The fine structure of $(v, 3)$ directed triple systems: $v \equiv 2 \pmod 3$*

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ABSTRACT: The fine structure of a directed triple system of index λ is the vector $(c_1, c_2, \cdots, c_\lambda)$, where c_i is the number of directed triples appearing precisely i times in the system. We determine necessary and sufficient conditions for a vector to be the fine structure of a directed triple system of index 3 for $v \equiv 2$ (mod 3).

1 Introduction and definitions

Let a, b and c be three distinct elements. A *transitive* or *directed triple* (a, b, c) is a set of three ordered pairs of the form $\{(a, b), (b, c), (a, c)\}$. A *directed triple system* of order v and *index* λ, or (v, λ)DTS, is a pair $(V, \mathcal{D})$ where V is a v-set of elements, and $\mathcal{D}$ is a collection of directed triples (called *blocks*) on V, with the property that every *ordered* pair (x, y) of elements of V appears in precisely λ of the directed triples. Directed triple systems have been studied extensively, often under the name "transitive triple systems". The necessary condition for a (v, λ)DTS to exist is simply that the number of ordered pairs $\lambda v(v-1)$ occurring in blocks be divisible by three. Hence, we require $v \equiv 0,\ 1 \pmod 3$ for $\lambda \equiv 1,\ 2 \pmod 3$, and we require only $v \neq 2$ for $\lambda \equiv 0 \pmod 3$. It is well-known that these conditions are also sufficient for the existence of (v, λ)DTSs (see Colbourn and Rosa [5] for a recent survey).

The *fine structure* of a directed triple system of index λ is the vector $(c_1, c_2, \cdots, c_\lambda)$, where c_i is the number of directed triples appearing pre-

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cisely i times in the system. Colbourn, Mathon, Rosa and Shalaby [3] determined the fine structure of threefold triple systems ($(v, 3, 3)$BIBDs) for $v \equiv 1$ or 3 (mod 6), and Colbourn, Mathon and Shalaby [4] determined the fine structure of threefold triple systems for $v \equiv 5$ (mod 6). In [6] the third author found the fine structure of *balanced ternary designs* with block size 3, index 3 and $\rho_2 = 3$. The necessary and sufficient conditions for the vector (c_1, c_2, c_3) to be the fine structure of a $(v, 3)$DTS with $v \equiv 0$ or 1 (mod 3) was settled by the third author in [7].

In this paper we study the fine structure of $(v, 3)$DTSs for $v \equiv 2$ (mod 3). Indeed, we determine the necessary and sufficient conditions for a vector to be the fine structure of a directed triple system of index 3 for $v \equiv 2$ (mod 3). Since any two of $\{c_1, c_2, c_3\}$ determine the third, we use a more convenient notation for the fine structure: (t, s) is said to be the fine structure of a $(v, 3)$DTS if $c_2 = t$ and $c_3 = \lfloor v(v-1)/3 \rfloor - s$. We first need to know the pairs (t, s) which can possibly arise as fine structures. We define $\text{Adm}(v) = \{(t, s) \mid 0 \leqslant t \leqslant s \leqslant \lfloor v(v-1)/3 \rfloor,\ s \notin \{0, 1, 2, 3, 4, 5\}\}$ and use the notation Fine(v) for the set of fine structures which actually arise in $(v, 3)$DTSs. We prove the following result:

Main Theorem Fine(v) =Adm(v) for all $v \equiv 2$ (mod 3).

We make use of *group divisible designs* and *directed triple systems with holes* in the next sections. A group divisible design, GDD$(K, \lambda, M; v)$, is a collection of subsets of size $k \in K$, called blocks, chosen from a v-set, where the v-set is partitioned into disjoint subsets (called groups) of size $m \in M$ such that each block contains at most one element from each group, and any two elements from distinct groups occur together in λ blocks. If $M = \{m\}$ and $K = \{k\}$, for convenience we write GDD$(k, \lambda, m; v)$. A $(v + h, \lambda)$*DTS with a hole of size* h is a pair $(V \cup H, \mathcal{D})$, where V is a v-set, H is a h-set, $V \cap H = \emptyset$, and $\mathcal{D}$ is a collection of directed triples on $V \cup H$, with the property that no ordered pair (x, y) with $x, y \in H$ appears in the directed triples and every other ordered pair (x, y) with $x, y \in V \cup H$ appears in precisely λ of the directed triples.

We will use the well-known construction outlined in the following theorem:

Theorem 1.1 If there exists a GDD$(k, 1, m; v)$, a $(u, 3)$DTS, $u \in \{k, m + h\}$, and an $(m + h, 3)$DTS with a hole h, then there exists a $(v + h, 3)$DTS.

2 Necessary conditions

In this section we show that for every $v \equiv 2$ (mod 3), Fine(v) $\subseteq$Adm(v).

Lemma 2.1 If $(t, s) \in \text{Fine}(v)$ then $0 \leqslant t \leqslant s \leqslant \lfloor v(v-1)/3 \rfloor$.

Proof: To see $t \leqslant s$, note that any ordered pair of elements which appears in doubly repeated triples cannot appear in triply repeated triples, and hence appears in non-repeated triples. So there must be at least t non-repeated triples. It follows that $3t \leqslant c_1 + 2c_2 = v(v-1) - 3c_3 = 2 + 3s$, or $t \leqslant s$. The other two inequalities are trivial. □

Before we show that if $(t, s) \in \text{Fine}(v)$ then $s \notin \{0, 1, 2, 3, 4, 5\}$ we need some more notation and a few results. If T is a set of triples with elements chosen from S, let r_x be the number of triples of T which contain x and let λ_{xy} be the number of triples of T which contain both x and y (in any order). Now suppose T is the set of triples which are not triply repeated in a $(v, 3)$DTS, with $v \equiv 2 \pmod 3$ and elements chosen from a set S, then it follows that

(1) $r_x \equiv 0 \pmod 3$ for all $x \in S$;

(2) $\lambda_{xy} = 0, 3$ or 6 for all $x, y \in S$.

Also, the following elementary results hold in any collection T of triples:

(3) $\sum_{x \in S} r_x = 3|T|$;

(4) $\sum_{x,y \in S} \lambda_{xy} = 3|T|$;

(5) $\sum_{1 \le i \le k} r_{x_i} \le |T| + \sum_{1 \le i < j \le k} \lambda_{x_i y_j}$ for any distinct $x_1, x_2, \ldots, x_k \in S$ (using the inclusion and exclusion principle);

(6) For each $x \in S$, $\sum_{y \in S} \lambda_{xy} = 2r_x$.

Lemma 2.2 If T satisfies (1)-(6) then for distinct $x_1, x_2, \ldots, x_k \in S$, with $\lambda_{x_i x_j} = 0$ for all $i \neq j$, $\sum_{i \in \{1,2,\ldots,k\}} r_{x_i} \neq |T| - 2$.

Proof: If $\sum_{i \in \{1,2,\ldots,k\}} r_{x_i} = |T| - 2$ then by (1) and (2) any element which occurs in the two triples which do not contain any of the x_i must occur 0 (mod 3) times in these two triples which is impossible. □

Lemma 2.3 If T satisfies (1)-(6) and $r_x = 3$ for some x then there is a set T' of triples satisfying (1)-(6) with $|T'| = |T| - 3$.

Proof: By (2), the three triples containing x must contain exactly the same elements and so the remaining $|T| - 3$ triples will satisfy (1)-(6). □

Lemma 2.4 If T satisfies (1)-(6), $r_x = 6$ and $\lambda_{xy} = 6$ for some x, y then there is a set T' of triples satisfying (1)-(6) with $|T'| = |T| - 6$.

Proof: The other elements in the six triples containing x and y must be u, u, u, v, v, v for some $u, v \in S$ (u, v not necessarily distinct). Hence it is easy to see that the remaining $|T| - 6$ triples will satisfy (1)-(6). □

Lemma 2.5 If $(t, s) \in \text{Fine}(v)$ then $s \notin \{0, 1, 2, 3, 4, 5\}$.

Proof: The result follows if we show that when T satisfies (1)-(6) then $|T| \neq 2, 5, 8, 11, 14, 17$.

It is obvious that $|T| \neq 2$ and if $|T| = 5$ then we must have $r_x = 3$ for all x which is impossible (by Lemma 2.3). Also, $|T| \neq 8$ since by Lemma 2.3 we cannot have $r_x = 3$ and by Lemma 2.2 we cannot have $r_x = 6$. If $|T| = 11$ then by Lemma 2.2 we cannot have $r_x = 9$ and by Lemma 2.3 we cannot have $r_x = 3$. Hence, $r_x = 6$ for all x. But we need $\sum_{x \in S} r_x = 33$, a contradiction.

If $|T| = 14$ then by Lemma 2.2 we cannot have $r_x = 12$. Also, by Lemma 2.3 we cannot have $r_x = 3$. Hence, for all x, $r_x = 6$ or 9. By (5), if there exist x, y with $r_x = r_y = 9$ then $\lambda_{xy} = 6$. If there exist x, y with $r_x = r_y = 6$ then $\lambda_{xy} = 3$; since by Lemma 2.4 we cannot have $\lambda_{xy} = 6$ and by Lemma 2.2 we cannot have $\lambda_{xy} = 0$. Similarly if there exist x, y with $r_x = 6$ and $r_y = 9$ then $\lambda_{xy} = 3$; since by Lemma 2.4 we cannot have $\lambda_{xy} = 6$ and by (5) we cannot have $\lambda_{xy} = 0$.

By (3) there are three possibilities to consider:

- $r_{x_1} = r_{x_2} = r_{x_3} = r_{x_4} = 9$ and $r_{x_5} = 6$, which is impossible by (4), since $6 \cdot 6 + 4 \cdot 3 \neq 42$;
- $r_{x_1} = r_{x_2} = 9$ and $r_{x_3} = r_{x_4} = r_{x_5} = r_{x_6} = 6$, which is impossible by (4), since $6 + 6 \cdot 3 + 8 \cdot 3 \neq 42$;
- $r_{x_1} = r_{x_2} = \ldots = r_{x_7} = 6$, which is impossible by (4), since $21 \cdot 3 \neq 42$.

If $|T| = 17$ then by Lemma 2.2 we cannot have $r_x = 15$. Also, by Lemma 2.3 we cannot have $r_x = 3$. Hence, for all x, $r_x = 6, 9$ or 12. By (5), there can be at most one x with $r_x = 12$ and if there exist x, y with $r_x = 12$ and $r_y = 9$ then $\lambda_{xy} = 6$. If there exist x, y with $r_x = 12$ and $r_y = 6$ then by (5) and Lemma 2.4, $\lambda_{xy} = 3$. If there exist x, y with $r_x = r_y = 9$ then by (5) $\lambda_{xy} \neq 0$ and so $\lambda_{xy} = 3$ or 6. If there exist x, y with $r_x = 9$ and $r_y = 6$ then by Lemma 2.2 and Lemma 2.4, $\lambda_{xy} = 3$. If there exist x, y with $r_x = r_y = 6$ then by Lemma 2.4 $\lambda_{xy} \neq 6$ and so $\lambda_{xy} = 0$ or 3.

By (3) there are five possibilities to consider:

- $r_{x_1} = 12, r_{x_2} = r_{x_3} = r_{x_4} = 9$ and $r_{x_5} = r_{x_6} = 6$, which is impossible since (6) tells us that $3+3+3+3+\lambda_{x_5,x_6} = 12$ and so $\lambda_{x_5,x_6} = 0$, but (5) tells us that $r_{x_1} + r_{x_5} + r_{x_6} = 12+6+6 \leq 3+3+\lambda_{x_5,x_6}+17$, and so $\lambda_{x_5,x_6} = 3$;

- $r_{x_1} = 12, r_{x_2} = 9$ and $r_{x_3} = r_{x_4} = \ldots = r_{x_7} = 6$, which is impossible by (6), since $6+3+3+3+3+3 \neq 24$;

- $r_{x_1} = r_{x_2} = \ldots = r_{x_5} = 9$ and $r_{x_6} = 6$, which is impossible by (6), since $5 \cdot 3 \neq 12$;

- $r_{x_1} = r_{x_2} = r_{x_3} = 9$ and $r_{x_4} = r_{x_5} = \ldots = r_{x_7} = 6$, which is impossible since (6) tells us that $\lambda_{x_1,x_2} + \lambda_{x_1,x_3} + 3+3+3+3 = 18$, and so $\lambda_{x_1,x_2} = \lambda_{x_1,x_3} = 3$ (and by symmetry $\lambda_{x_2,x_3} = 3$ also). But then (5) tells us that $r_{x_1} + r_{x_2} + r_{x_3} = 9+9+9 \leq 17+3+3+3$.

- $r_{x_1} = 9$ and $r_{x_2} = r_{x_3} = \ldots = r_{x_8} = 6$, which is impossible by (6), since $7 \cdot 3 \neq 18$.

□

Combining Lemmas 2.1 and 2.5, we have the main result of this section:

Lemma 2.6 For all $v \equiv 2 \pmod 3$, $\mathrm{Fine}(v) \subseteq \mathrm{Adm}(v)$.

3 Small cases

In this section we show that $\mathrm{Fine}(v) = \mathrm{Adm}(v)$ for $v = 5$, 8, 11, 14 and 17. The necessary small designs were obtained computationally, using a variation of a hill-climbing algorithm.

Lemma 3.1 $\mathrm{Fine}(v) = \mathrm{Adm}(v)$ for $v = 5$ and 8.

Proof: See [1] for a $(v,3)$DTS of type $(t,s) \in \mathrm{Adm}(v)$, $v = 5$ and 8. Now the result follows by Lemma 2.6. □

Lemma 3.2 There exist

1. $(5,3)$DTSs with a hole of size 2 of types (t,s), where $(t,s) \in \{(a,b) : 0 \leq a \leq b \leq 6\} \setminus \{(0,1),(1,1),(0,2),(1,2),(1,3),(3,3),(0,4)\}$.

2. $(8,3)$DTSs with a hole of size 2 of types $(0,0)$, $(18,0)$ and $(0,18)$.

3. $(11,3)$DTSs with a hole of size 5 of types $(0,0)$, $(30,0)$ and $(0,30)$.

Proof: See [1] for these designs. □

Lemma 3.3 If $(t,s) \in \text{Fine}(v)$ then $(t,s) \in \text{Fine}(2v+1)$ and $(t,s) \in \text{Fine}(2v+4)$.

Proof: Apply Lemmas 1.1 and 1.2 of [5]. □

Lemma 3.4 $\text{Fine}(v) = \text{Adm}(v)$ for $v = 11$, 14 and 17.

Proof: For $v = 11$ ($v = 17$) we apply Theorem 1.1 with the following ingredients: a GDD$(3,1,3;9)$ (a GDD$(3,1,3;15)$), a $(3,3)$DTS, a $(5,3)$DTS and a $(5,3)$DTS with a hole of size two. The result is a $(11,3)$DTS (or $(17,3)$DTS). Using different types for the ingredients we can find a $(11,3)$DTS (a $(15,3)$DTS) for all types $(t,s) \in \text{Adm}(11)$ ($(t,s) \in \text{Adm}(17)$), except for $(t,s) \in \{(0,7),(1,7),\cdots,(7,7),(9,9)\}$ (for Fine(3), see [7]). For these remaining cases, if $v = 11$ see [1] and if $v = 17$ apply Lemma 3.3 with $v = 8$. So Fine(11) =Adm(11) (Fine(17) =Adm(17), respectively).

For $v = 14$ we apply Theorem 1.1 with a GDD$(4,1,3;12)$, a $(4,3)$DTS, a $(5,3)$DTS and a $(5,3)$DTS with a hole of size two. Using different types for the ingredients we can find a $(14,3)$DTS for all types $(t,s) \in \text{Adm}(14)$, except for $(t,s) \in \{(0,7),\ (1,7),\cdots,(7,7),(0,8),(1,8)\}$ (for Fine(4), see [7]). For the remaining cases see [1]. So Fine(14) =Adm(14), by Lemma 2.6. □

4 Recursive construction

In this section we show that $\text{Fine}(v) = \text{Adm}(v)$ for all $v \equiv 2 \pmod 3$, $v \geqslant 20$. First we need two lemmas which are used to show that we have enough different types of ingredient designs available for use in recursive constructions. Lemma 4.1 is for the $v \equiv 2 \pmod 6$ case and Lemma 4.2 is for the $v \equiv 5 \pmod 6$ case.

Lemma 4.1 Let u, c_2 and c_3 be three non-negative integers such that $u \geq 3$, $0 \leq c_2 + c_3 \leq \alpha$ and $c_3 \notin \{\alpha, \alpha-1, \cdots, \alpha-5\}$, where $\alpha = 12u^2 + 6u$. Then there exist non-negative integer vectors (a_1, a_2, a_3), (b_1, b_2, b_3) and (c_2', c_3') such that:

(1) $a_1 + a_2 + a_3 = 6u(u-1)$;

(2) $b_1 + b_2 + b_3 = u - 1$;

(3) $0 \leq c_2 + c_3 \leq 18$, $c_3 \notin \{13, 14, \cdots, 18\}$; and

(4) $(c_2, c_3) = a_1(0,0) + a_2(2,0) + a_3(0,2) + b_1(0,0) + b_2(18,0) + b_3(0,18) + (c_2', c_3')$.

Proof: The proof is by induction. If $(c_2, c_3) = (0,0)$ we take the vectors $(6u(u-1), 0, 0)$, $(u-1, 0, 0)$ and $(0,0)$. Now suppose that the statement is true for the vector (c_2, c_3). So there exist vectors (a_1, a_2, a_3), (b_1, b_2, b_3) and (c'_2, c'_3) which satisfy (1), (2), (3) and (4). We prove that the statement is true for the vector (c_2+1, c_3), $c_2+1+c_3 \leq \alpha$, and the vector (c_2, c_3+1), $c_2+c_3+1 \leq \alpha$ and $c_3+1 \notin \{\alpha, \alpha-1, \cdots, \alpha-5\}$. Table 4.1 takes care of the vector (c_2+1, c_3) and Table 4.2 takes care of the vector (c_2, c_3+1) if $c'_2 + c'_3 < 18$. When $c'_2 + c'_3 = 18$ first we find the vectors (a'_1, a'_2, a'_3), (b'_1, b'_2, b'_3) and (c''_2, c''_3) which satisfy (1), (2), (3) and (4), moreover $c''_2 + c''_3 < 18$. Then we use Table 4.2. Let $c'_2 + c'_3 = 18$. If $a_1 \neq 0$ we take the vectors (a_1-1, a_2+1, a_3), (b_1, b_2, b_3) and (c'_2-2, c'_3). If $a_1 = 0$, $b_1 \neq 0$ and $a_2 \geq 8$ we take the vectors (a_1+8, a_2-8, a_3), (b_1-1, b_2+1, b_3) and (c'_2-2, c'_3). If $a_1 = 0$, $b_1 \neq 0$ and $a_2 < 8$ we take the vectors (a_1+8, a_2+1, a_3-9), (b_1-1, b_2, b_3+1) and (c'_2-2, c'_3). Finally, if $c'_2 + c'_3 = 18$ and $a_1 = b_1 = 0$ then $c_2 + c_3 + 1 = 2(6u(u-1)) + 18(u-1) + 18 + 1 > \alpha$. □

If	we take
$c'_2 + c'_3 < 18$	(a_1, a_2, a_3), (b_1, b_2, b_3) and (c'_2+1, c'_3)
$c'_2 + c'_3 = 18$, $a_1 \neq 0$	(a_1-1, a_2+1, a_3), (b_1, b_2, b_3) and (c'_2-1, c'_3)
$c'_2 + c'_3 = 18$, $a_1 = 0$, $b_1 \neq 0$, $a_3 \geq 9$	(a_1+8, a_2+1, a_3-9), (b_1-1, b_2, b_3+1) and (c'_2-1, c'_3)
$c'_2 + c'_3 = 18$, $a_1 = 0$, $b_1 \neq 0$, $a_3 < 9$	(a_1+8, a_2-8, a_3), (b_1-1, b_2+1, b_3) and (c'_2-1, c'_3)
$c'_2 + c'_3 = 18$, $a_1 = b_1 = 0$	$c_2 + c_3 + 1 = 2(6u(u-1)) + 18(u-1) + 18 + 1$ $> \alpha$

Table 4.1

If	we take
$c'_2 + c'_3 < 18$, $c'_3 \leq 11$	(a_1, a_2, a_3), (b_1, b_2, b_3) and (c'_2, c'_3+1)
$c'_2 + c'_3 < 18$, $c'_3 = 12$, $a_1 \neq 0$	(a_1-1, a_2, a_3+1), (b_1, b_2, b_3) and (c'_2, c'_3-1)
$c'_2 + c'_3 < 18$, $c'_3 = 12$, $a_1 = 0$, $b_1 \neq 0$, $a_3 \geq 3$	(a_1+3, a_2, a_3-3), (b_1-1, b_2, b_3+1) and $(c'_2, 1)$
$c'_2 + c'_3 < 18$, $c'_3 = 12$, $a_1 = 0$, $b_1 \neq 0$, $a_3 < 3$	(a_1+3, a_2-9, a_3+6), (b_1-1, b_2+1, b_3) and $(c'_2, 1)$
$c'_2 + c'_3 < 18$, $c'_3 = 12$, $a_1 = b_1 = 0$, $a_2 \neq 0$	(a_1, a_2-1, a_3+1), (b_1, b_2, b_3) and (c'_2+2, c'_3-1)
$c'_2 + c'_3 < 18$, $c'_3 = 12$, $a_1 = b_1 = a_2 = 0$, $b_2 \neq 0$	(a_1, a_2+8, a_3-8), (b_1, b_2-1, b_3+1) and (c'_2+2, c'_3-1)
$c'_2 + c'_3 < 18$, $c'_3 = 12$, $a_1 = b_1 = a_2 = b_2 = 0$	$c_3 + 1 = 2(6u(u-1)) + 18(u-1) + 12 + 1$ $= \alpha - 5$

Table 4.2

Using a method of proof similar to that of Lemma 4.1 we also have the following.

Lemma 4.2 Let u, c_2 and c_3 be three non-negative integers such that $u \geq 3$, $0 \leq c_2 + c_3 \leq \alpha$ and $c_3 \notin \{\alpha, \alpha-1, \cdots, \alpha-5\}$, where $\alpha = 12u^2 + 18u + 6$. Then there exist non-negative integer vectors (a_1, a_2, a_3), (b_1, b_2, b_3) and (c_2', c_3') such that:

(1) $a_1 + a_2 + a_3 = 6u(u-1)$;

(2) $b_1 + b_2 + b_3 = u - 1$;

(3) $0 \leq c_2 + c_3 \leq 36$ and $c_3 \notin \{31, 32, \cdots, 36\}$; and

(4) $(c_2, c_3) = a_1(0,0) + a_2(2,0) + a_3(0,2) + b_1(0,0) + b_2(30,0) + b_3(0,30) + (c_2', c_3')$.

Lemma 4.3 If $v = 6u + 2 \pmod 6$, $u \geqslant 3$, then Fine(v) =Adm(v).

Proof: Apply Theorem 1.1 with a GDD$(3,1,6;6u)$ which exists (see for example [2]), a $(3,3)$DTS, an $(8,3)$DTS and an $(8,3)$DTS with a hole of size two. The result is a $(6u+2,3)$DTS. Using different types for the ingredients and applying Lemma 4.1 we find that Adm(v) $\subseteq$ Fine(v). Therefore Fine(v) = Adm(v) by Lemma 2.6. □

Lemma 4.4 If $v = 6u + 5 \pmod 6$, $u \geqslant 3$, then Fine(v) =Adm(v).

Proof: Apply Theorem 1.1 with a GDD$(3,1,6;6u)$, a $(3,3)$DTS, an $(11,3)$DTS and an $(11,3)$DTS with a hole of size five. The result is a $(6u+5,3)$DTS. Using different types for the ingredients and applying Lemma 4.2 we find that Adm(v) $\subseteq$ Fine(v). Therefore Fine(v) = Adm(v) by Lemma 2.6. □

Combining the results of this and the previous section, we have:

Theorem 4.5 Fine(v) =Adm(v) for all $v \equiv 2 \pmod 3$. □

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