Dimensions for Cographs

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Abstract

Cographs—complement-reducible graphs—can be viewed as intersection graphs (of k-dimensional boxes), as intersections of graphs (of P_4, C_4-free graphs), and as common tieset graphs of two-terminal graphs. This approach connects cographs with other topics such as chordal, interval, and series-parallel graphs, and it provides a natural dimension for cographs.

The frequently-studied family of *cographs* was introduced by Corneil, Lerchs and Stewart-Burlingham [2] as the family of *complement-reducible graphs*: graphs that can be reduced to edgeless graphs by repeatedly taking complements within components. The following are among the many characterizations of a graph G being a cograph in [1, 2, 3, 8]:

- G is *P_4-free* (meaning that no induced subgraph of G is isomorphic to P_4, the 4-vertex path).
- In every induced sugraph of G, every maximal complete subgraph and every maximal independent set of G have exactly one vertex in common.
- Every *nontrivial* (meaning $\ncong K_1$) induced subgraph of G contains vertices u, w that have exactly the same neighbors (except possibly for u and w themselves).
- For every nontrivial induced subgraph G' of G, either G' or its complement $\overline{G'}$ is not connected.
- Every connected induced subgraph of G has diameter at most two.
- G can be generated from trivial graphs by a sequence of disjoint-unions-and-joins.

The graph G shown in Figure 1 is a cograph: G'_1 and G'_2 are the two components of $\overline{G}$; $\overline{G'_1} \cong K_1 \cup K_1$ and $\overline{G'_2} \cong K_1 \cup P_3$; and so on.

From the point of view of the present paper, P_4, C_4*-free graphs* (meaning graphs with no induced subgraph isomorphic to P_4 or C_4) are the most primitive

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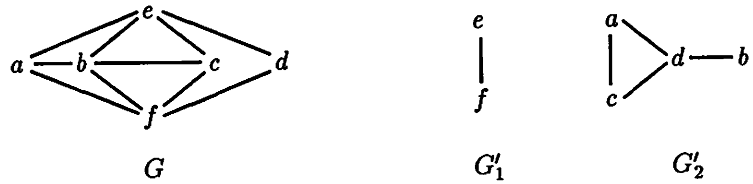

Figure 1: A cograph G and the components G'_1 and G'_2 of its complement.

cographs. Such graphs, introduced by Wolk in 1961 [15, 16], have a multitude of other names—including 'trivially perfect,' 'quasi-threshold,' 'nested interval,' 'domination reducible,' and 'hereditary upper bound' graphs (see [11])—each with a corresponding characterization. P_4, C_4-free graphs are easily seen to be *chordal graphs*, meaning that they are C_k-free for all $k \geq 4$; indeed, P_4, C_4-free graphs are precisely the chordal cographs.

A graph G is the *intersection of graphs* $G_1, \ldots, G_k$ if each $V(G_i) = V(G)$ and $E(G)$ is the intersection of $E(G_1), \ldots, E(G_k)$. In [13], a graph G is defined to be *k-chordal* if it is the intersection of k chordal graphs; thus chordal graphs are precisely the 1-chordal graphs. Let $K_{d[2]}$ denote the *d-dimensional generalized octahedron*, meaning the complete d-partite graph $K_{2,\ldots,2}$ with $d \geq 2$.

Theorem 1 *The following are equivalent for any cograph G:*
(1) *G is the intersection of at most k P_4, C_4-free graphs.*
(2) *G is k-chordal.*
(3) *G is $K_{(k+1)[2]}$-free.*

Proof. Implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) hold for all graphs: the first since P_4, C_4-free graphs are chordal, and the second since [13] shows that $K_{(k+1)[2]}$ is not k-chordal (it is $(k+1)$-chordal).

To show that (3) implies (1), suppose G is a $K_{(k+1)[2]}$-free cograph. Let $G'_1, G'_2, \ldots$ be the nontrivial components of $\overline{G}$. For each G'_i, let $G'_{i1}, G'_{i2}, \ldots$ be the nontrivial components of the complements of the components of $\overline{G'_i}$, noting that each G'_{ij} will be an induced subgraph of G'_i. Similarly, let $G'_{ij1}, G'_{ij2}, \ldots$ be the nontrivial components of the complements of the components of $\overline{G'_{ij}}$, each an induced subgraph of G'_{ij}, and so on. Let $\mathcal{G}'$ be the collection of all these subgraphs $G'_i, G'_{ij}, \ldots$, and let $G'_{(1)}, \ldots, G'_{(\ell)}$ be the minimal members of $\mathcal{G}'$. Then $\ell \leq k$ since taking one edge from each of the $G'_{(i)}$'s would correspond to a $K_{\ell[2]}$ back in G. For each $i \leq \ell$, define G_i to be the union of G with the edgeset of all members of $\mathcal{G}'$ that are incomparable to $G'_{(i)}$. Clearly $G = G_1 \cap \cdots \cap G_\ell$ and each G_i is P_4-free. If any G_i were to contain an induced cycle a, b, c, d, a, then edges ac and bd would be in incomparable $G'_{(j)}$'s, and so at least one of them would be in G_i, a contradiction. Thus each G_i is P_4, C_4-free. □

There are non-cographs that satisfy (2) but not (1): e.g., P_4 for $k = 1$ and C_5 for $k = 2$. (For the latter, if G is the cycle a, b, c, d, e, a, then G is 2-chordal since it is the intersection of the chordal graphs G_1 with $E(G_1) = E(G) \cup \{ac, ad\}$ and G_2 with $E(G_1) = E(G) \cup \{bd, be\}$; but inserting edges into G to make P_4, C_4-free graphs graphs G_1, G_2 would require three new edges for each, and so one of the five edges of $\overline{G}$ will have to be used in both G_1 and G_2, making $G_1 \cap G_2 \neq G$.) Similarly, there are non-cographs that satisfy (3) but not (2): e.g., C_5 for $k = 1$ and Figure 1 of [13] for $k = 2$.

Since every graph is $K_{(k+1)[2]}$-free for some k, Theorem 1 provides a concept of 'dimension' for cographs. As a corollary, every cograph is the intersection of P_4, C_4-free graphs. The converse fails since P_4 can easily be formed from the intersection of two P_4, C_4-free graphs.

A *nested interval representation* is a family of intervals of the real line such that two intervals in the family have a nonempty intersection only when one of the two is contained in the other. Skrien [14] characterized P_4, C_4-free graphs as the intersection graphs of nested interval representations. A *k-dimensional box* is the cartesian product of intervals $[a_i, b_i]$ for $1 \leq i \leq k$.

Corollary 2 *For any cograph G, conditions* (1)–(3) *above are equivalent to the following*:

(4) *G is the intersection graph of k-dimensional boxes in* $\mathbb{R}^k$ *where the projection of the boxes onto each of the k axes is a nested interval representation.*

Proof. The equivalence (1) $\Leftrightarrow$ (4) holds for all graphs by the nested interval characterization of P_4, C_4-free graphs of [14]. □

Figure 2 shows a 2-dimensional box representation for the cograph G from Figure 1, along with its nested interval representation projections.

The complements-within-components definition of cographs provides a parsing of a cograph into smaller cographs. Figure 3 shows the parse tree T that results from the cograph G of Figure 1; the root is labeled s if G is connected p if G is disconnected, and then s's and p's alternate in lower levels. This also corresponds to the disjoint unions-and-joins characterization of cographs, with the p-vertices of the tree corresponding to unions and the s-vertices to joins. The s and p notation reflects that such a tree also corresponds to a two-terminal series-parallel graph, as we describe next. This sort of graph is well-known to be intimately related to cographs—see for instance [1, 7]—but with a significantly different feel.

A *two-terminal graph* N is a multigraph (without loops) with two distinguished vertices (*terminals*—the vertices drawn as black disks in Figure 3). A *tieset* of N is a path connecting the terminals (for instance abe or de in Figure 3), and a *cutset* of N is a minimal set of edges whose removal would disconnect the terminals (for instance $\{a, c, d\}$ or $\{b, d\}$ in Figure 3). Alternatively, N can be viewed as a multigraph with a distinguished edge (typically not drawn) joining

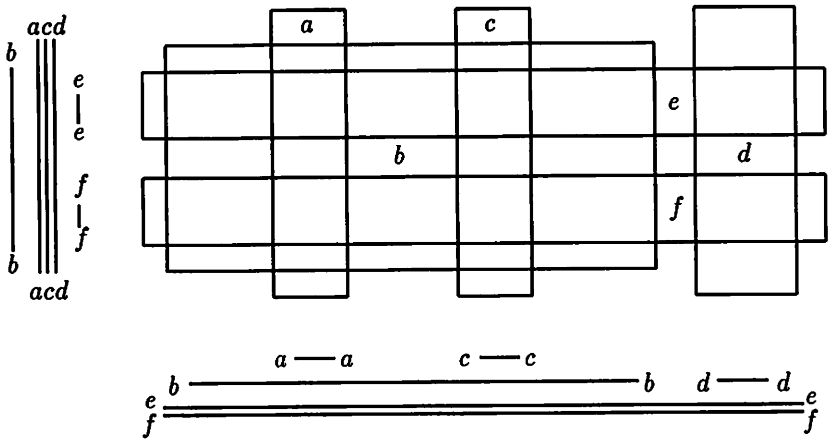

Figure 2: A 2-dimensional box representation for G from Figure 1.

the terminals, and then the tiesets and cutsets of N correspond, respectively, to cycles and cutsets containing that distinguished edge.

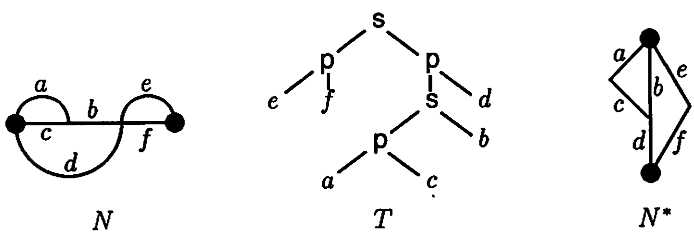

Figure 3: A two-terminal series-parallel graph N corresponding to the cograph G from Figure 1, its tree T, and its dual graph N^*.

A two-terminal graph N is *series-parallel* if it can be built up by, alternately, connecting smaller two-terminal graphs $N_1, N_2, \ldots$ in 'series' ($i \neq j$ implying that each edge of N_i will be in a common tieset with each edge of N_j) and in 'parallel' ($i \neq j$ implying that each edge of N_i will be in a common cutset with each edge of N_j). In Figure 3 for instance, N consists of two subgraphs connected in series, with the larger of these consisting of two subgraphs (one consisting of edges a, b and c, and the other consisting of edge d) connected in parallel. Among various characterizations, [4] shows that N is series-parallel if and only if a new edge can be inserted to join the terminals without there being any subgraph homeomorphic to K_4, and [6] shows this is equivalent to each cutset of N meeting each tieset at exactly one edge.

This relationship between cographs and two-terminal series-parallel graphs is known to correspond to a cograph G being the *common tieset graph* of a two-terminal series-parallel graph N, meaning that $V(G) = E(N)$ with two vertices are adjacent in G if and only if the corresponding edges are in a common tieset of N. For instance, the cograph G of Figure 1 is the common tieset graph of the series-parallel graph N of Figure 3. Indeed, the maximal complete subgraphs of G correspond to the tiesets of N, and the maximal independent subgraphs of G correspond to the cutsets of N.

Since it is easy to see that the common tieset graph of *any* two-terminal graph must be P_4-free, the following simplier (but apparently unnoticed) result actually holds.

Proposition *A graph is a cograph if and only if it is the common tieset graph of a two-terminal graph.* □

Figure 4 shows how a two-terminal graph that is not series-parallel can correspond to the same cograph as a two-terminal series-parallel graph.

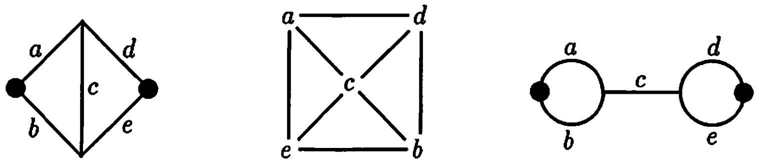

Figure 4: A two-terminal, non-series-parallel graph, the cograph that is its common tieset graph, and a two-terminal series-parallel graph with the same common tieset graph.

Define the p-*width* of such a parse tree T as above to be the maximum number of noncomparable p-vertices (two, for the tree in Figure 3). Every such two-terminal series-parallel N has a *dual graph* N^*, also series-parallel, in which the tiesets of N become the cutsets of N^* and vice versa. Therefore if G is the common tieset graph of a series-parallel graph N, then the complement of G is the common tieset graph of N^*. Notice that N in Figure 3 has a tieset that contains every vertex, while N^* requires two tiesets to cover its vertices.

Corollary 3 *For any cograph G with corresponding two-terminal series-parallel graph N, the following are equivalent to conditions* **(1)–(4)** *above:*

(5) *The tree T corresponding to N has* p-*width at most k.*

(6) *The vertices of the dual graph N^* can be covered with k tiesets.*

Proof. Both (5) and (6) can be viewed as analogous to (3): Suppose $G_1, \ldots, G_k$ are the k subgraphs that correspond to k noncomparable p-vertices of T. For each $i \in \{1, \ldots, k\}$, let v_i and w_i be vertices of G_i from different children of the ith p-vertex. Then $\{v_1, w_1, \ldots, v_k, w_k\}$ will induce a subgraph of G isomorphic to $K_{k[2]}$ and each pair v_i, w_i will correspond to edges connected in series in N^* that will require a separate tieset in the vertex cover. Conversely, each subgraph of G isomorphic to $K_{k[2]}$ will correspond to vertices $v_1, w_1, \ldots, v_k, w_k$ below k noncomparable p-vertices in T. □

Because the concepts of series-parallel and duality are naturally developed within matroid theory, the appearance of vertices in condition (6) is somewhat unexpected. Perhaps this is why the special series-parallel graphs in the $k = 1$ case—what might be called 'hamiltonian' two-terminal series-parallel graphs—do not seem to have been studied, while the correspondingly special cographs—the P_4, C_4-free graphs—have been studied from so many directions.

Corollary 4 is included for completeness; those unfamiliar with 'dimension-k chordal graphs' can omit the rest of this paper. In [12], a graph is defined to be *dimension-k chordal* if every induced subgraph of G has the kth Betti number of the simplicial complex of its maximal complete subgraphs equal to zero. Relatedly, [10] characterizes dimension-k chordal graphs as intersection graphs of subgraphs of certain 'clique representations' (a greedy generalization of the well-known clique trees of chordal graphs) that involve no polyhedra of dimension greater than k; [9] discusses the $k = 2$ case in detail.

Corollary 4 *For any cograph G, conditions* (1)–(6) *above are equivalent to the following*:

(7) *G is dimension-k chordal.*

Proof. The proof of [10, Theorem 2] shows that (2) ⇒ (7) for all graphs, and [12, Theorem 4b] shows that (7) ⇒ (3) for all graphs. □

There are examples of non-cographs that satisfy (7) but not (2) (e.g., Figure 1 of [13] for $k = 2$; when $k = 1$, the conditions are both equivalent to G being chordal) and examples of non-cographs that satisfy (3) but not (7) (e.g., C_5 for $k = 1$ and the isocahedron for $k = 2$).

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