Induced-Paired Domination in Graphs

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Abstract

For a graph $G = (V, E)$, a set $S \subseteq V$ is a *dominating set* if every vertex in $V - S$ is adjacent to at least one vertex in S. A dominating set $S \subseteq V$ is a paired-dominating set if the induced subgraph $\langle S \rangle$ has a perfect matching. We introduce a variant of paired-domination where an additional restriction is placed on the induced subgraph $\langle S \rangle$. A paired-dominating set S is an induced-paired dominating set if the edges of the matching are the induced edges of $\langle S \rangle$, that is, $\langle S \rangle$ is a set of independent edges. The minimum cardinality of an induced-paired dominating set of G is the induced-paired domination number $\gamma_{ip}(G)$. Every graph without isolates has a paired-dominating set, but not all these graphs have an induced-paired dominating set. We show that the decision problem associated with induced-paired domination is NP-complete even when restricted to bipartite graphs and give bounds on $\gamma_{ip}(G)$. A characterization of those triples (a, b, c) of positive integers $a \le b \le c$ for which a graph has domination number a, paired-domination number b, and induced-paired domination c is given. In addition, we characterize the cycles and trees that have induced-paired dominating sets.
Key words: Domination, paired-domination number, paired-domination.

1 Introduction

Let $G = (V, E)$ be a graph with order n. The *open neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V | uv \in E(G)\}$ and the *closed neighborhood* $N[v] = N(v) \cup \{v\}$. For a set S, let its closed neighborhood $N[S] = \bigcup_{v \in S} N[v]$. A set S is a *dominating set* if $N[S] = V$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G.

We call a dominating set with minimum cardinality a *γ-set*. The *independent domination number* $i(G)$ is the minimum cardinality of a dominating set of G that is independent; and we call an independent dominating set with minimum cardinality an *i-set*. For a complete review on the topic of domination, see [6, 7].

For an application of domination where security is a concern, we let a vertex represent a location to be guarded or protected and an edge represent a viewing path. In this sample application, a γ-set represents a minimum number of guards necessary to insure that every vertex (location) in the system is guarded. Adding an additional constraint that each guard have a partner led Haynes and Slater [8] to introduce the concept of paired-domination. A set $S \subseteq V$ is a *paired-dominating set* if S is a dominating set and the induced subgraph $\langle S \rangle$ has a perfect matching. The *paired-domination number* $\gamma_p(G)$ is the minimum cardinality of a paired-dominating set of G and a paired-dominating set with minimum cardinality is called a *γ_p-set*. Note that $\gamma_p(G)$ is an even number. Paired-domination is also studied in [2, 5, 9].

In the guard application, we can think of a paired-dominating set as a set of guards able to secure each location and satisfy the requirement that each guard be assigned an adjacent guard as a designated backup. Note that in a paired-dominating set, a guard may be adjacent to guards other than his/her designated backup. Hence communication (signals) between partnered guards is not necessarily private, that is, a signal between partnered guards may be received by another adjacent guard.

Motivated by the potential for problems with interference in communication between a guard and its designated backup, we introduce induced-paired domination. A set $S \subseteq V$ is an *induced-paired dominating set* if it is a dominating set and the induced subgraph $\langle S \rangle$ is a set of independent edges, that is, $|S| = 2t$ and $\langle S \rangle = tK_2$. The minimum cardinality of an induced-paired dominating set of G is the *induced-paired domination number* $\gamma_{ip}(G)$ and an induced-paired dominating set with minimum cardinality is called a *γ_{ip}-set*. In our guard example, an induced-paired dominating set represents a configuration of security guards in which each guard is assigned one other as a designated backup (as in a paired-dominating set), but to avoid conflicts (such as radio interference) between a guard and his/her backup, we require that the backup for each guard be unique. Since among the guards only designated partners are adjacent to each other, we reduce the possibility of conflicts in communication.

Clearly, a graph with an induced-paired dominating set has no isolated vertices. Haynes and Slater [8] observed that any graph without isolated vertices has a paired-dominating set. However, not all isolate-free graphs have an induced-paired dominating set. For example, the cycle C_5 has a paired-dominating set but no induced-paired dominating set since four vertices are necessary in any paired-dominating set and every set of four vertices induces a P_4.

Our observations follow directly from the definitions.

Observation 1 *For every graph G with a γ_{ip}-set, $\gamma(G) \leq \gamma_p(G) \leq \gamma_{ip}(G)$.*

A *support vertex* is a vertex that is adjacent to an endvertex.

Observation 2 *If v is a support vertex of G, then v is in every γ_{ip}-set of G.*

In Section 2 we show that the decision problem associated with induced-paired domination is NP-complete and give bounds on γ_{ip}. In Section 3 we characterize those triples (a, b, c) of positive integers $a \leq b \leq c$ for which there is a graph G having $\gamma(G) = a$, $\gamma_p(G) = b$, and $\gamma_{ip}(G) = c$. In Section 4 those cycles, paths, and trees that have γ_{ip}-sets are characterized and additional bounds on γ_{ip} are given.

2 Induced-paired Domination Number

2.1 Complexity Results

It was shown in [8] that the paired-domination problem is NP-complete. As expected, the induced-paired domination problem is also NP-complete. In fact, using a technique similar to methods of McRae [10], we show that INDUCED-PAIRED DOMINATING SET (IPDS) is NP-complete even when restricted to bipartite graphs.

We use the following well-known NP-complete problem (see [3]).

EXACT COVER BY 3 SETS (X3C)

INSTANCE: A finite set X with $|X| = 3q$ and a collection of 3-element subsets of X.

QUESTION: Does C contain an exact cover for X, that is, a subcollection $C' \subseteq C$ such that every element of X occurs in exactly one member of C'? (Note that if C' exists, then its cardinality is precisely q.)

Next we define IPDS and give a polynomial time reduction of X3C to IPDS.

INDUCED-PAIRED DOMINATING SET (IPDS)

INSTANCE: Graph $G = (V, E)$ and (even) positive integer $k \leq |V|$.

QUESTION: Is $\gamma_{ip}(G) \leq k$?

Theorem 3 *The induced-paired dominating set problem IPDS is NP-complete, even when restricted to bipartite graphs.*

Proof. IPDS is obviously in NP. We will construct a bipartite graph $G = (V, E)$ and a positive integer k from an instance of X3C, such that the X3C instance will have an exact cover if and only if G has an induced-paired dominating set of cardinality at most k. Let $X = \{x_1, x_2, ..., x_{3q}\}$ and $C = \{C_1, C_2, ..., C_t\}$ be an arbitrary instance of X3C. Construct G by creating a vertex x_i for each $x_i \in X$ and a $P_4 = c_j, d_j, e_j, f_j$ for each subset $C_j \in C$. Add communication edges

$$E' = \cup_{j=1}^{t} \{c_j x_i | x_i \in C_j\}.$$

Let $k = 4q + 2(t - q) = 2q + 2t$.

First assume that C' is an exact cover for C. Let $S_1 = \{c_j x_i | C_j \in C'$ and x_i is selected from C_j to be paired with $c_j\} \cup \{e_j f_j | C_j \in C'\}$ and $S_2 = \{d_j e_j | C_j \notin C'\}$. Then it is easy to verify that $S = S_1 \cup S_2$ is an induced-paired dominating set for G with $|S| = 4q + 2(t - q)$. Note that $t \geq q$ since C' is an exact cover.

Now assume that S with $|S| \leq 2q + 2t$ is an induced-paired dominating set for G. In each P_4, either $\{d_j, e_j\} \in S$ or $\{e_j, f_j\} \in S$ to dominate f_j for $1 \leq j \leq t$. Therefore at least two vertices from each $P_4 - \{c_j\}$ must be in S, that is, $2t$ vertices of S come from these P_4's. Note that if $c_j \in S$, it must be paired with some $x_i \in C_j$. Since $|S| \leq 2q + 2t$, we can have at most q of these pairs. Furthermore, these pairs $\{c_j, x_i\}$ must dominate all the x_i's since S is an induced-paired dominating set. Let $C' = \{C_j | c_j \in S\}$. Since each c_j is adjacent to exactly three x_i vertices in G and we have at most q of these c_j in S, each vertex x_i is adjacent to exactly one $c_j \in S$. Thus C' is an exact cover for C. □

2.2 Bounds on $\gamma_{ip}(G)$

We now turn our attention to bounds on $\gamma_{ip}(G)$ for those graphs G that have a γ_{ip}-set. Our first two propositions are straightforward and the proofs are omitted.

Proposition 4 *If a graph G has a γ_{ip}-set, then*

$$2 \leq \gamma_{ip}(G) \leq n$$

and these bounds are sharp.

Proposition 5 *For any graph G, $\gamma_{ip}(G) = n$ if and only if $G = mK_2$.*

Haynes and Slater [8] showed that $\gamma_p(G) \geq \frac{n}{\Delta(G)}$ which yields a lower bound on $\gamma_{ip}(G)$ also.

Proposition 6 *If a graph G has a γ_{ip}-set, then*

$$\gamma_{ip}(G) \geq \frac{n}{\Delta(G)}.$$

Let G' be the graph shown in Figure 1.

Theorem 7 *If G is a connected graph with a γ_{ip}-set and order $n \geq 3$, then $\gamma_{ip}(G) \leq n-1$ with equality if and only if $G \in \{P_3, C_3, P_5, G'\}$.*

Proof. Since G is connected, Proposition 5 gives that $\gamma_{ip}(G) \leq n-1$. It is easy to see that if G is one of the graphs in the theorem, then $\gamma_{ip}(G) = n-1$. For the sufficiency, let S be a γ_{ip}-set such that $\gamma_{ip}(G) = n-1$. If $\gamma_{ip}(G) = 2$, then G is either a P_3 or C_3 and if $\gamma_{ip}(G) = 4$, then $G = P_5$. Assume $\gamma_{ip}(G) \geq 6$ and let x be the vertex not in S. Let $S_p = \{\{u_i, v_i\} | u_i, v_i$ are the vertices paired in $S\}$. Again the connectivity of G and the restriction that S induces a set of independent edges imply that for each pair $\{u_i, v_i\}$, at least one of u_i and v_i is adjacent to x. Hence, G must be the graph G' shown in Figure 1. If $s = 0$, then $t \geq 3$ and $\{u_1, x\}$ is an induced-paired dominating set for G', a contradiction. If $s = 1$, then $t \geq 2$ and again $\{u_1, x\}$ is an induced-paired dominating set for G', a contradiction. Thus, $s \geq 2$. □

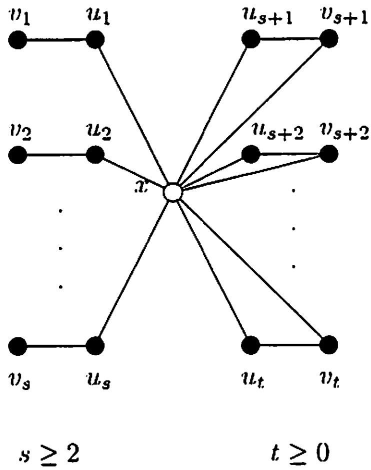

Figure 1: Graph G'.

Corollary 8 *For any graph G, $\gamma_{ip}(G) = n - 1$ if and only if $G = H \cup rK_2$ for $H \in \{P_3, C_3, P_5, G'\}$.*

We note that this upper bound holds and is sharp for paired-domination also. Haynes and Slater [8] added a restriction on minimum degree and showed that for any graph G with $\delta(G) \geq 2$, $\gamma_p(G) \leq 2n/3$. However, this bound will not suffice for $\gamma_{ip}(G)$. In fact, it can be shown that even with the restriction on minimum degree, $\gamma_{ip}(G)$ can be made arbitrarily larger than $\gamma_p(G)$ and larger than $2n/3$. Consider the graph in Figure 2. The vertices in a γ_{ip}-set are shaded. Note that w is not in any γ_{ip}-set since if it were, then some a_j would not be dominated. But then a_j, $1 \leq j \leq s$, must be paired with either b_j or c_j and u_i, v_i, for $1 \leq i \leq t$, must be in every γ_{ip}-set of G. Therefore, $\gamma_{ip}(G) = 2s + 2t$, $n = 3s + 1 + 2t$, and $2n/3 = (6s + 2 + 4t)/3 < 2s + 2t$ if $t > 1$. Moreover, $\gamma_p(G) = 2s$ where a γ_p-set is $\{a_i, b_i | 2 \leq i \leq s\} \cup \{b_1, w\}$. Therefore, $\gamma_{ip}(G)$ can made arbitrarily larger than $\gamma_p(G)$ by increasing t.

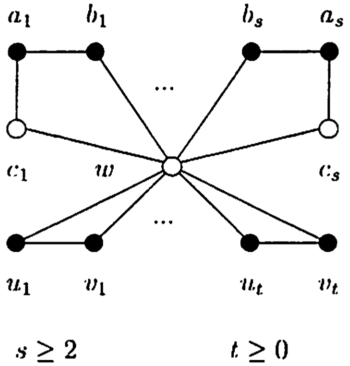

Figure 2: Graph G with $\gamma_{ip}(G) \geq \frac{2n}{3}$.

3 Existence Results

Here we characterize those triples (a, b, c) of positive integers $a \leq b \leq c$ for which there is a graph G with $\gamma(G) = a$, $\gamma_p(G) = b$, and $\gamma_{ip}(G) = c$. Haynes and Slater [8] showed that if $\gamma(G) = a$ and $\gamma_p(G) = b$, then $b \leq 2a$. We consider now the possibilities in two cases $b \leq 2a - 2$ and $b = 2a$.

3.1 $b \leq 2a - 2$

First we show that there is a graph G having $\gamma(G) = a$, $\gamma_p(G) = b$, and $\gamma_{ip}(G) = c$ for any triple of positive integers (a, b, c) where $a \leq b \leq c$, b is even, and $b \leq 2a - 2$.

Theorem 9 *For any triple (a, b, c) of positive integers such that $a \leq b \leq c$, b is even, and $b \leq 2a-2$, there exists a graph G having $\gamma(G) = a$, $\gamma_p(G) = b$, and $\gamma_{ip}(G) = c$.*

Proof. Let a, b, and c be integers where $a \leq b \leq c$, b is even, and $b \leq 2a-2$. For $a < b \leq c$ or $a = b = c$, we show that the graph G in Figure 3 has $\gamma(G) = a$, $\gamma_p(G) = b$, and $\gamma_{ip}(G) = c$. Let S be a set containing x, all the support vertices of G, and one additional vertex from each C_4 subgraph. Since each C_4 subgraph requires at least two vertices to dominate it and each endvertex or its neighbor must be in any dominating set, it is straightforward to see that S is a γ-set for G and $|S| = 1 + b - a + 1 + 2(2a - b - 2)/2 = a$. Next we form a γ_p-set for G as follows. Begin with S and pair the adjacent support vertices. Then pair x with a neighbor in some C_4 subgraph. Now since x can only be paired with one vertex, we must add to S an additional vertex from each of the remaining C_4 subgraphs in order to pair the vertices. Thus we have a γ_p-set and $\gamma_p(G) = \gamma(G) + (b - a + 1 - 1) = b$. Notice that x cannot be in any γ_{ip}-set. Hence to form an induced-paired dominating set S', the support vertices paired as above, two adjacent vertices (paired) from each of the $C_4 - x$ subgraphs, and the remaining two vertices (paired) from each of the K_3 subgraphs containing x must be in S', implying that $|S'| = \gamma_{ip}(G) = \gamma_p(G) + 2(c - b)/2 = c$. This construction has restrictions that either $a = b = c$ or $b \geq a + 1$.

To complete the proof, let $a = b < c$. Using a similar argument, it is straightforward to show that the graph H shown in Figure 4 has $\gamma(H) = \gamma_p(H) = 2(a-2)/2) + 2 = a$ and $\gamma_{ip}(H) = a - 2 + 2 + c - a = c$ for any triple of positive integers (a, b, c) where $a = b < c$, b is even, and $b \leq 2a - 2$. □

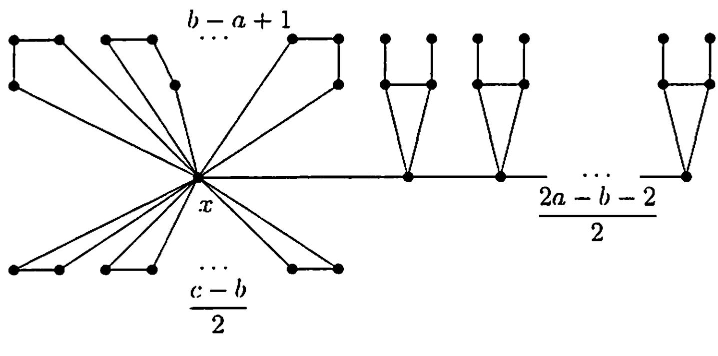

Figure 3: Graph G with $\gamma(G) = a$, $\gamma_p(G) = b$, and $\gamma_{ip}(G) = c$.

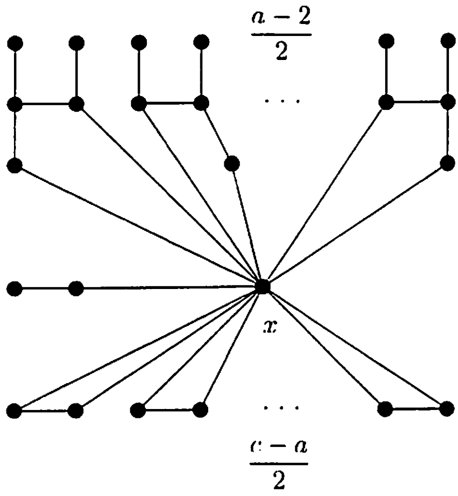

Figure 4: Graph H with $\gamma(H) = a = \gamma_p(H)$, and $\gamma_{ip}(H) = c$.

3.2 $b = 2a$

First we give some more terminology and a couple of known results that will be useful. The *private neighborhood* of a vertex v with respect to a set S is $N[v] - N[S - \{v\}]$. A vertex in the private neighborhood of v is called a *private neighbor* of v with respect to S. Bollobás and Cockayne [1] established the following property of minimum dominating sets.

Theorem 10 [1] *If G is a graph with no isolated vertices, then G has a γ-set S such that each vertex of S has a private neighbor in $V - S$.*

Here we are concerned with graphs G for which $\gamma(G) = a$ and $\gamma_p(G) = b = 2a$. Haynes and Slater [8] determined a property of these graphs.

Theorem 11 [8] *If a graph G has $\gamma_p(G) = 2\gamma(G)$, then every γ-set of G is an i-set of G.*

Using these results, we prove the following.

Theorem 12 *If G has no induced C_5 and $\gamma_p(G) = 2\gamma(G)$, then G has a γ_{ip}-set and $\gamma_{ip}(G) = \gamma_p(G) = 2\gamma(G)$.*

Proof. Let G be a graph with no induced C_5 subgraph and $\gamma_p(G) = 2\gamma(G)$. If any γ_p-set is a γ_{ip}-set, then we are finished. Thus, suppose no γ_p-set is an induced matching. From Theorem 11, we have that every γ-set is a an i-set. Furthermore, from Theorem 10 we know that at least one of these γ-sets, say S, is a private dominating set, that is, each vertex in S has a private neighbor in $V - S$. Let $S = \{u_1, u_2, ..., u_\gamma\}$ and choose a private neighbor $v_i \in V - S$ for each $u_i \in S$ such that the number of edges in $\langle\{v_1, v_2, ..., v_\gamma\}\rangle$ is minimized. Let S' be this set of private neighbors. Now $P = S \cup S'$ is a γ_p-set of G. If there are no edges in $\langle S'\rangle$, then P is also a γ_{ip}-set and the theorem holds, so assume $v_1v_2 \in E(G)$. If $Q = P - \{u_1, v_1, u_2, v_2\} \cup \{v_1, v_2\}$ dominates G, then Q is a paired-dominating set with cardinality less that $\gamma_p(G)$, a contradiction. Hence there exists $w \in N(u_1) \cup N(u_2)$ that is not dominated by $P - \{u_1, u_2\}$. If w is adjacent to both u_1 and u_2, then $\langle u_1, w, u_2, v_1, v_2\}\rangle$ is an induced C_5, contradicting our assumption that G is C_5-free. Therefore, without loss of generality, let w be a private neighbor of u_1 with respect to S. But then $(S' - \{v_1\} \cup \{w\}) \cup S$ is a paired-dominating set of G formed from S and a private neighbor for each vertex in S such that $\langle(S' - \{v_1\}) \cup \{w\}\rangle$ has fewer edges than $\langle S'\rangle$, contradicting our choice of S'. □

Let G be a graph with $\gamma(G) = a$ and $\gamma_p(G) = 2a$. If G has no induced C_5, then Theorem 12 holds implying that G has an induced-paired dominating set and $\gamma_{ip}(G) = c = \gamma_p(G) = b = 2\gamma(G) = 2a$. If G has an induced C_5, then we must consider the following possibilities:

(1) G with no γ_{ip}-set,

(2) G has a γ_{ip}-set and $\gamma_{ip}(G) > \gamma_p(G)$, or

(3) G has a γ_{ip}-set and $\gamma_{ip}(G) = \gamma_p(G)$.

The cycle C_5 is an example of (1). For an example of graphs satisfying (2) and (3), see Figure 5.

4 Characterizations of Graphs with γ_{ip}-sets

The paired-domination number for paths P_n and cycles C_n, $\gamma_p(P_n) = \gamma_p(C_n) = 2\lceil n/4\rceil$, is given in [8]. This result can be extended to induced-paired domination for all cycles except C_5 and all paths as shown below. The straightforward proofs are omitted.

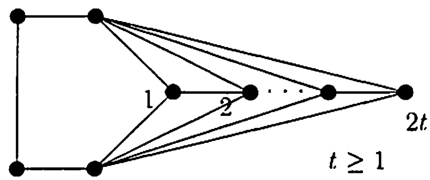

Figure 5: A graph G with $\gamma_p(G) = 2\gamma(G) = 4 \leq \gamma_{ip}(G) = 2 + 2t$.

Theorem 13 *(1) Every cycle C_n, $n \neq 5$, has an induced-paired dominating set and $\gamma_{ip}(C_n) = 2\lceil n/4 \rceil$.*

(2) Every nontrivial path P_n has an induced-paired dominating set and $\gamma_{ip}(P_n) = 2\lceil n/4 \rceil$.

4.1 Trees

From Observation 2, it is apparent that not all trees have γ_{ip}-sets. For example, the tree in Figure 6 has no γ_{ip}-set, since each support vertex must be in every γ_{ip}-set and the graph induced by these three vertices is a P_3. We proceed, in a manner similar to that of Gavlas and Schultz [4], to constructively characterize the trees T which have a γ_{ip}-set.

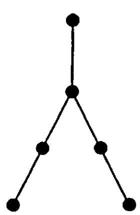

Figure 6: A tree with no induced-paired dominating set.

Theorem 14 *A nontrivial tree T has an induced-paired dominating set S if and only if T can be recursively obtained from P_2, that is $P_2 = T_1, T_2, ..., T_k = T$ where each T_i has an induced-paired dominating S_i, and T_{i+1} is obtained from T_i using one of the following operations:*

1. *Add a new vertex adjacent to a vertex in $S_i \subseteq V(T_i)$ and let $S_{i+1} = S_i$.*
2. *Identify vertex u of a path $P_3 = u, w, x$ with a vertex $u' \notin S_i$ and let $S_{i+1} = S_i \cup \{w, x\}$.*
3. *Identify vertex u of a path $P_4 = u, v, w, x$ with a vertex $u' \notin S_i$ and let $S_{i+1} = S_i \cup \{w, x\}$.*

Proof. Assume T is a tree obtained recursively from $P_2 = T_1, T_2, ..., T_k = T$ where each T_i has an induced-paired dominating set S_i and T_{i+1} is obtained from T_i using operation 1, 2, or 3. It is easy to see that S_k is an induced-paired dominating set for T.

Next we proceed by induction on the order of T. If T is a tree of order $n = 2$, then $T = P_2$ and can be obtained from P_2 using the given operations. Assume that any tree T' of order $n' < n$, with an induced-paired dominating set S', can be obtained recursively from P_2 using the three operations. Let T be a tree with order $n \geq 3$ and an induced-paired dominating set S.

Case 1: The tree T has an endvertex $x \notin S$. Then x must be adjacent to a vertex in S. Thus $T' = T - \{x\}$ is a tree of order $n' < n$, which has an induced-paired dominating set $S' = S$, and T' can be recursively obtained from P_2. Therefore, T can be recursively obtained from P_2 by using operation 1 on T'.

Case 2: All endvertices of T are in S. This implies that any suppport vertex is adjacent to exactly one endvertex and is paired with it in S. Furthermore, any longest path in T must have at least five vertices. Now consider a longest path P in T, where $P = xwvuv_1v_2...v_kz$ as shown in Figure 7. Necessarily, $x \in S$, $w \in S$, and $v \notin S$. Suppose deg $w > 2$. Then any vertex a adjacent to w must be an endvertex of T or P is not a longest path. But this contradicts that w is adjacent to exactly one endvertex. Hence deg $w = 2$. If deg $v = 2$ and $u \in S$, then $T' = T - \{w, x\}$ is a tree with $S' = S - \{w, x\}$ that can be recursively obtained from P_2 using the given operations. Thus T can be recursively obtained from P_2 using operation 2 on T'. Otherwise, if $u \notin S$, then $T' = T - \{v, w, x\}$ is a tree with $S' = S - \{w, x\}$ that can be recursively obtained from P_2 using the given operations and T can be recursively obtained from T' using operation 3. If deg $v > 2$, then any vertex b adjacent to v cannot be an endvertex since $v \notin S$, and hence cannot be paired with b. Then since P is a longest path, $c \in N(b)$ must be an endvertex. Thus $\{b, c\} \subseteq S$. So once again the tree $T' = T - \{b, c\}$ is a tree with $S' = S - \{b, c\}$ that can be recursively obtained from P_2 using the three operations. Furthermore, the tree T can be obtained recursively from P_2 by using operation 2 on T'. Hence, a tree T with an induced-paired dominating set can be recursively obtained from P_2 using only the three operations given in the theorem. □

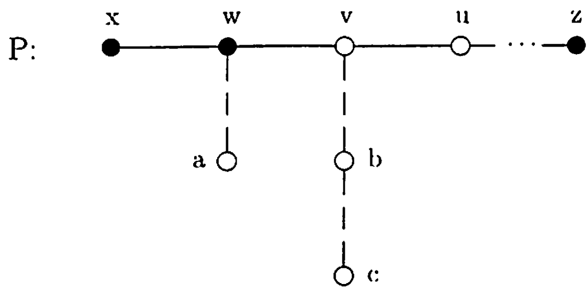

Figure 7: A longest path.

For example, in Figure 8, we give a tree that was constructed using the three operations described in Theorem 14 (the shaded vertices represent the induced-paired dominating set determined by the construction). The tree was constructed from a $P_2 = ab$ in which both vertices must both be in S. We then use operation 1 to add the new vertex u_1. Next we use operation 3 to add the vertices $\{v_1, w_1, x_1\}$, followed by operation 1 to add the vertex u_2. Operation 2 was used to add the vertices $\{w_2, x_2\}$. Finally, the tree was completed by using operation 1 to add the vertex u_3. Note that the order of the operations to produce this tree is not unique. Also, notice that the induced-paired dominating set determined by the construction is a γ_{ip}-set, but this is not necessarily the case in general, that is, the construction may produce an induced-paired dominating set that does not have minimum cardinality.

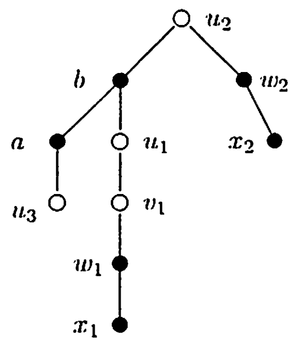

Figure 8: A tree constructed using Theorem 14.

4.2 Caterpillars

A *caterpillar* is a tree with the property that the removal of its endvertices results in a path. The resulting path $u_1u_2...u_s$ is referred to as the *spine* of the caterpillar and the endvertices are called the *legs* of the caterpillar. A sequence of nonnegative integers $(t_1, t_2, ..., t_s)$ where t_i is the number of endvertices (legs) adjacent to u_i for $s \geq 2$ is associated with T. Both this sequence and its reverse sequence define T. The *code* C of the caterpillar is the larger of these two sequences. For example, the code of the caterpillar in Figure 9 is (23021).

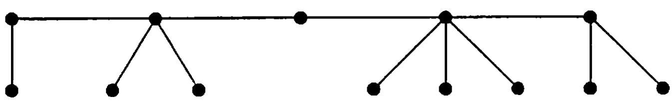

Figure 9: A caterpillar.

Although we have given a constructive characterization for trees, we now give a more descriptive characterization for caterpillars in terms of their codes. To aid in the presentation of the characterization, we introduce some additional notation. Let R_s denote a caterpillar with $s \geq 2$ vertices on its spine. For convenience, we write the code C of R_s as a sequence of zeros and variables where a variable represents a nonzero entry. For a variable a representing a positive integer in the code, let v_a denote the vertex of R_s associated with a. Also, let v_a^L be the vertex to the left of v_a on the spine, v_a^R be the vertex to the right of v_a, and v_a' be an endvertex adjacent to v_a. A *block of zeros* is a maximal set of consecutive zeros in C. Let z_i denote the number of zeros in the ith block of zeros. Thus for the code $(2\underbrace{000}35\underbrace{000000}21)$, $z_1 = 3$, and $z_2 = 6$.

Let $F = F_1 \cup F_2 \cup F_3$ be the collection of subcodes where

$$F_1 = \{(abc) | a, b, c \geq 1\}$$

$$F_2 = \{(ab000cd) | a, b, c, d \geq 1\}$$

$$F_3 = \{(ab000c_1000c_2...000c_t000de) | a, b, c_i, d, e \geq 1, t \geq 1\}.$$

We are now ready for the characterization.

Theorem 15 *A caterpillar R_s has a γ_{ip}-set if and only if its code does not contain a subcode from F.*

Proof. If a caterpillar with code C has a γ_{ip}-set S, then it follows from Theorem 14 that C has no subcode from F. Next let R_s be a caterpillar with code C that does not contain any subcode from F. We will show that R_s has an induced-paired dominating set S. Since C has no subcode from F, C does not have three consecutive nonzero integers. If $z_i = 3$ for each zero block i of C (and C has no subcode from F), then C is one of the following codes:

$$\{(c_1000c_2000...c_{M-1}000c_M) | c_i \geq 1, M > 1\}$$

$$\{(c_1000c_2000...c_j000ab000c_{j+1}000...c_{M-1}000c_M) | \; a, b, c_i \geq 1, j \geq 1, M \geq 1\}$$

$$\{(ab000c_1000c_2...000c_M) | a, b, c_i \geq 1, M \geq 1\}$$

$$\{(c_1000c_2...000c_M000ab) | a, b, c_i \geq 1, M \geq 1\}.$$

For each code, we give an induced-paired dominating set S for R_s.

For code $C = (c_1000c_2000...c_{M-1}000c_M)$, let $S = \{v_{c_1}, v_{c_1}^R, v_{c_i}^L, v_{c_i} | 2 \leq i \leq M\}$.

For code $C = (c_1000c_2000...c_j000ab000c_{j+1}000...c_{M-1}000c_M)$, let $S = \{v_{c_i}, v_{c_i}^R | 1 \leq i \leq j\} \cup \{v_{c_i}^L, v_{c_i} | j+1 \leq i \leq M\} \cup \{v_a, v_b\}$.

For code $C = (ab000c_1000c_2...000c_M)$, let $S = \{v_{c_i}^L, v_{c_i} | 1 \leq i \leq M\} \cup \{v_a, v_b\}$.

For code $C = (c_1000c_2...000c_M000ab)$, let $S = \{v_{c_i}, v_{c_i}^R | 1 \leq i \leq M\} \cup \{v_a, v_b\}$.

Thus, if $z_i = 3$ for all zero blocks i in C, we are finished. Hence, either $z_i \neq 3$ for all zero blocks i or there exists $z_i = 3$ and $z_j \neq 3$ for some zero blocks i and j.

Case 1: Assume $z_i \neq 3$ for all zero blocks i in C. Again, we determine an induced-paired dominating set S. By Observation 2, every vertex v_a associated with a nonzero entry $a \in C$ must be in S. Consider such vertices and pair all the adjacent ones in S. Now for the remaining unpaired $v_a \in S$, add v_a' to S and pair it with v_a. Next consider the zero block i of C. If $z_i = 1$ or $z_i = 2$, then the vertices associated with the ith block are dominated by S. If $z_i \geq 4$, then the vertices associated with the z_i zeros in the ith block form a P_{z_i} path. Let v_L, v_R be the two endvertices in the induced subgraph $\langle P_{z_i} \rangle$. Consider the path $P_{z_i} - \{v_L v_R\}$. Since $z_i \geq 4$, $P_{z_i} - \{v_L v_R\}$ is a path with at least two vertices (v_L and v_R are dominated by S). By Theorem 13, $P_{z_i} - \{v_L v_R\}$ has an induced-paired dominating set S_i. Adding the vertices of S_i to S for each $z_i \geq 4$ yields an induced-paired dominating set for R_s.

Case 2: Assume $z_i = 3$ and $z_j \neq 3$ for zero blocks i and j in C. Again we build an induced-paired dominating set S. First select all subcodes C_j of C such that C_j is a code of a caterpillar, C_j has at least one zero block, every zero block i in C_j has $z_i = 3$, and C_j is maximal in length. Assume there are $k \geq 1$ of these subcodes and dominate each of these subcaterpillars with codes C_j, $1 \leq j \leq k$, as we did above for caterpillars with such codes. Let S be the union of these induced-paired dominating sets. Next select all subcodes C_j of C (that have not been previously selected) such that C_j is a code of a caterpillar, every zero block i in C_j has $z_i \neq 3$ and C_j is maximal in length (note that C_j does not necessarily have a zero block, that is, $C_j = (ab)$). Dominate these caterpillars as we did in Case 1 and add these dominating sets to S. Finally, if for any nonzero entry a, v_a is not already in S, then add v_a and v'_a to S (pairing them).

Note that S induces a set of independent edges and the only vertices that may not be dominated correspond to zero blocks separating the subcodes whose vertices are dominated by S. Consider such a block i. By the way we selected the subcodes, $z_i \neq 3$. Furthermore, the vertices associated with the first and last zero in each block are dominated by S. Hence if $z_i = 1$ or $z_i = 2$, we are finished. If $z_i \geq 4$, dominate the path as we did in Case 1. Adding these dominating sets to S gives an induced-paired dominating set for R_s. □

Finally, we give bounds on $\gamma_{ip}(R_s)$.

Theorem 16 *For any caterpillar R_s which has an induced-paired dominating set,*

$$2\left\lceil \frac{s+2}{4} \right\rceil \leq \gamma_{ip}(R_s) \leq s+1$$

and these bounds are sharp.

Proof. For the lower bound, let P_{s+2} be a longest path of R_s (this path contains all vertices on the spine and two endvertices). Since $\gamma_{ip}(P_{s+2}) = 2\left\lceil \frac{s+2}{4} \right\rceil$ by Theorem 13 and adding endvertices cannot decrease the induced-paired domination number, we have $\gamma_{ip}(R_s) \geq 2\left\lceil \frac{s+2}{4} \right\rceil$. To see that this bound is sharp, let $R_s = P_n, n \equiv 0 \pmod 4$.

For the upper bound, let S be an induced-paired dominating set for R_s. Since a vertex on the spine dominates more vertices than an endvertex dominates, $\gamma_{ip}(R_s)$ is maximum for a caterpillar with the code $C = (a_1 0 a_2 0 \ldots a_{t-1} 0 a_t)$ where $t = \dfrac{s+1}{2}$. By Observation 2, $v_{a_i} \in S$. Furthermore, v'_{a_i} is paired with v_{a_i} in S. So $|S| = 2\left(\frac{s+1}{2}\right) = s+1$. Therefore, $\gamma_{ip}(R_s) \leq 2\left(\frac{s+1}{2}\right) = s+1$. □

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