On the Hull Number of a Graph

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ABSTRACT

For two vertices u and v of a connected graph G, the set $H(u,v)$ consists of all those vertices lying on a $u-v$ geodesic in G. Given a set S of vertices of G, the union of all sets $H(u,v)$ for $u,v \in S$ is denoted by $H(S)$. A convex set S satisfies $H(S) = S$. The convex hull $[S]$ is the smallest convex set containing S. The hull number $h(G)$ is the minimum cardinality among the subsets S of $V(G)$ with $[S] = V(G)$. When $H(S) = V(G)$, we call S a geodetic set. The minimum cardinality of a geodetic set is the geodetic number $g(G)$. It is shown that every two integers a and b with $2 \leq a \leq b$ are realizable as the hull and geodetic numbers, respectively, of some graph. For every nontrivial connected graph G, we find that $h(G) = h(G \times K_2)$. A graph F is a minimum hull subgraph if there exists a graph G containing F as induced subgraph such that $V(F)$ is a minimum hull set for G. Minimum hull subgraphs are characterized.

1 Introduction

The *distance* $d(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest $u-v$ path in G. See the books [1, 5] for graph theory notation and terminology. A $u-v$ path of length $d(u,v)$ is called a $u-v$ *geodesic*. The set $H(u,v)$ consists of of all vertices lying on some

$u - v$ geodesic of G, while for $S \subseteq V(G)$,

$$H(S) = \bigcup_{u,v \in S} H(u,v).$$

The set S is *convex* if $H(S) = S$. The *convex hull* $[S]$ is the smallest convex set containing S. Obviously, $[S]$ is the intersection of all convex sets containing S. The convex hull $[S]$ of S can also be formed from the sequence $\{H^k(S)\}$, $k \geq 0$, where $H^0(S) = S, H^1(S) = H(S)$, and $H^k(S) = H(H^{k-1}(S))$ for $k \geq 2$. From some term on, this sequence must be constant. Let p be the smallest number such that $H^p(S) = H^{p+1}(S)$. Then $H^p(S)$ is the convex hull $[S]$. This sequential construction was utilized in [6] in connection with the " geodetic iteration number ".

A set S of vertices of G is called a *hull set* of G if $[S] = V(G)$, and a hull set of minimum cardinality is a *minimum hull set of* G. If S is a hull set of G and $u, v \in S$, then each vertex of every $u - v$ geodesic of G belongs to $H(S)$. This observation as stated next will be used on several occasions.

Lemma 1.1 *Let S be a minimum hull set of a connected graph G and let $u, v \in S$. If w ($\neq u, v$) lies on a $u - v$ geodesic in G, then $w \notin S$.*

The cardinality of a minimum hull set in G is called the *hull number* $h(G)$. This number was introduced by Everett and Seidman [4], who characterized graphs having some particular hull numbers and who obtained a number of bounds for the hull numbers of graphs. The hull number of median graphs was determined by Mulder [7]. (A connected graph G is a *median graph* if, for every three vertices u, v, w of G, there is a unique vertex lying on a geodesic between each pair of u, v, w.)

Clearly, $2 \leq h(G) \leq n$ for every connected graph G of order $n \geq 2$. For example, in the graph G of Figure 1, Let $S_1 = \{u, z\}$. Since $[S_1] = S_1$ which is a proper subset of $V(G)$, it follows that S_1 is not a hull set of G. On the other hand, let $S_2 = \{x, y\}$. Since $H(S_2) = \{x, y, u, v, w\}$ and $H(H(S_2)) = V(G)$, we have $[S_2] = V(G)$ and so $h(G) = 2$.

Since the hull number of a disconnected graph is the sum of the hull numbers of its components, we are only concerned with connected graphs. While the graph G of Figure 1 has the smallest possible hull number for a nontrivial connected graph, we note that for each integer n, there is only one connected graph of order n having the largest possible hull number, namely n, and that is the complete graph K_n.

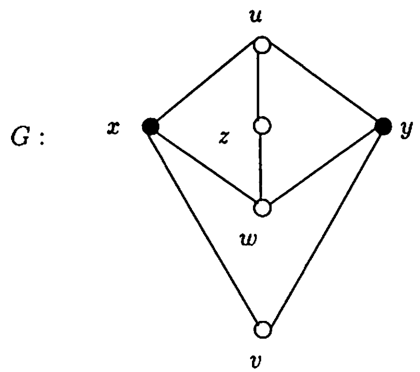

Figure 1: The graph G

2 Relating the Hull Number to the Geodetic Number

A related concept was introduced in [2, 3]. A set S of vertices of G is a *geodetic set* if $H(S) = V(G)$. A geodetic set of minimum cardinality is a *minimum geodetic set*, and this cardinality is the *geodetic number* $g(G)$.

Obviously, $h(G) \leq g(G)$ for every nontrivial connected graph G. The hull and geodetic numbers of the graph G_1 of Figure 1 are both 2 as $\{u, v\}$ is both a minimum hull set and minimum geodetic set of G_1. While the hull number of the graph G_2 of Figure 1 is 2 and $\{x, y\}$ is a minimum hull set, its geodetic number is 3 and $\{x, y, z\}$ is a minimum geodetic set.

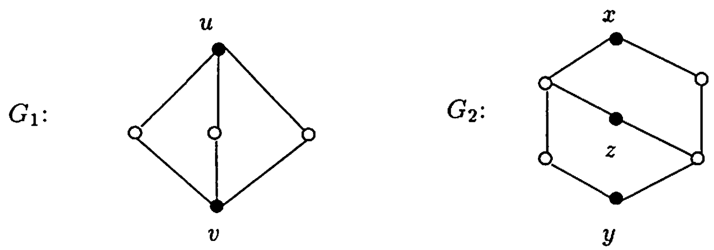

Figure 2: The graphs G_1 and G_2

Next, we show that every pair a, b of integers with $2 \leq a \leq b$ are realizable as the hull and geodetic numbers, respectively, of some graph. To show this, we first state a lemma, which is an immediate consequence of the trivial observation that each end-vertex v of a graph is an end-vertex of every geodesic containing v.

Lemma 2.1 *Each end-vertex of a graph G belongs to every hull set and every geodetic set of G.*

Theorem 2.2 *For every pair a, b of integers with $2 \leq a \leq b$, there exists a connected graph G such that $h(G) = a$ and $g(G) = b$.*

Proof. If $a = b$, then $G = K_a$ has the desired properties. Thus we may consider that $a < b$. We construct a graph G with the required hull and geodetic numbers. For each i with $1 \leq i \leq b - a$, let G_i be the graph of Figure 3.

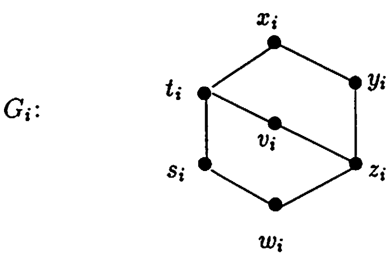

Figure 3: The graph G_i

Then G is obtained from the graphs G_i by adding vertices u_j for $1 \leq j \leq a - 1$ and edges (1) $x_i x_{i+1}$ and $w_i w_{i+1}$ for $1 \leq i \leq b - a - 1$ and (2) $w_{b-a} u_j$ for $1 \leq j \leq a - 1$. The graph G is shown in Figure 4.

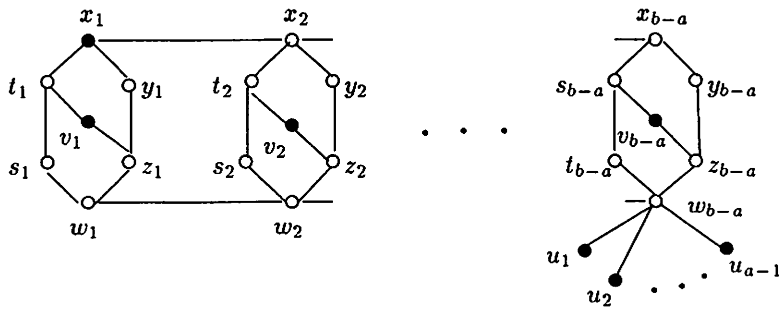

Figure 4: The graph G with $h(G) = a$ and $g(G) = b$

We first show that $h(G) = a$. By Lemma 2.1, the end-vertices u_1, u_2, $\cdots$, u_{a-1} of G belong to every hull set of S of G and so $h(G) \geq a - 1$. Since $[\{u_1, u_2, \cdots, u_{a-1}\}] \neq V(G)$, it follows that $h(G) \geq a$. Let $S_1 = \{x_1, u_1,$

$u_2, \cdots, u_{a-1}\}$. Since $H(S_1) = V(G) - \{v_1, v_2, \cdots, v_{b-a}\}$ and $H(H(S_1)) = V(G)$, it follows that $[S_1] = V(G)$. Therefore, $h(G) = |S_1| = a$.

Next we show that $g(G) = b$. Since

$$S_2 = \{x_1, u_1, u_2, \cdots, u_{a-1}, v_1, v_2, \cdots, v_{b-a}\}$$

is a geodetic set, $g(G) \leq b$. Suppose, to the contrary, that there exists a geodetic set W of G with $|W| < b$. Certainly, $\{u_1, u_2, \cdots, u_{a-1}\} \subset W$. We claim that W contains at least one vertex in each G_i for all i with $1 \leq i \leq b-a$. Suppose, to the contrary, that for some i, $V(G_i) \cap W = \emptyset$. Observe that v_i does not lie on any $x-y$ geodesic in G for $x, y \notin V(G_i)$. This implies that $v_i \notin H(W)$, which contradicts the fact that W is a geodetic set. Therefore, W contains at least one vertex from each G_i and so $|W| \geq (a-1)+(b-a) = b-1$. If $|W| = b-1$, then W contains exactly one vertex from each G_i $(1 \leq i \leq b-a)$. Since v_i only lies on those geodesics having v_i as one of its end-vertices or having both end-vertices belonging to G_i, it follows that $v_i \in W$. This implies that $W = \{u_1, u_2, \cdots, u_{a-1}, v_1, v_2, \cdots, v_{b-a}\}$. Since $x_1 \notin H(W)$, we have a contradiction. Therefore, $g(G) = b$. ∎

The hull number of a graph has certain properties that are also possessed by the geodetic number of a graph. To show these, we need some additional definitions. For a vertex v of G, the *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the *radius* $rad\ G$ and the maximum eccentricity is its *diameter* $diam\ G$. Two vertices u and v with $d(u,v) = diam\ G$ are called *antipodal vertices*. One might be led to believe that every minimum hull set must contain some pair of antipodal vertices, but this is not so, as the graph in Figure 1 shows.

In [2] it was shown that if G is a connected graph of order $n \geq 2$ and diameter d, then $g(G) \leq n-d+1$. The following theorem is an immediate consequence of this result and was also established in [4].

Theorem 2.3 *If G is a connected graph of order $n \geq 2$ and diameter d, then $h(G) \leq n-d+1$.*

The proofs of the next two theorems are similar to those for the geodetic number given in [2] and are omitted.

Theorem 2.4 *For positive integers r, d, and $k \geq 2$ with $r \leq d \leq 2r$, there exists a connected graph G with $rad\ G = r$, $diam\ G = d$, and $h(G) = k$.*

Theorem 2.5 *If n, d, and k satisfy $2 \leq d < n$, $2 \leq k < n$, and $n-d-k+1 \geq 0$, then there exists a graph of order n, diameter d, and hull number k.*

3 The Hull Number of a Cartesian Product

We now determine the hull number of the Cartesian product of a nontrivial connected graph with K_2. We first consider the hull numbers of the cycles C_n of order $n \geq 3$. When n is even, the set of any two antipodal vertices of C_n is a hull set. But when n is odd, no two vertices form a hull set. Since there exists a 3-vertex hull set,

$$h(C_n) = \begin{cases} 2 & n \text{ even} \\ 3 & n \text{ odd} \end{cases}$$

It is easy to verify that the hull number of the Cartesian product $C_n \times K_2$ is exactly that of C_n. In fact, we shall see that every nontrivial connected graph G has this property. Some notation and definitions are useful to help simplify our proof of this fact. Let $G \times K_2$ be formed from two copies G_1 and G_2 of G with $V(G_i) = V_i$, $i = 1, 2$. For a set S of vertices in $G \times K_2$, let $\pi(S)$ be the *projection* of S onto V_1, that is, $\pi(S)$ is the union of those vertices of G_1 belonging to S and those in G_1 corresponding to the vertices of G_2 that are in S. Certainly, $\pi(S) \subseteq \pi(H(S))$. First, we state a lemma.

Lemma 3.1 *Let $G \times K_2$ be formed from two copies G_1 and G_2 of G. If $S \subseteq V(G \times K_2)$, then $H(\pi(S)) = \pi(H(S))$.*

Proof. First, we show that $H(\pi(S)) \subseteq \pi(H(S))$. Let $v_1 \in H(\pi(S))$. Since $\pi(S) \subseteq \pi(H(S))$, we may regard that $v_1 \notin \pi(S)$. Then v_1 lies on some $x_1 - y_1$ geodesic P_1 in G_1, where $v_1 \neq x_1, y_1$ and $x_1, y_1 \in S_1$. We consider three cases.

Case 1. $x_1, y_1 \in S$. Then P_1 is the same $x_1 - y_1$ geodesic in $G \times K_2$ that contains v_1. So $v_1 \in \pi(H(S))$.

Case 2. Exactly one of x_1 and y_1 belongs to S, say $x_1 \in S$ and $y_1 \notin S$. Let $y_2 \in S$ be the corresponding vertex of y_1 and P be the $x_1 - y_2$ path obtained by adding y_1y_2 and y_2 to P_1. Then P contains v_1, where $x_1, y_2 \in S$. Therefore, $v_1 \in \pi(H(S))$.

Case 3. $x_1, y_1 \notin S$. Let x_2, y_2 and v_2 be the vertices corresponding to x_1, y_1, and v_1 in G_2, respectively, and let P_2 be the corresponding $x_2 - y_2$ geodesic of P_1 in G_2. Since $v_2 \in V(P_2) \subseteq H(S)$, it follows that v_1 belongs to $\pi(H(S))$.

Therefore, $H(\pi(S)) \subseteq \pi(H(S))$.

Next, we show that $\pi(H(S)) \subseteq H(\pi(S))$. Let $v_1 \in \pi(H(S))$. If $v_1 \in \pi(S)$, then $v_1 \in H(\pi(S))$. So we assume that $v_1 \notin \pi(S)$. We consider two cases.

Case 1. $v_1 \in H(S) \cap V_1$. Then v_1 lies on some $x - y$ geodesic P in $G \times K_2$, where $x, y \in S$. Since $v_1 \in V_1$, at least one of x and y belongs to V_1, which implies that at least one of x and y belongs to $\pi(S)$. If $x, y \in \pi(S)$, then

P is an $x - y$ geodesic in G_1 that contains v_1. So $v_1 \in H(\pi(S))$. Thus exactly one of x and y belongs to $\pi(S)$, say $x \in \pi(S)$ and $y \notin \pi(S)$. Let $y_1 \in \pi(S)$ be the corresponding vertex of y in G_1. Then v_1 lies on some $x - y_1$ geodesic in G_1, where $x, y_1 \in \pi(S)$. Hence $v_1 \in H(\pi(S))$.

Case 2. $v_1 \notin H(S) \cap V_1$. Let v_2 be the corresponding vertex of v_1 in V_2. Then v_2 lies on some $x - y$ geodesic P in $G \times K_2$, where $x, y \in S$. Since $v_2 \in V_2$, at least one of x and y belongs to V_2. If $x, y \in V_2$, then P is an $x - y$ geodesic in G_2 that contains v_2. Then the corresponding path P_1 of P in G_1 contains v_1, which implies that v_1 belongs to $H(\pi(S))$. Thus exactly one of x and y belongs to V_2, say $x \notin V_2$ and $y \in V_2$. Let $y_1 \in \pi(S)$ be the corresponding vertex of y in G. Then v_1 lies on an $x - y_1$ geodesic in G_1. Since $x, y_1 \in \pi(S)$, it follows that $v_1 \in H(\pi(S))$. ■

Corollary 3.2 *Let $G \times K_2$ be formed from two copies G_1 and G_2 of G. If S is a hull set of $G \times K_2$, then $\pi(S)$ is a hull set of G_1.*

Proof. Since S is a hull set of $G \times K_2$, it follows that $[S] = V_1 \cup V_2$. Hence there exists a positive integer k such that $H^k(S) = V_1 \cup V_2$. So $\pi(H^k(S)) = V_1$. By k applications of Lemma 3.1, we have $H^k(\pi(S)) = \pi(H^k(S)) = V_1$. Therefore, $\pi(S)$ is a hull set of G_1. ■

Theorem 3.3 *For every nontrivial connected graph G,*

$$h(G) = h(G \times K_2)$$

Proof. By applying Corollary 3.2 to the case where S is a minimum hull set of $G \times K_2$, we have that $h(G) \leq h(G \times K_2)$.

It remains to show that $h(G \times K_2) \leq h(G)$. Let $G \times K_2$ be formed from two copies G_1 and G_2 of G. Suppose that $S_1 = \{u_1, u_2, \cdots, u_k\}$ is a minimum hull set of G_1 and $S_2 = \{v_1, v_2, \cdots, v_k\}$ is the set corresponding to S_1 in G_2. Let $S = \{v_1, u_2, \cdots, u_k\}$. Thus $|S| = |S_1|$. We show that S is hull set of $G \times K_2$, which will imply that $h(G \times K_2) \leq h(G)$.

Next we show that $S_1 \cup S_2 \subseteq H(S)$, which will imply that S is a hull set of $G \times K_2$. Since $S \subseteq H(S)$, we need only show that $u_1 \in H(S)$ and $S_2 - \{v_1\} \subseteq H(S)$. Let P_1 be a $u_2 - u_1$ geodesic and let P be the path obtained from P_1 by adding u_1v_1 and v_1. Then P is an $u_2 - v_1$ geodesic in $G \times K_2$ containing u_1. Since $u_2, v_1 \in S$, it follows that $u_1 \in H(S)$. For $2 \leq i \leq k$, let P_i be a $v_i - v_1$ geodesic in G_2 and let P_i' be obtained from P_i by adding u_iv_i and u_i. Then P_i' is a $u_i - v_1$ geodesic in $G \times K_2$ containing v_i. Since $u_i, v_1 \in S$, we have that $S_2 - \{v_1\} \subset H(S)$ and so S is a hull set of $G \times K_2$. ■

4 Minimum Hull Subgraphs

In this section, we present a characterization of graphs of order n having hull number $n-1$. If v is a vertex of a graph G such that $\langle N(v)\rangle$ is complete, then v is an end-vertex of every geodesic containing v. This observation extends Lemma 2.1 and gives us the following result, due to Everett and Seidman [4] in the case of hull sets. Corollary 4.1 is an immediate consequence of Theorem A.

Theorem A If v is a vertex of a graph G such that $\langle N(v)\rangle$ is complete, then v belongs to every hull set and every geodetic set of G.

Corollary 4.1 *The hull and geodetic numbers of a tree T are the number of end-vertices in T. In fact, the set of all end-vertices of T is the unique hull set and the unique geodetic set of T.*

By Corollary 4.1, the star $K_{1,n-1}$ of order $n \geq 3$, which can also be expressed as $K_1 + \overline{K}_{n-1}$, has hull number $n-1$. Our characterization of graphs of order n having hull number $n-1$ shows that the class of stars can be generalized to produce all graphs having hull number $n-1$. First, we present a lemma this is an immediate consequence of Theorem 2.3.

Lemma 4.2 *If G is a connected graph of order $n \geq 2$ such that $h(G) = n-1$, then G has diameter 2.*

We now present a characterization of graphs with hull number $n-1$.

Theorem 4.3 *A connected graph G of order $n \geq 3$ has hull number $n-1$ if and only if*

$$G = (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_r}) + K_1$$

where r (≥ 2), $n_1, n_2, \cdots, n_r$ are positive integers with $n_1 + n_2 + \cdots + n_r = n-1$.

Proof. By Theorem A, $G = (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_r}) + K_1$ has hull number $n-1$ and, in fact, the vertex set of $K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_r}$ is the unique minimum hull set of G.

For the converse, assume that G is a connected graph of order $n \geq 3$ such that $h(G) = n-1$. By Lemma 4.2, *diam* $G = 2$. Let S be a minimum hull set of G, where $V(G) - S = \{v\}$. Then $H(S) = V(G)$, which implies that S is also a minimum geodetic set of G.

We claim that v is adjacent to every vertex in S. If $x, y \in S$ and $xy \notin E(G)$, then since *diam* $G = 2$, there exists a vertex of G mutually adjacent to x and y. By Lemma 1.1, this vertex cannot be in S, so v is adjacent to x and y. Hence, if v is not adjacent to some vertex u in S, then u must be adjacent to all other vertices of S. Since S is a geodetic set, however, v lies on some $s - t$ geodesic, necessarily of length 2, where $s, t \in S$. Since $us, ut \in E(G)$, it follows that $u \notin S$ by Lemma 1.1, which is a contradiction. Hence, as claimed, v is adjacent to all vertices of S.

To show that $G = (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_r}) + K_1$, it only remains to verify that if $xy, yz \in E(G)$, where $x, y, z \in S$, then $xz \in E(G)$. However, this follows again by Lemma 1.1. ■

Theorem 4.3 can also be derived from a theorem of Everett and Seidman [4].

Since $h(G) \leq g(G)$ and the graph K_n is the only graph with geodetic number n, we have the next observation.

Corollary 4.4 *A connected graph G of order $n \geq 3$ has geodetic number $n - 1$ if and only if*

$$G = (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_r}) + K_1$$

where r (≥ 2), $n_1, n_2, \cdots, n_r$ satisfy $n_1 + n_2 + \cdots + n_r = n - 1$.

We now introduce a concept that will turn out to be closely connected to the result already stated in this section. A graph F is called a *minimum hull subgraph* if there exists a graph G containing F as an induced subgraph such that $V(F)$ is a minimum hull set. For example, consider the graphs F and G in Figure 8. Since $S = \{u, v, w\}$ is a minimum hull set of G, and F is an induced subgraph of G, it follows that F is a minimum hull subgraph of the graph G. Also, by Theorem 4.3, for positive integers $n_1, n_2, \cdots, n_r$, where $r \geq 2$, the graph $K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_r}$ is a minimum hull subgraph. We shall see shortly that this example illustrates the general situation.

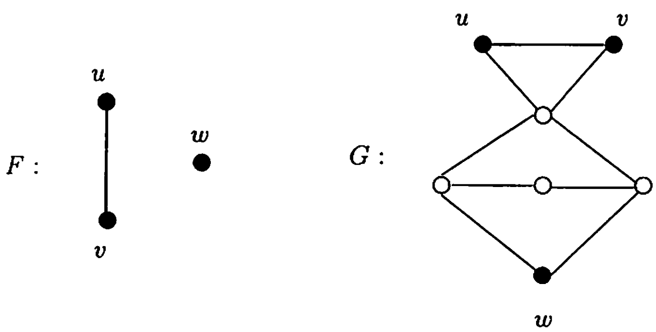

Figure 5: A minimum hull subgraph F

A related concept was defined in [2]. A graph F is a *minimum geodetic subgraph* if there exists a graph G containing F as an induced subgraph such that $V(F)$ is a minimum geodetic set in G. Theorem B, which was verified in [2], gives a characterization of minimum geodetic subgraphs.

Theorem B A nontrivial graph F is a minimum geodetic subgraph if and only if every vertex of F has eccentricity 1 or no vertex of F has eccentricity 1.

We now determine exactly which graphs are minimum hull subgraphs.

Theorem 4.5 *A nontrivial graph F is a minimum hull subgraph of some connected graph if and only if every component of F is complete.*

Proof. First, let F be a minimum hull subgraph of a graph G. Assume, to the contrary, that F contains a component that is not complete. Then there exist $u, v \in V(F)$ such that $d_F(u, v) = 2$. Let w be a vertex of F lying on some $u - v$ geodesic in F. Then $w \in V(F)$ by Lemma 1.1, producing a contradiction.

We now verify the converse. Let F be a connected graph, every component of which is complete. If F is connected, then $V(F)$ is the minimum hull subgraph of F itself. Otherwise, $F = K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_r}$ for positive integers $n_1, n_2, \cdots, n_r$, where $r \geq 2$. Let $G = K_1 + F$. Then as we have seen, $V(F)$ is the unique minimum hull subgraph of G. ∎

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