Cops, Robber, and Photo Radar.

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Abstract

This game is a mixture of Searching and Cops and Robber. The Cops have partial information provided by sensing devices called photo radar. The Robber has perfect information. We give bounds on the number of photo radar units required by one Cop to capture a Robber on a tree and, with less tight bounds, on a copwin graph.

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The game of Cops and Robber is played on a reflexive graph, that is a graph with loops at every vertex. The Cops choose vertices to occupy then the Robber chooses a vertex. The two sides then move alternately where a move is to slide along an edge. There is perfect information, that is each side is always aware of the position of the other. The loops are a technical device which allows any subset of the Cops and the Robber to pass. The Cops win if any of the Cops and the Robber occupy the same vertex at the same time. Graphs in which one Cop suffices to win are called *copwin* graphs and are characterized in [3] and [5].

In this paper, a variation of the game of Cops and Robber is introduced. The Cops no longer have perfect information, but rather can only get information about the Robber's position through the use of sensing devices known as photo radar. As in the original game, it is assumed that the Robber has perfect information.

Suppose the game is being played on a graph G. Photo radar units are placed on the edges of G. These units alert the Cops if the Robber moves along an edge equipped with a photo radar unit. The units also indicate the direction in which the Robber is moving. The minimum number of photo radar units required by a single Cop to guarantee the capture of the

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Robber on G will be referred to as the *photo radar number* of G, and will be denoted $pr(G)$. In general, one can ask for the least number of photo radar units needed if there are k Cops.

A graph H is a *retract* of a graph G if there is an edge-preserving map $f : G \to H$ such that $f(x) = x$ for all $x \in H$. We use $a \sim b$ to indicate that vertex a is adjacent to vertex b and $a \simeq b$ if a is adjacent or equal to b.

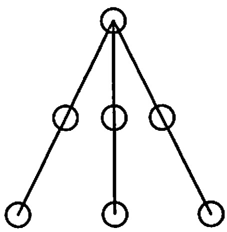

Figure 1: The tree T_1.

Lemma 1 *Let T_1 be the tree shown in Figure 1. A single Cop playing without photo radar cannot guarantee the capture of a Robber on T_1.*

Proof. The tree T_1, given in Figure 1, has three branches. The Cop having searched down a branch and returned to the root has a choice of taking one of two branches. In the four moves that it takes to search this branch and return to the root the Robber has the time to move between the other two branches. The Cop is now faced with having to decide which of two branches to search. Thus there is no guaranteed win by the Cop. □

Theorem 2 *For all finite, positive integers n, there exists a graph G such that $pr(G) > n$.*

Proof. Consider a tree similar to T_1 shown in Figure 1 with $n+3$ branches rather than 3. Regardless of how n photo radar units are placed there is still a subgraph with three branches. The Robber restricts himself to playing on this subgraph and by Lemma 1 the Cop does not have a winning strategy. □

Consider a tree T with n vertices. Now $pr(T) \leq n - 1$ since the photo radar units can be placed one to an edge. If the Robber doesn't move then visiting all the vertices ensures a Cop win. If the Robber moves, the game is equivalent to Cop and Robber since the Cop will always know the location of the Robber. This bound can be improved.

Let G be a graph. An edge is *free* if it has no photo radar and a path P of G is said to be a *freepath* if every edge of P is free.

Let T be a tree. Let T_a be the tree T rooted at vertex a. An *a-branch* of T_a is a path of T with a as one end vertex. We define $k(T_a)$ as the minimum number of edges having photo radar such that the free edges form freepaths and each maximal freepath is on an a-branch. Let $T' = T \setminus \{a \in V(T) : a \text{ is a leaf}\}$ and set $k_T = \min\{k(T'_a) : a \in V(T')\}$.

Theorem 3 *Let T be a finite tree. Then $pr(T) \leq k_T$.*

Proof. Let T be a tree and let a be a vertex for which $k(T'_a) = k_T$. Draw a planar representation of T with a as the root vertex at the top and all edges directed downward. Also, since every vertex is incident with at most two edges of a freepath in T' we can assume that any edge of a freepath of T' emanating from a vertex is the leftmost edge. We can also place any leaves so as to be the next edges (in a counterclockwise direction). We refer to a freepath together with adjacent leaves as a *free area.*

There are two phases to the strategy.

Firstly, the Cop does a depth first search of T' except when he comes to a stem v, he visits any leaves adjacent to v. The Robber cannot move to v without being caught on the Cop's next move. The Cop always enters at one end of a freepath (never in the middle) and exits at the bottom without leaving the freepath, except for leaves. The Robber can never move past the Cop, thus once a free area has been searched by the Cop, he is assured that the only way a Robber could be on that free area is if he has used an edge with a photo radar unit. Thus if the Robber stays on the free area then he will be caught in this phase. If he does move off then he will be detected by a photo radar unit and the Cop will always know the free area in which the Robber is located.

The second phase starts when the Robber is detected by a photo radar unit. The Cop moves up the tree until he is on a vertex which lies above the free area on which the Robber is currently located. Assuming that the Robber is not caught in this maneuver, the Cop then starts down the a-branch that contains the Robber until he enters the same free area as the Robber. (Note that the Robber can move to a different free area but this move will be detected by the photo radar units and the Cop will always move so as to be above the Robber.) By moving down the freepath and visiting adjacent leaves, the Cop will either catch the Robber or force him to leave the free path moving down the tree and below the Cop. The Robber will eventually be caught on a leaf if not sooner. □

Consider the graph T_2 shown in Figure 2. If there is only one photo radar unit and it is not on the dashed edge then there will be a tree isomorphic to T_1 so one Cop will not suffice. If the unit is on the dashed edge then the Cop can force the Robber to move across this edge. So the Cop knows in which portion of T the Robber is located. However, the Cop still has a choice of two branches to search. In the four moves it takes to search one branch, the Robber has time to move to the other side of T along the edge with a photo radar unit. This can continue indefinitely with the result that the Robber is not caught. So $pr(G) > 1$. Note: $k(T_b') = 2$, and $k(T_c') = 3$, and therefore $k_T = 2$; so indeed $pr(T) = 2$.

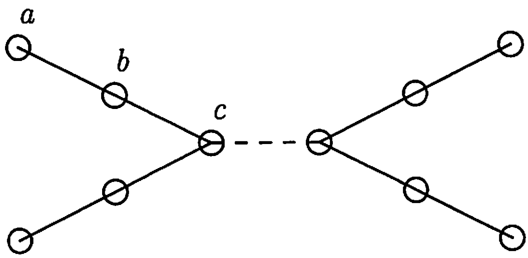

Figure 2: The graph T_2.

We wish to extend this strategy so that it can be used on a copwin graph. This is the subject of Theorem 8. However, we first need the characterization of copwin graphs the proof of which can be found in [3, 5].

Theorem 4 *A finite graph is copwin if and only if there is an ordering* $(v_1, v_2, \ldots, v_n)$ *of the vertices of* G *such that for each* $v_i \in V(G)$*, there exists a vertex* $u \in \{v_i, v_{i+1}, \cdots, v_n\}$ *such that* $N[v_i] \subseteq N[u]$ *in the subgraph induced by* $\{v_i, v_{i+1}, \cdots, v_n\}$.

This ordering is known as a *copwin ordering.* The vertex v_n is referred to as the *start vertex* of the ordering.

Consider a finite copwin graph G. Define the induced subgraphs $G_{i+1} = G_i \setminus \{v_i\}$ where $G_1 = G$, and let $f_i : G_i \rightarrow G_{i+1}$ be the retraction map from G_i to G_{i+1}. We note that f_i is a one-point retraction. We define $F_i = f_i \circ f_{i-1} \circ \cdots \circ f_1$.

Fix a copwin ordering of G, and construct a spanning tree S of G such that the root of the spanning tree is the start vertex of the copwin ordering. This spanning tree shall be referred to as a *copwin spanning tree.* Let $x_1, x_2 \in V(G)$. We say that $x_1 \succeq x_2$ if $F_i(x_2) = x_1$ for some i and $x_1 \succ x_2$ if $x_1 \neq x_2$. (See Figure 3.)

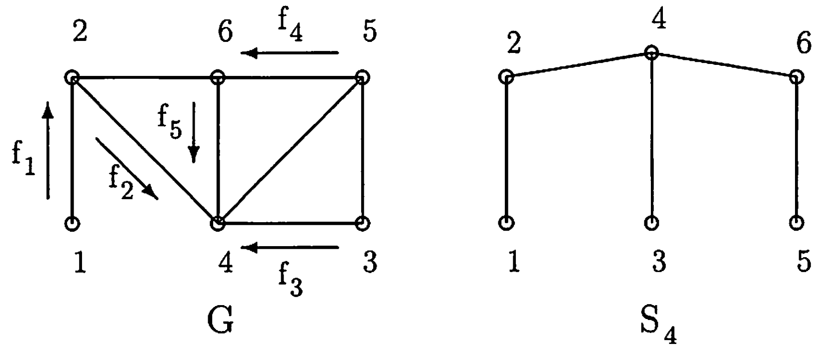

Figure 3: A copwin ordering with corresponding copwin spanning tree S_4.

We would like to mimic the proof of the tree result, the problem that arises here is that the Robber can move from one a-branch to another by using an edge not in the tree. The next two lemmas allow us to deal with that problem when it arises in Theorem 8.

Let A and B be two v-branches of a copwin spanning tree S_v. Suppose $a \in A$ and a is adjacent to some vertices of B. We take $b \in B$ to be the lowest vertex in B that is adjacent to a and write $a \rightarrow b$. Under most circumstances, the Cop will move from a to b.

Lemma 5 *Let G be a copwin graph with copwin spanning tree S_v, and let A and B be two v-branches of S_v. If there exists vertices $x \in A$ and $y \in B$, $x \simeq y$ then for all $a \succeq x$ there exists $b \succeq y$ such that $a \simeq b$.*

Proof. For every f_i, a vertex and its image lie on the same v-branch. Let j be the least index such that $F_j(x) = a$, note that $F_j(y)$ is still on B and so $a = F_j(x) \simeq F_j(y)$, proving the lemma. □

Lemma 6 *Let G be a copwin graph with copwin spanning tree S_v, and let A and B be two v-branches of S_v. Let $a, x \in A$ and $b, y \in B$ with $x \prec a$, $x \sim y$ and $a \rightarrow b$, then either $y \prec b$ or $y \sim a$.*

Proof. If $y \prec b$ then we are finished. Suppose now $y \succeq b$. Let $a' \in A$ such that $a' \sim a$ and $a' \prec a$. Let k be the greatest index such that $F_k(x) = a'$, then $F_{k+1}(x) = a$. If $F_{k+1}(b) \preceq y$ then $F_{k+1}(y) = y$ and so $y = F_{k+1}(y) \sim F_{k+1}(x) = a$, that is $y \sim a$. If $F_{k+1}(b) \succ y$, then since $F_{k+1}(a) = a, a \sim F_i(b)$ $i = 1, 2, \ldots, k+1$. That is, $a \sim y = F_j(b)$, $j \leq k+1$.
□

Corollary 7 *Let G be a copwin graph with copwin spanning tree S_v. If both the Cop and the Robber are on a v-branch A with the Cop above the Robber, and the Robber moves to another v-branch then the Cop can move to the same v-branch still above or on the same vertex as the Robber.*

Proof: Suppose that the Cop is above the Robber on the same v-branch. Lemma 5 shows that if the Robber moves from one v-branch to another the Cop can also move to the same v-branch. Lemma 6 shows that the Cop will either capture the Robber when moving from v-branch to v-branch or stay above him on the new v-branch. □

Hence we need only consider the Robber moving from x to y with the Cop on a where $a \succ x$, $b \succ y$ and $a \rightarrow b$. Let G be a copwin graph. We define $K_G = \min_{S_v}\{k(S_v) : S_v$ is a copwin spanning tree with root $v\}$.

Theorem 8 *Let G be a finite copwin graph. Then*

$$pr(G) \leq |E(G)| - [(n-1) - K_G].$$

Proof. Let S_v be a copwin spanning tree at which K_G is attained. The Cop begins on the start vertex v. Draw the tree as in Theorem 3, however we do not worry about the leaves.

The Cop traverses the tree in a depth first search so as to visit all vertices of G as in the proof of Theorem 3. If the Robber never moves off a freepath then he will be caught during this phase. If the Robber moves off a freepath, he will be detected by a photo radar unit. The Cop moves to the lowest vertex in S_v which is above the freepath containing the Robber. (Since the Robber can still move off this freepath, the Cop may end up at v.)

The Cop descends the v-branch leading to the freepath containing the Robber. If the Robber does not leave this path, he will be caught. If he does leave, then either he descends down the tree and the Cop continues his descent toward the Robber, or the Robber moves to another v-branch and the Cop, by Corollary 7, can always move to the same v-branch. Lemma 6 shows that the Robber can never get above the Cop so it remains to show that the Robber can not force repetitions of positions. Note that $F_i(x) = F_{i+1}(x)$ except for that vertex $v_i \in G_i \setminus G_{i+1}$.

Let A_i, $i = 1, 2, \ldots, n$ be v-branches of S_v. Assume that the Cop has moved above the Robber and the Cop knows which freepath the Robber is on. Consider the consecutive corresponding moves $x_1, x_2, \ldots x_{n+1}$ and $c_1, c_2, \ldots, c_{n+1}$ by the Robber and Cop respectively where $c_i, x_i \in A_i$ and $c_i \succ x_i$ for all i. Also note that $c_i \rightarrow c_{i+1}$ and $x_i \sim x_{i+1}$, $i = 1, 2, \ldots n$. (See Figure 4.) Since $x_i \prec c_i$ then there exists $c_i' \sim c_i$ with $c_i' \prec c_i$ for all i. Let j_i be the least index such that $F_{j_i}(x_i) = c_i'$. Since $c_i \rightarrow c_{i+1}$ it follows that $F_{j_i}(x_{i+1}) \succeq c_{i+1}$ since $F_{j_i+1}(x_i) = c_i$. Thus $F_{j_i+1}(x_{i+1}) = F_{j_i}(x_{i+1})$. Consequently $j_i \succ j_{i+1}$ so $j_n \prec j_1$. (Recall $x_n \prec c_n$ by Lemma 5.)

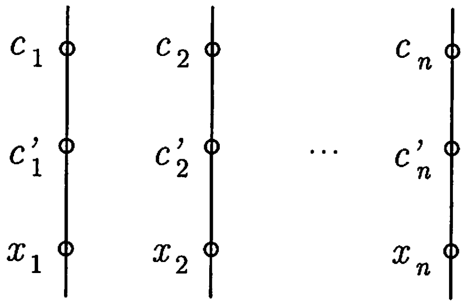

Figure 4: The consecutive moves.

Suppose $x_{n+1} \prec c'_1$. Then $F_{j_n}(x_{n+1}) \preceq F_{j_1}(x_1) = c'_1$. But then $F_{j_n+1}(x_n) = c_n$ by the definition of j_n, $F_{j_n+1}(x_{n+1}) = F_{j_n}(x_{n+1}) \preceq c'_1$ and $F_{j_n}(x_{n+1}) \sim c_n$. This contradicts $c_n \to c_{n+1}$.

The one remaining case is $c_1 \preceq x_{n+1} \preceq c_{n+1}$. (If $x_{n+1} \succeq c_{n+1}$, the Robber will be caught on the next move by Lemma 6.) Since $F_{i_1}(x_1) = c'_i$ it follows that $F_{i_n}(x_{n+1}) = x_{n+1}$. But we have $F_{i_n+1}(x_n) = c_n$ and $F_{i_n+1}(x_{n+1}) = F_{i_n}(x_{n+1}) \sim F_{i_n+1}(x_n) = c_n$. This contradicts $c_n \to c_{n+1}$.

So whenever the Cop moves to a v-branch, he is on the same v-branch as the Robber, but he is strictly lower than his last position on the v-branch. Therefore since the graph is finite, the Robber is eventually caught. □

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