). Denote the remaining $(d-1)$ out-neighbours of w_1, w_2 and w_3 by a_i, b_i, and c_i $(i = 1, 2, \ldots d-1)$ respectively. By Lemmas 4 and 5 and bearing in mind that all vertices in G are selfrepeats, we can show that the remaining $(d-1)$ out-neighbours of v_1, v_2 and v_3 are as follows.

$$N^+(v_1)\backslash\{w_1\} = \{b_1, \ldots, b_m, c_{m+1}, \ldots, c_{d-1}\}$$

$$N^+(v_2)\backslash\{w_2\} = \{c_1, \ldots, c_m, a_{m+1}, \ldots, a_{d-1}\}$$

$$N^+(v_3)\backslash\{w_3\} = \{a_1, \ldots, a_m, b_{m+1}, \ldots, b_{d-1}\}$$

where $2 \leq m \leq d-3$.

Now let us view the digraph G from the two vertices v_1 and w_2. To reach a_1 from w_2 we should go through b_k for some k, $k \in \{m+1, \ldots, d-1\}$. This implies that $a_1 (\neq v_3)$ is a second repeat of v_3, which is a contradiction. □

Next we shall apply Lemmas 4, 5 and 6 to $(d, 2)$-digraphs with only selfrepeat vertices for $d = 2, 3$ and 4 respectively. In these digraphs, if $N^+(x) \cap N^+(y) \neq \emptyset$ for some x and y in G then $N^+(x) = N^+(y)$. This is exactly the characterization of a line digraph [11]. Therefore, for $d = 2, 3$ and 4 if G is a $(d, 2)$-digraph with only selfrepeat vertices then G must be the line digraph of K_{d+1}. (Note that for $d = 2$ the proof is straighforward; for the case $d = 3$ this has been proved previously in [3], but it is quite a long proof.)

Lemma 7 *If G is a $(5, 2)$-digraph with only selfrepeat vertices then G is the line digraph of K_6.*

Proof By Lemmas 4, 5 and 6, it suffices to show that there are no two vertices x and y in G such that $|N^+(x) \cap N^+(y)| = 1$. For a contradiction assume we have such vertices x and y. Denote all different out-neighbours of x and y by v_i and w_i ($i = 1, 2, 3, 4$) respectively. By Lemma 6, we have arcs $(v_i, w_i) \in G$ for $i = 1, 2, 3, 4$. Clearly, every out-neighbour (other than w_1) of v_1 must be not in $N^+(w_1)$. Therefore, if $a \in N^+(v_1)$ and $a \neq w_1$ then $a \in N^+(\{w_2, w_3, w_4\})$. Then there exists $j \neq 1$ such that there are two vertices of $N^+(w_j)$ which are both out-neighbours of v_1. This is a contradiction by Lemma 6. □

If we apply Theorem 2, Lemma 3 and consider the above results then we get

Theorem 3 *For $d = 2, 3, 4$ and 5 there are no $(d, 2)$-digraphs with a self-repeat other than the line digraphs of K_{d+1}.*

Let us consider a $(d, 2)$-digraph G with selfrepeats but not only self-repeats, $6 \leq d \leq 12$. Using Theorems 1 and 2, and Lemma 3, the only possible permutation cycle structures of $N^+(v)$ for a fixed selfrepeat v of G are listed in Table 1.

By checking the condition (3) of Theorem C using a computer, we found that none of these structures are possible. Thus we have

Theorem 4 *For $d = 2, 3, ..., 12$, if a $(d, 2)$-digraph G contains a selfrepeat then all the vertices of G are selfrepeats.*

In this section, we have seen the characterization of $(d, 2)$-digraphs with selfrepeats for small degrees, namely that such digraphs contain only selfrepeat vertices. One of the digraphs with this characterization (for any d) is the line digraph of K_{d+1}. However, we do not know whether such digraphs are the only ones fulfilling this condition. For degrees 2,3,4 and 5 we have shown that the line digraph of K_{d+1} is the only such $(d, 2)$-digraph.

In conclusion, let us give some open problems:

Degree	Permutation cycle structures
6	(1 2 3 4 5)(6)
7	(1 2 3 4 5)(6)(7)
8	(1 2 3 4 5)(6)(7)(8) (1 2 3 4 5 6 7)(8)
9	(1 2 3 4)(5 6 7 8)(9) (1 2 3 4 5)(6)(7)(8)(9) (1 2 3 4 5 6 7)(8)(9)
10	(1 2 3 4)(5 6 7 8)(9)(10) (1 2 3 4 5)(6)(7)(8)(9)(10) (1 2 3 4 5 6 7)(8)(9)(10) (1 2 3 4 5 6 7 8 9)(10)
11	(1 2 3 4)(5 6 7 8)(9)(10)(11) (1 2 3 4 5)(6 7 8 9 10) (11) (1 2 3 4 5)(6)(7)(8)(9)(10)(11) (1 2 3 4 5 6 7)(8)(9)(10)(11) (1 2 3 4 5 6 7 8 9)(10)(11)
12	(1 2 3 4)(5 6 7 8)(9)(10)(11)(12) (1 2 3 4 5)(6 7 8 9 10)(11)(12) (1 2 3 4 5)(6)(7)(8)(9)(10)(11)(12) (1 2 3 4 5 6 7)(8)(9)(10)(11)(12) (1 2 3 4 5 6 7 8 9)(10)(11)(12) (1 2 3 4 5 6 7 8 9 10 11) (12)

Table 1: The possible permutation cycle structures of $N^+(v)$

Problem 3 *Is it true that a $(d, 2)$-digraph G with every vertex a selfrepeat (i.e., every vertex lies on a cycle C_2) implies G is the line digraph of K_{d+1}?*

We know that in the line digraph of K_{d+1} for any $d \geq 2$ every vertex is a selfrepeat. On the other hand, for $k \geq 3$ there are no (d, k)-digraphs with every vertex on a C_k [2]. Alternatively,

Problem 4 *Find $(d, 2)$-digraphs with every vertex a selfrepeat other than the line digraph of K_{d+1}.*

Problem 5 *For $d \geq 13$, are there any $(d, 2)$-digraphs with selfrepeats other than those with every vertex a selfrepeat?*

In this section, we have shown that the answer is 'No' for degree up to 12. Finally,

Problem 6 *For $d \geq 4$ are there any $(d, 2)$-digraphs without selfrepeats?*

We note that there are two non-isomorphic $(2, 2)$-digraphs without selfrepeats and no $(3, 2)$-digraph without selfrepeats [3].

Acknowledgement. We wish to thank the referee for his/her thoughtful suggestions which improved the readability of the paper.

References

[1] E.T. Baskoro, M. Miller and J. Plesník, On the structure of diregular digraphs with order close to the Moore bound, *Graphs Combin.*, to appear.

[2] E.T. Baskoro, M. Miller, J. Plesník and Š. Znám, Digraphs of degree 3 and order close to the Moore bound, *J. Graph Theory* 20 (1995) 339-349.

[3] E.T. Baskoro, M. Miller, J. Plesník and Š. Znám, Regular digraphs of diameter 2 and maximum order, *Australas. J. Combin.* 9 (1994) 291-306. Errata, 12 (1995) 305.

[4] J. Bosák, Directed graphs and matrix equations, *Math. Slovaca* 28 (1978) 189-202.

[5] W.G. Bridges and S. Toueg, On the impossibility of directed Moore graphs, *J. Combin. Theory Ser. B* 29 (1980) 339-341.

[6] F. Comellas and M.A. Fiol, Vertex symmetric digraphs with small diameter,*Discrete Appl. Math.* 58 (1995) 1-12.

[7] M.A. Fiol, The connectivity of large digraphs and graphs, *J. Graph Theory* 17 (1993) 31-45.

[8] M.A. Fiol, I. Alegre and J.L.A. Yebra, Line digraph iteration and the (d,k) problem for directed graphs, *Proc. 10th Symp. Comp. Architecture*, Stockholm (1983) 174-177.

[9] M.A. Fiol and J.L.A. Yebra, Dense bipartite digraphs, *J. Graph Theory* 14 (1990) 687-700.

[10] M.A. Fiol, J.L.A. Yebra and I. Alegre, Line digraphs iterations and the (d,k) digraph problem, *IEEE Trans. Comput.*, C-32 (1984) 400-403.

[11] C. Heuchenne, Sur une certaine correspondance entre graphes, *Bull. Soc. Roy. Sci. Liège* 33 (1964) 743-753.

[12] M. Imase and M. Itoh, A design for directed graphs with minimum diameter, *IEEE Trans. Comput.* C-32 (1983) 782-784.

[13] M. Miller and I. Fris, Minimum diameter of diregular digraphs of degree 2, *Comput. J.* 31 (1988) 71-75.

[14] M. Miller and I. Fris, Maximum order digraphs for diameter 2 or degree 2, *Pullman volume of Graphs and Matrices, Lecture Notes in Pure and Appl. Math.* 139 (1992) 269-278.

[15] M. Miller, Digraph covering and its application to two optimisation problems for digraphs, *Australas. J. Combinat.* 3 (1991) 151-164.

[16] J. Plesník and Š. Znám, Strongly geodetic directed graphs, *Acta F. R. N. Univ. Comen. - Mathematica XXIX* (1974) 29-34.

[17] S. Toueg and K. Steiglitz, The design of small-diameter network by local search, *IEEE Trans. Comput.* C-28 (1974) 537-542.

[18] J. Xu, An inequality relating the order, maximum degree, diameter and connectivity of a strongly connected digraph, *Acta Math. Appl. Sinica* 8 (1992) 144-152.