Square-celled animal reconstruction problems

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Abstract

We consider reconstruction problems involving *square-celled animals* and other, similar, problems. Our main results, Corollary 3.2 and Theorem 3.3, give positive answers to the problems raised at the end of [4] by Harary and Manvel.

1 Introduction

A *square-celled animal* is a finite set of rookwise-connected squares (called *cells*) which form a simply connected region in the plane. Here we mean connected in the usual sense, for example the animal in figure 1 is rookwise connected but does not form a simply connected region in the plane. The deletion of a single cell from an animal leaves a collection of cells which are agreed to be called a *sub-animal*, even though the collection itself may not form an animal but may consist of as many as four separate animals. Two animals will be considered *isomorphic* if one can be superimposed on the other by a suitable translation and rotation but reflections are not allowed. To keep our notation standard we have substituted the word *isomorphic* for what is termed *same* in [4]. An animal which does not form a simply connected region in the plane is referred to as a *holey animal*.

The multiset of all sub-animals, given up to isomorphism, of an animal is called its *deck*. An animal is *reconstructible* if there is no non-isomorphic animal sharing exactly the same deck. The main result of [4] was the following theorem.

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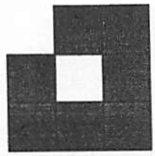

Figure 1: The smallest holey square-celled animal has 7 cells.

Theorem 1.1 (Harary and Manvel [4]) *Every square-celled animal A is reconstructible from its multiset of cell-deleted sub-animals.*

The paper of Harary and Manvel concluded with the following open problem for which they expected an affirmative answer to the first part.

Problem 1.2 (Harary and Manvel [4]) *Are animals which are not simply connected, (i.e. holey), reconstructible? Further, determine the minimum number of cell-deleted animals which enable the reconstruction of A.*

2 Main tools

Throughout G denotes a permutation group on a finite or infinite set Ω. Our notation for permutation groups is that of Wielandt [10]. Two Ω-subsets Δ_1 and Δ_2 are said to be *G-isomorphic*, $\Delta_1 \approx_G \Delta_2$, if $\exists g \in G$ such that $\Delta_1^g = \Delta_2$. A *G-hypomorphism* between two Ω-subsets Δ_1 and Δ_2 is a bijection $h : \Delta_1 \to \Delta_2$ such that for every $\alpha \in \Delta_1$ we have

$$\Delta_1 \backslash \{\alpha\} \approx_G \Delta_2 \backslash \{h(\alpha)\},$$

see [3] and also [1], [2] and [9] for equivalent definitions. Two Ω-subsets Δ_1 and Δ_2 are said to be *G-hypomorphic*, $\Delta_1 \sim_G \Delta_2$, if there exists a G-hypomorphism between them. Thus, two Ω-subsets Δ_1 and Δ_2 are G-hypomorphic if and only if the multiset of G-isomorphism classes of point-deleted subsets of Δ_1 and Δ_2 are the same.

*An Ω-subset Δ is G-***reconstructible** *if every set G-hypomorphic to it is G-isomorphic to it, i.e. $\Delta_1 \sim_G \Delta_2 \Rightarrow \Delta_1 \approx_G \Delta_2$.*

When the group G in question is clear we omit the prefix G and refer simply to the concepts *isomorphic*, *hypomorphic*, *hypomorphism* and *reconstructible*.

More generally we consider k-deck reconstruction. Let Δ be an Ω-subset of cardinality m and k an integer satisfying $1 \le k < m$. The *k-deck* of Δ is the multiset of all Δ-subsets of cardinality $m - k$ given up to isomorphism.

Then Δ is reconstructible from its k-deck if any other Ω-subset with the same k-deck is isomorphic to Δ. The following theorem which generalizes a result of Nash-Williams from graph theory can be found in [1].

Theorem 2.1 (Alon et al. [1]) *Let G be a permutation group on a possibly infinite set Ω. Let Δ be an Ω-subset of cardinality m which is not reconstructible from its k-deck. Suppose, further, that there is a Δ-subset S of cardinality $|S| = t$ with finite stabilizer $|G_S| < \infty$ where $m - k \geq t$. Then there is a set T of cardinality $|T| \geq m - k + 1$ satisfying $S \subseteq T \subseteq \Delta$, and there is an $\varepsilon \in \{0, 1\}$ such that for every set K satisfying $S \subseteq K \subseteq T$ and $|K| \equiv \varepsilon \pmod 2$, there is a $g \in G$ with $T \cap \Delta^g = K$.*

For $k = 1$ the following theorem is the analogue of a theorem of Lovász for graphs, (for details see Corollary 7.2 in [2] by Babai).

Theorem 2.2 *Let G be a permutation group on a possibly infinite set Ω. Let Δ be a finite Ω-subset which is not k-reconstructible. Then for every $\Gamma \subseteq \Delta$ with $|\Gamma| \leq |\Delta| - k$ it follows that*

$$|\{g \in G : \Gamma^g \subseteq \Delta\}| \geq 2^{|\Delta|-|\Gamma|-k}.$$

Proof: Set $|\Delta| = m$. Clearly we can assume that $|G_\Gamma| < \infty$ otherwise the result is clear. By Theorem 2.1 there exists T satisfying $|T| \geq m - k + 1$ and $\Gamma \subseteq T \subseteq \Delta$. Further, there exists $\varepsilon \in \{0, 1\}$ such that for every K satisfying $\Gamma \subseteq K \subseteq T$ and $|K| \equiv \varepsilon \pmod 2$ there is a $g \in G$ with

$$T \cap \Delta^g = K.$$

Clearly corresponding to each such K the $g \in G$ in the above is distinct. Further, $\Gamma^{g^{-1}} \subseteq K^{g^{-1}} \subseteq \Delta$. Hence,

$$|\{g \in G : \Gamma^g \subseteq \Delta\}| \geq |\{K : \Gamma \subseteq K \subseteq T; |K| \equiv \varepsilon \pmod 2\}|.$$

Since $|\Gamma| \leq m - k$ and $|T| \geq m - k + 1$ it follows that the number on the right is at least $1/2 \cdot 2^{m-k+1-|\Gamma|} = 2^{m-|\Gamma|-k}$, as claimed. □

3 Results and theorems

In this section we shall look at a more general reconstruction problem than the one considered in the introduction. Let $\mathcal{G}$ be the following group of permutations of the infinite square grid in the plane: $\mathcal{G}$ translates the grid arbitrarily and also rotates it through multiples of 90 degrees around the centre of any square. For brevity, we refer to configurations of m squares in the plane as *m-figs*. Deleting a square from an m-fig results in an $(m-1)$-fig also referred to as a *sub-fig*. It is a simple matter to check that the concept of reconstruction in the permutation group sense introduced in section 2 agrees with that of the first section for animals. That is to say, two animals A and B are non-reconstructible as animals if and only if they are non-reconstructible under $\mathcal{G}$ when considered as m-figs.

We need the following definition. Let A be an m-fig. We define the *dimension* of A, denoted $\dim(A)$, to be the pair (α, β) with $\alpha \geq \beta$, where the smallest rectangle enclosing A in the plane is of size $\alpha \times \beta$ or $\beta \times \alpha$. Clearly the dimension is a $\mathcal{G}$-orbit invariant, i.e. isomorphic configurations have the same dimension.

Theorem 3.1 *Let m and k be given integers. If $m-k \geq 7$ then all m-figs are reconstructible from their k-decks.*

Proof: Assume for a contradiction that $m-k \geq 7$ and Δ is an m-fig which is not k-reconstructible. Let Δ have dimension (α, β), then Δ sits inside some $\alpha \times \beta$ or $\beta \times \alpha$ rectangle. Let a, b, c, d be any points of Δ which lie on the 4 outside edges, (not necessarily distinct, e.g. if Δ has "corners"). Then put $\Gamma := \{a, b, c, d\}$, so $|\Gamma| \leq 4$. We now invoke Theorem 2.2 which states that

$$|\{g \in \mathcal{G} : \Gamma^g \subseteq \Delta\}| \geq 2^{m-|\Gamma|-k}.$$

Now clearly, $|\{g \in \mathcal{G} : \Gamma^g \subseteq \Delta\}| \leq 4$. Also $m-|\Gamma|-k \geq 3$, a contradiction. The result follows. □

Corollary 3.2 *Let A be any m-fig. Then A is reconstructible from its multiset of 7-figs. Thus an m-fig is reconstructible if $m \geq 8$. In particular, all holey animals are reconstructible.*

Proof: The first two parts are clear. It follows that any holey animal of size m is reconstructible if $m \geq 8$. There is only one holey animal with 7 cells and none with fewer cells. □

Remarks: We note that Theorem 1.1 can be deduced from Corollary 3.2: It follows immediately from Corollary 3.2 that any animal with more than 7 cells is reconstructible. The remaining cases are easily checked by hand.

The above Corollary establishes an affirmative answer to the first part of Problem 1.2 as conjectured in [4]. For consideration of the second part of Problem 1.2 we need more specific arguments.

If A and B are m-figs define $r(A, B)$ to be the cardinality of the intersection of the decks of A and B. (A similar definition was given for graphs in [6]). That is, the number of common sub-figs (up to isomorphism) of A and B. Then the second part of Problem 1.2 is to establish $r(A, B)$ for any two m-figs A and B with m arbitrary. We have the following bound for $r(A, B)$.

Theorem 3.3 *Let m be any natural number. If A and B are non-isomorphic m-figs then*

$$r(A, B) \leq 8.$$

Proof: Let Y be an arbitrary m-fig. It is easy to note that at most 4 sub-figs have dimension different to that of Y and all others have the same dimension as Y. Thus, if $\dim(A) \neq \dim(B)$ then $r(A, B) \leq 4$. Henceforth we assume that $\dim(A) = \dim(B)$.

Define $X_A = [\{A\backslash\{a\} : a \in I\}]$ to be a maximal (w.r.t. size) multiset of sub-figs of A which B shares up to isomorphism. (Note, $I \subseteq A$ is just an index set and is not necessarily unique). So for example, $|X_A| = r(A, B)$. Then we note the following: Let a and a' be distinct elements of I. If for some $b, b' \in B$ and $g, h \in \mathcal{G}$ we have

$$A\backslash\{a\} = (B\backslash\{b\})^g$$

and

$$A\backslash\{a'\} = (B\backslash\{b'\})^h$$

then $h \neq g$. Since otherwise, assuming $g = h$, we have

$$B^g = (B\backslash\{b\})^g \cup b^g = A\backslash\{a\} \cup b^g$$

and

$$B^g = (B\backslash\{b'\})^g \cup b'^g = A\backslash\{a'\} \cup b'^g.$$

This implies that the two sets $\{a', b^g\}$ and $\{a, b'^g\}$ are equal. Then $a = b^g$ since we are assuming that $a' \neq a$. But this implies that $B^g = A$, a contradiction. So corresponding to each element of X_A we get a distinct element of the group $\mathcal{G}$.

Then we note, in the same way as at the beginning of the proof, $|\{C \in X_A : \dim(C) \neq \dim(A)\}| \leq 4$. Hence we now assume that $\dim(C) = \dim(A)$, for some $C \in X_A$. It follows that for some $g \in \mathcal{G}$ both A and B^g

sit inside the same rectangle of size $\alpha \times \beta$ or $\beta \times \alpha$ for some integers α and β.

If $\alpha \neq \beta$ then there are only two distinct elements of $\mathcal{G}$ fixing this rectangle, namely the identity and rotation by 180 degrees (if both α and β are odd) or the identity and rotation by 180 degrees followed by a unique translation (otherwise). Thus it follows that there are at most two $C \in X_A$ with $\dim(C) = \dim(A)$. Hence we have $|X_A| \leq 6$ when $\alpha \neq \beta$ and the result follows in this case.

We now consider the case when $\alpha = \beta$. Assume that $C \in X_A$ has the same dimension as A. The elements of $\mathcal{G}$ taking distinct elements of the deck of B^g to such C of the same dimension as A must be either rotations or rotations plus unique translations. In particular, there are at most four such group elements. Thus, $|\{C \in X_A : \dim(C) = \dim(A)\}| \leq 4$ and so $|X_A| \leq 8$. This completes the proof. □

Remarks: Let A be an m-fig. Theorem 3.3 says that given any 9 or more sub-figs of A we can uniquely reconstruct A. With more elaborate arguments it seems possible to reduce the bound of Theorem 3.3 to 6. Experimental evidence would suggest that $r(A, B)$ could be as small as 2. However, the precise value is still open. Similarly, it is not clear if the bound of Theorem 3.1 could be improved. As pointed out to me by Frank Harary, Theorem 3.3 (proved from first principles) essentially gives a new proof of Corollary 3.2 (derived from Theorem 3.1, proved with some serious machinery).

It should also be noted that a continuous version of the problems just considered was considered in both [1] and [7]. Specifically they examined the following problem.

Problem 3.4 ([1] and [7]) *Is a finite set A of points in R^n or on the unit sphere S^n, given up to isometry, reconstructible from all its subsets of cardinality $|A| - k$, the k-deck of A ?*

For R^n and $k = 1$ the question is settled positively but for other values of k the question is less clear, for more details see [1]. Unfortunately it seems that the methods used here are not applicable to this continuous case.

Closing remarks: There are many obvious variants on the reconstruction of square-celled animals. Instead of using squares one may consider equilateral triangles or hexagons, (see e.g. [5]). It is clear than the methods used here are applicable to these problems also. For instance we can show that any triangular celled animal is reconstructible.

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