# The Lattice Polynomial of a Graph

Jonathan Wiens
and
Kara L. Nance
*Department of Mathematical Sciences*
*University of Alaska Fairbanks*
*Fairbanks, AK 99775 USA*

April 5, 2000

ABSTRACT

This paper defines a new graph invariant by considering the set of connected induced subgraphs of a graph and defining a polynomial whose coefficients are detemined by this partially ordered set of subgraphs. We compute the polynomial for a variety of graphs and also determine the effects on the polynomial of various graph operations.

## 1 Introduction

Let G be a finite simple graph. Throughout this paper we shall use the notation of [1]. $V(\mathrm{G})$ is the set of vertices of G and $|\mathrm{G}|$ is the order of G and is the number of vertices of G. $E(\mathrm{G})$ is the set of edges of G and we shall denote the edge connecting vertices $v$ and $w$ by $vw$.

We are interested in the relationship between G and the set of connected induced subgraphs of G. Recall that if G is a graph and H is a subgraph of G then we say H is an induced subgraph of G if every edge in G connecting vertices in H is also an edge of H. For instance, $\mathrm{K}_4$, the complete graph on 4 vertices does NOT contain $C_4$, the cycle graph consisting of 4 vertices, as an induced subgraph. It is very important to note that virtually all of the subgraphs we shall consider will be induced subgraphs.

If G is a graph, then let $P(\mathrm{G})$ be the set of all connected induced subgraphs of G, and call it the induced subgraph poset. For the sake of convenience we shall say that the empty graph is also connected. Equivalently, $P(\mathrm{G})$ is the collection of all connected (or empty) subsets of $V(\mathrm{G})$,

**ARS COMBINATORIA 57(2000), pp. 139-149**

the vertex set of G. (This definition extends trivally to multigraphs, but extending our results to this more general setting will needlessly complicate many of our statements without changing their substance, so we shall remain in the simple graph setting.)

We can define a partial order on $P(\mathrm{G})$ by inclusion: if $\mathrm{H}, \mathrm{K} \in P(\mathrm{G})$ then we say $\mathrm{K} \leq \mathrm{H}$ if and only if $V(\mathrm{K}) \subseteq V(\mathrm{H})$. $P(\mathrm{G})$ has minimal element $\emptyset$ (which we usually denote by 0) and if G is connected then $P(\mathrm{G})$ has G as its maximal element. We shall write $\mathrm{H} < \mathrm{K}$ or $\mathrm{H} \subset \mathrm{K}$ only if $V(\mathrm{H})$ is a proper subset of $V(\mathrm{K})$. We shall define our graph invariant via the Möbius function on this partially ordered set (or poset).

**Definition 1.1** *If $P$ is a finite poset with minimal element* 0, *then a Möbius function on $P$ is an integer-valued function on $P$ such that* $\mu(0) = 1$ *and* $\sum_{0 \leq \mathrm{K} \leq \mathrm{H}} \mu(\mathrm{K}) = 0$ *for each* $\mathrm{H} \in P$.

The Möbius function may be recursively computed and is clearly uniquely defined. More precisely:

**Theorem 1.2** *If $P$ is a finite partially ordered set with minimal element 0, and if $f$ is an integer-valued function on $P$ such that* $f(0) = 1$ *and for each* H *in $P$ we have* $\sum_{0 \leq \mathrm{K} \leq \mathrm{H}} f(\mathrm{K}) = 0$ *then $f$ is the Möbius function on $P$.*

$P(\mathrm{G})$ has a natural ranking, given by the number of vertices in each induced subgraph. We use this ranking to define our polynomial.

**Definition 1.3** *Let* G *be any finite graph,* $P(\mathrm{G})$ *be the set of connected induced subgraphs of* G *and $\mu$ the Möbius function on* $P(\mathrm{G})$. *The lattice polynomial of* G *is*

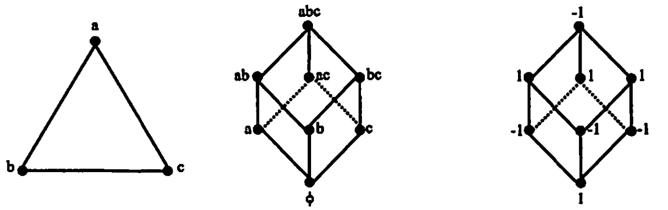
$$\pi(\mathrm{G}, t) = \sum_{\mathrm{H} \in P(\mathrm{G})} \mu(\mathrm{H}) t^{|\mathrm{H}|}$$

This polynomial should more properly be called the characteristic polynomial of the poset $P(\mathrm{G})$, but since the characteristic polynomial of a graph has been previously defined in graph theory, we use the term lattice polynomial. We also adopt the following (somewhat redundant) notation and define $\mu(\mathrm{G})$ to equal the coefficient of $t^{|\mathrm{G}|}$ in $\pi(\mathrm{G}, t)$. If G is connected then $\mu(\mathrm{G})$ is just the value of the Möbius function at the maximal element G of $P(\mathrm{G})$, while if G is not connected then $\mu(\mathrm{G}) = 0$. This convention will allow us (when convenient) to write $\pi(\mathrm{G}, t) = \sum_{\mathrm{K} \subseteq \mathrm{G}} \mu(\mathrm{K}) t^{|\mathrm{K}|}$ where the sum is taken over *all* induced subgraphs of G.

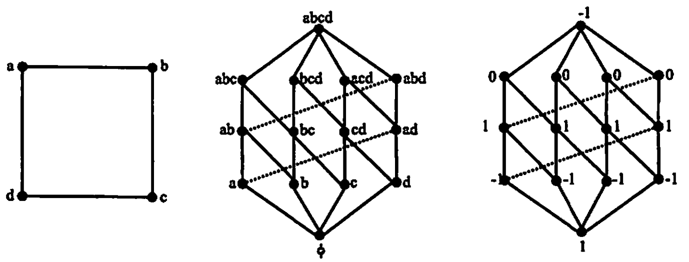
Clearly both $\pi(\mathrm{G},t)$ and $\mu(\mathrm{G})$ are invariants of the graph type. The goal of this paper is to explore the properties of these two invariants. In particular we shall determine the effects that certain graph operations such as join and addition or contraction of edges and vertices have on the lattice polynomial.

# 2 Empirical Results

In this section we give several examples of the lattice polynomial on small graphs, and derive a few of its properties. Figure 2.1 shows the induced subgraph poset and the values of the Möbius function on the poset for the graphs $K_3$ and $C_4$. Here we visualize the induced subgraph poset as a (directed) graph whose edges (directed upward) record inclusion of the connected induced subgraphs.




$K_3$ $\quad$ $P(K_3)$ $\quad$ $\pi(K_3,t) = 1 - 3t + 3t^2 - t^3$




$C_4$ $\quad$ $P(C_4)$ $\quad$ $\pi(C_4,t) = 1 - 4t + 4t^2 - t^4$

FIGURE 2.1 *The lattice polynomial of $K_3$ and $C_4$.*

A glance at the definition of möbius function shows that if $P(\mathrm{G})$ has a maximal element, then the sum of the coefficients of $\pi(\mathrm{G},t)$ is zero, i.e. $\pi(G,1) = 0$ if G is connected. Our study of the lattice polynomial begins with a simple observation concerning graphs which are not connected.

**Proposition 2.1** *If $G$ is a disjoint union of the graphs $\mathrm{G}_1$ and $\mathrm{G}_2$ then $\pi(\mathrm{G},t) = \pi(\mathrm{G}_1,t) + \pi(\mathrm{G}_2,t) - 1$ and hence the number of connected components of* G *is* $1 - \pi(G,1)$

**Proof:** $P(G_1 \cup G_2)$ is just $P(G_1)$ and $P(G_2)$ joined at the minimal element. Hence for degree $k > 0$ the coefficient of $t^k$ is

$$\sum_{\substack{H \in P(G) \\ |H|=k}} \mu(H) = \sum_{\substack{H \in P(G_1) \\ |H|=k}} \mu(H) + \sum_{\substack{H \in P(G_2) \\ |H|=k}} \mu(H).$$

and so $\pi(G,t) = \pi(G_1,t) + \pi(G_2,t) - 1$. Clearly, then, if G is a disjoint union of the connected graphs $G_1, \ldots, G_n$ then

$$\pi(G,t) = -(n-1) + \sum_{k=1}^{n} \pi(G_k,t)$$

Since 1 is a root of the lattice polynomial for connected graphs we have $1 - \pi(G,1) = n$. ∎

In light of this proposition, we shall concentrate on connected graphs. For these graphs the sum of the coefficients of $\pi$ is zero, a fact we shall use repeatedly in our later results. Before revealing these results in Section 3, we shall list the lattice polynomials of some small graphs. ($E_n$ is the graph with $n$ vertices and no edges, $P_n$ is the path of length $n$, $C_n$ is the circle graph with $n$ vertices, $K_n$ is the complete graph on $n$ vertices and $K_{m,n}$ is the complete bipartite graph with $m+n$ vertices.)

1. $\pi(E_n,t) = 1 - nt$
2. $\pi(K_1,t) = 1 - t$
3. $\pi(K_2 = P_1, t) = 1 - 2t + t^2$
4. $\pi(K_3 = C_3, t) = 1 - 3t + 3t^2 - t^3$
5. $\pi(P_2,t) = 1 - 3t + 2t^2$
6. $\pi(K_4,t) = 1 - 4t + 6t^2 - 4t^3 + t^4$
7. $\pi(C_4 = K_{2,2}, t) = 1 - 4t + 4t^2 - t^4$
8. $\pi(P_3,t) = 1 - 4t + 3t^2$
9. $\pi(K_{2,3},t) = 1 - 5t + 6t^2 - 3t^4 + t^5$

Clear patterns emerge in the above list and we shall later provide the general formulae for the lattice polynomials of these types of graphs. If we denote by $|H|$ the number of induced subgraphs of a graph G which are isomorphic to the graph H, then a review of the values of $\mu$ for all graphs on 5 or fewer vertices shows that

$$\pi(G,t) = 1 - |V(G)|t + |E(G)|t^2 - |K_3|t^3 + (|K_4| - |C_4|)\,t^4 + \cdots$$

As a result we see that several graph properties can be determined by the lattice polynomial, including:

1. If the coefficient of $t^3$ is non-zero then the chromatic number of G is at least three, since G contains a subgraph isomorphic to $K_3$.
2. If the coefficient of $t^4$ is positive then the chromatic number of G is at least four, since G must contain a subgraph isomorphic to $K_4$.

The situation for graphs with 5 or more vertices is much more complex. Figure 2.2 shows all graphs with five vertices having a non-zero value for $\mu$.

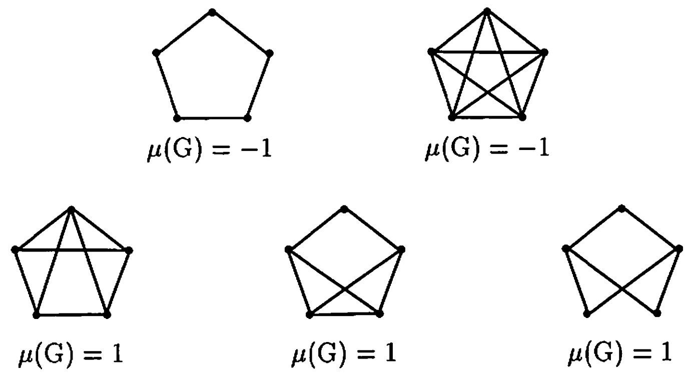



FIGURE 2.2 *A complete list of graphs with 5 vertices and* $\mu(G) \neq 0$.

This classification allows us to add the following to our list of graph properties reflected in the lattice polynomial.

3. If the coefficient of $t^4$ is positive or the coefficient of $t^5$ is positive then G is not outerplanar. This follows, since these conditions require that G contain a subgraph isomorphic to $K_4$ or $K_{2,3}$, which cannot occur in an outerplanar graph (See [1] p. 36 problem 92.)

Further analysis on graphs of 5 or more vertices would reveal additional graph properties contained in the lattice polynomial.

# 3 Properties of the Lattice Polynomial

In this section we discuss several graph operations and their effects on the lattice polynomial.

If G and H are disjoint graphs, then the join of G and H, denoted G+H, is the graph consisting of G ∪ H with an edge connecting each vertex of G with each vertex of H. For instance, $K_{m,n} = E_m + E_n$. Borrowing terminology from topology, call $E_1 + G$ the cone of G and denote it by cone(G). One can easily see that cone$(K_n) = K_{n+1}$.

**Theorem 3.1** *If* G *is any graph then* $\mu(\text{cone(G)}) = -\mu(\text{G})$ *and*

$$\pi(\text{cone(G)}, t) = (1-t)\pi(\text{G}, t).$$

**Proof:** We shall prove the result by induction on $n = |\text{G}|$. The cases $n = 0, 1$ are trivial, so assume $n > 1$. If H is a connected induced subgraph of cone(G) then H is either a connected induced subgraph of G or the cone of a not-necessarily connected induced subgraph of G (noting that the added vertex is the cone of the empty graph), hence

$$\pi(\text{cone(G)}, t) = \sum_{\text{H} \subseteq \text{G}} \mu(\text{H})t^{|\text{H}|} + \sum_{\text{H} \subset \text{G}} \mu(\text{cone(H)})t^{|\text{H}|+1} + \mu(\text{cone(G)})t^{|\text{G}|+1}$$

$$= \pi(\text{G}) - \pi(\text{G})t + \Big(\mu(\text{G}) + \mu(\text{cone(G)})\Big)t^{n+1}$$

and since the cone of G is connected, $t = 1$ is a root of $\pi(\text{cone(G)}, t)$ and we see that $\mu(cone(\text{G})) = -\mu(\text{G})$ and $\pi(\text{cone(G)}, t) = (1-t)\pi(\text{G}, t)$. ■

As an immediate corollary we have

**Corollary 3.2** $\pi(K_n + G, t) = (1-t)^n \pi(\text{G}, t)$, *hence* $\pi(K_n, t) = (1-t)^n$ *and* $\mu(K_n) = (-1)^n$.

However, in general it is not true that $\pi(\text{G} + \text{H}, t) = \pi(\text{G}, t) \cdot \pi(\text{H}, t)$ as one can see by considering $E_2 + E_2 = C_4$.

Next we consider a particular kind of gluing operation. Let G and H be graphs each of which contains a subgraph isomorphic to $K_n$. We again borrow terminology from topology and denote by $\text{G} \cup_{K_n} \text{H}$ the graph obtained by gluing (or identifying) these two subgraphs with each other. Obviously there are many ways of gluing the two $K_n$ together, but since the lattice polynomial does not distinguish these, we shall not worry about the particular way in which they are glued together and shall retain this ambiguous notation.

**Theorem 3.3** $\pi(\text{H}_1 \cup_{K_n} \text{H}_2, t) = \pi(\text{H}_1, t) + \pi(\text{H}_2, t) - \pi(K_n, t)$ *and hence* $\mu(\text{H}_1 \cup_{K_n} \text{H}_2) = 0$ *if* $|\text{H}_1|, |\text{H}_2| > n$.

**Proof:** In the following proof we use $\text{K} \subset \text{G}$ to mean K is a proper induced subgraph of G, while $\text{K} \subseteq \text{G}$ is used when $\text{K} = \text{G}$ is allowed. Theorem 2.1 shows that we need only consider connected graphs. Denote $\text{H}_1 \cup_{K_n} \text{H}_2$ by G. First note that if either $\text{H}_1$ or $\text{H}_2$ is isomorphic to $K_n$ there is nothing to prove.

We prove the result by induction on $n$. The base case is $n = 1$. We prove the base case by induction on $k = |G| - n$. Since neither $H_1$ nor $H_2$ is a $K_n$, the base case is $k = 2$. If $k = 2$ then

$$\begin{aligned}\pi(G,t) &= \sum_{K\subseteq H_1} \mu(K)t^{|K|} + \sum_{K\subseteq H_2} \mu(K)t^{|K|} - \sum_{K\subseteq H_1\cap H_2} \mu(K)t^{|K|} + \mu(G)t^{|G|} \\ &= \pi(H_1,t) + \pi(H_2,t) - \pi(K_n,t) + \mu(G)t^{|G|}\end{aligned}$$

Since $G, H_1, H_2$ and $K_n$ are all connected, 1 is a root of each of their lattice polynomials and hence $\mu(G) = 0$.

If $k > 2$ then

$$\begin{aligned}\pi(G,t) &= \sum_{K\subseteq H_1} \mu(K)t^{|K|} + \sum_{K\subseteq H_2} \mu(K)t^{|K|} - \sum_{K\subseteq H_1\cap H_2} \mu(K)t^{|K|} \\ &\quad + \sum_{K\subset G, K\not\subseteq H_1, K\not\subseteq H_2} \mu(K)t^{|K|} + \mu(G)t^{|G|}\end{aligned}$$

By induction on $k$ the second to last sum is zero hence

$$\pi(G,t) = \pi(H_1,t) + \pi(H_2,t) - \pi(K_n,t) + \mu(G)t^{|G|}$$

and, as before, it follows that $\mu(G) = 0$.

Now assume $n > 1$ and the result holds for $n - 1$. We prove this by induction on $k = |G| - n$ where $k = 2$ is again the base case. If $k = 2$ then

$$\begin{aligned}\pi(G,t) &= \sum_{K\subseteq H_1} \mu(K)t^{|K|} + \sum_{K\subseteq H_2} \mu(K)t^{|K|} - \sum_{K\subseteq H_1\cap H_2} \mu(K)t^{|K|} + \mu(G)t^{|G|} \\ &= \pi(H_1,t) + \pi(H_2,t) - \pi(K_n,t) + \mu(G)t^{|G|}\end{aligned}$$

and it follows that $\mu(G) = 0$.

If $k > 2$ then

$$\begin{aligned}\pi(G,t) &= \sum_{K\subseteq H_1} \mu(K)t^{|K|} + \sum_{K\subseteq H_2} \mu(K)t^{|K|} - \sum_{K\subseteq H_1\cap H_2} \mu(K)t^{|K|} \\ &\quad + \sum_{\substack{K\subset G, K\not\subseteq H_1 \\ K\not\subseteq H_2}} \mu(K)t^{|K|} + \mu(G)t^{|G|} \\ &= \pi(H_1,t) + \pi(H_2,t) - \pi(K_n,t) + \mu(G)t^{|G|}\end{aligned}$$

since by induction on $k$ or on $n$ the second to last sum is zero. Again, since G is connected it follows that $\mu(G) = 0$. ∎

This result does not hold if one tries to extend it to gluing along other types of subgraphs. For instance $C_4$ can be thought of as two $P_2$'s glued along the end vertices, yet $\pi(C_4,t) \neq 2\pi(P_2,t) - \pi(E_2,t)$.

Theorem 3.3 allows us to show that adding a leaf to a graph has a predictable effect on the lattice polynomial. Recall that a leaf is a vertex of degree 1.

**Proposition 3.4** *If G is a graph containing at least three vertices and which contains a leaf then* $\mu(\mathrm{G}) = 0$. *Furthermore if H is any graph and* G *is obtained from H by adding a leaf then* $\pi(\mathrm{G}, t) = \pi(\mathrm{H}, t) - t + t^2$.

**Proof:** If G is obtained by adding a leaf to H then $\mathrm{G} = \mathrm{H} \cup_{K_1} K_2$ and hence $\pi(\mathrm{G}, t) = \pi(\mathrm{H}, t) + \pi(\mathrm{K}_2, t) - \pi(\mathrm{K}_1, t)$. The result follows. ∎

This proposition has a simple corollary.

**Corollary 3.5** *If T is a tree with n vertices then* $\pi(T, t) = 1 - nt + (n-1)t^2$ *and hence* $\mu(T) = 0$ *if* $|T| > 2$. *Furthermore,* $\pi(P_n, t) = 1 - (n+1)t + nt^2$.

**Proof:** Since a tree with $n$ vertices may be built by successively adding leaves to a smaller tree, it follows that $\pi(T, t) = (1 - t) - (n - 1)(t - t^2)$. ∎

**Corollary 3.6** $\pi(C_n, t) = 1 - nt + nt^2 - t^n$.

**Proof:** All of the connected induced proper subgraphs of $C_n$ are trees and hence the coefficients of $t^3$ through $t^{n-1}$ in $\pi(C_n, t)$ are zero. ∎

One can combine Corollary 3.6 with Theorem 3.1 to see that the lattice polynomial for $W_n = \mathrm{cone}(C_n)$, the Wheel graph with $n$-spokes, is $\pi(W_n, t) = 1 - (n+1)t + 2nt^2 - nt^3 - t^n + t^{n+1}$.

**Corollary 3.7** *If G is a maximal outerplanar graph with n vertices then*

$$\pi(\mathrm{G}, t) = 1 - nt + (2n-3)t^2 - (n-2)t^3.$$

**Proof:** A maximal outerplanar graph with $n$ vertices can be constructed by successive gluing a $K_3$ onto an outer edge of a smaller maximal outerplanar graph. Hence $\pi(\mathrm{G}, t) = (1-t)^3 + (n-3)\left((1-t)^3 - (1-t)^2\right) = 1 - nt + (2n-3)t^2 - (n-2)t^3$. ∎

Next, we discuss the effect of deleting an edge from a graph. As we already know, this would decrease the coefficient of $t^2$ in the lattice polynomial by one, but the effect on higher terms of the polynomial is much more subtle. Two situations where the effect on the polynomial is known are given below. One should compare these results with those for the Tutte polynomial given in [1], pp 335-358.

Recall that an edge $e$ of a graph G is a cut edge if $\mathrm{G} \setminus e$ (the graph obtained by deleting the edge $e$ from G) is not connected.

**Corollary 3.8** *If G is a graph and e is a cut edge of G then*

$$\pi(G, t) = \pi(G \setminus e, t) + t^2.$$

**Proof:** Write G as $H_1 \cup_{K_1} K_2 \#_{K_1} H_2$ where the $K_2$ is the cut edge, then

$$\begin{aligned} \pi(G,t) &= \pi(H_1,t) + (1-t)^2 + \pi(H_2,t) - 2(1-t) \\ &= \pi(H_1,t) + \pi(H_2,t) - 1 + t^2 \end{aligned}$$

and so $\pi(G,t) = \pi(G \setminus e, t) + t^2$. ■

If $v$ is a vertex of a graph G, then the neighborhood of $v$, denoted $\Gamma(v)$ is the set of all vertices adjacent to $v$. If $vw$ is an edge of a graph G and $\Gamma(v) \cup \{v\} = \Gamma(w) \cup \{w\}$ then we shall say that $v$ and $w$ have $k$ common neighbors where $k = |\Gamma(v) \setminus \{w\}|$. For instance, any two vertices of $K_n$ have $n-2$ common neighbors.

**Lemma 3.9** *Let G be a graph and $e = vw$ an edge of G such that $v$ and $w$ have common neighbors. If H is a connected induced subgraph of G properly containing $v$ and $w$ then $H \setminus e$ is a connected induced subgraph of $G \setminus e$.*

**Proof:** $H \setminus e$ is clearly an induced subgraph of $G \setminus e$, so that we need only show it is connected. If $H \setminus e$ were not connected then $e$ would be a cut edge of H. This is clearly impossible, for since H is connected there exists a third vertex, $x$ of H adjacent to either $v$ or $w$ and hence both. Thus $xv$ and $xw$ are edges of H and so $e$ is not a cut vertex of H. ■

If $P$ is a partially ordered set and $H, K \in P$ with $H \leq K$ then the interval $[H, K] = \{L \in P \mid H \leq L \leq K\}$.

**Theorem 3.10** *If G is a graph and $e = vw$ is an edge of G such that $v$ and $w$ have $k$ common neighbors, then*

$$\pi(G,t) = \pi(G \setminus e, t) + t^2(1-t)^k.$$

**Proof:** Lemma 3.9 shows that every element of $P(G)$ is also an element of $P(G \setminus e)$ except for the edge $e$. Denote by $S$ the set of neighbors of $v$ and $w$, i.e. $S = \Gamma(v) \setminus \{w\} = \Gamma(w) \setminus \{v\}$. Let $\mu_G$ be the Möbius function on $P(G)$ and define the function $f$ on $P(G \setminus e)$ by

$$f(H) = \begin{cases} \mu_G(H) + (-1)^{|V(H) \cap S|-1} & \text{if } u, v \subset V(H) \subseteq S \\ \mu_G(H) & \text{otherwise} \end{cases}$$

We shall show that $f$ is a Möbius function on $P(G \setminus e)$.

Clearly $\mu(\emptyset) = 1$, so suppose $H \in P(G \setminus e)$. The induced subgraphs of $G \setminus e$ which do not contain both $v$ and $w$ are exactly the induced subgraphs of G which do not contain both $v$ and $w$, hence

$$\sum_{\substack{0 \leq K \leq H \\ K \in P(G \setminus e)}} f(K) = \sum_{\substack{0 \leq K \leq H \\ K \in P(G)}} \mu_G(K) - \mu_G(e) + \sum_{\substack{\{u,v\} \subset V(K) \\ \subseteq S \cap V(H)}} (-1)^{|V(K) \cap S|-1}$$

$$= \sum_{\substack{0 \le K \le H \\ K \in P(G)}} \mu_G(K) - \sum_{\substack{\{u,v\} \subseteq V(K) \\ \subseteq S \cap V(H)}} (-1)^{|V(K) \cap S|}$$

But it is easy to see that the last summand is the sum of the values of a Möbius function on the boolean sublattice $[e, H \cap S]$ of $P(G)$ and hence is zero (see [2] Prop. 2.44), so that $\sum_{\substack{0 \le K \le H \\ K \in P(G \setminus e)}} f(K) = 0$ and $f$ is a Möbius function on $P(G \setminus e)$. As a result

$$\begin{aligned} \pi(G \setminus e, t) &= \sum_{\substack{0 \le K \le H \\ K \in P(G)}} \mu_G(K) t^{|K|} - t^2 - \sum_{\substack{\{u,v\} \subset V(K) \\ \subseteq S \cap V(H)}} (-1)^{|K|} t^{|K|} \\ &= \pi(G, t) - t^2 \sum_{\substack{\{u,v\} \subseteq V(K) \\ \subseteq S \cap V(H)}} (-1)^{|K \cap S| - 2} t^{|K| - 2} \\ &= \pi(G) - t^2 (1 - t)^k \end{aligned}$$

and the result is proven. ■

If G is a graph then let cyl(G) $= E_2 + G$ and call it the cylinder of G. Since the cylinder of a graph is the two-fold cone of a graph with a deleted edge we have:

**Corollary 3.11** *$\pi(\text{cyl}(G, t)) = (1-t)^2 \pi(G, t) - t^2 (1-t)^{|G|}$ and $\mu(\text{cyl}(G)) = \mu(G) - (-1)^{|G|}$. Hence if $\mu(G) = 0$ then $\mu(\text{cyl}(G)) \neq 0$.*

One should now consider the 5 graphs on five vertices given in figure 2.2. Note that those other than $K_5$ and $C_5$ are the cylinders of the graphs with 3 vertices having a zero value for $\mu$.

We can now consider complete bipartite graphs. Recall that $K_{m,n} = E_m + E_n$. Since connected induced subgraphs of complete bipartite graphs are also complete bipartite graphs, we have the following theorem.

**Theorem 3.12** *If $m \ge 2$ and $n \ge 2$ then $\mu(K_{m,n}) = (-1)^{m+n+1}$ and*

$$\pi(K_{m,n}, t) = (1 - mt)(1 - t)^n + (1 - nt)(1 - t)^m - (1 - t)^{m+n}$$

**Proof:** Note that Proposition 3.5 gives the result in the case $m = 1$ or $n = 1$. While Corollary 3.11 gives the result if $m = 2$ or $n = 2$. We prove the result by induction on $m$. Now assume $m > 2$. We shall prove this by induction on $n$, so assume $n > 2$. Now each of the connected induced subgraphs are also complete bipartite, and $K_{k,l}$ occurs $\binom{m}{k}\binom{n}{l}$ times. So that

$$\pi(K_{m,n}, t) = 1 - (m+n)t + mnt^2 + \sum_{k=2, l=2}^{m,n} \mu(K_{k,l}) \binom{m}{k}\binom{n}{l} t^{k+l}$$

$$
\begin{aligned}
&= (1-mt)(1-nt) + \left(\mu(K_{m,n}) + (-1)^{m+n}\right) \\
&\quad + \sum_{k=2,l=2}^{m,n} (-1)^{k+l+1} \binom{m}{k}\binom{n}{l} t^{k+l} \\
&= (1-mt)(1-nt) + \left(\mu(K_{m,n}) + (-1)^{m+n}\right) t^{m+n} \\
&\quad - \sum_{k=2}^{n} \binom{m}{k} (-t)^k \sum_{l=2}^{n} \binom{n}{l} (-t)^l \\
&= (1-mt)(1-nt) + \left(\mu(K_{m,n} + (-1)^{m+n})\right) t^{m+n} \\
&\quad - \Big[(1-t)^m - (1-mt)\Big]\Big[(1-t)^n - (1-nt)\Big] \\
&= (1-mt)(1-t)^n + (1-nt)(1-t)^n - (1-t)^{m+n} \\
&\quad + \left(\mu(K_{m,n}) + (-1)^{m+n}\right) t^{m+n}
\end{aligned}
$$

Now since the graph is connected and $t = 1$ is a root of the above polynomial we see that $\mu(K_{m,n}) = (-1)^{m+n+1}$ and the proof is complete. ■

The above theorems allow us to compute the lattice polynomial for quite a few graphs. The interested reader can verify the following results, which are just a few of the many one can obtain.

1. $\pi(P_2 + \mathrm{G}, t) = (1-t)^3 \pi(\mathrm{G}, t) - t^2(1-t)^{|\mathrm{G}|+1}$ and hence $\mu(P_2 + \mathrm{G}) = \mu(\mathrm{G}) + (-1)^{|\mathrm{G}|}$.

2. $\pi(C_4 + \mathrm{G}, t) = (1-t)^4 \pi(\mathrm{G}, t) - 2t^2(1-t)^{|\mathrm{G}|+2}$ and hence $\mu(C_4 + \mathrm{G}) = \mu(\mathrm{G}) - 2(-1)^n$.

In a forthcoming paper we shall delve more deeply into the operations of edge deletion and contraction and discuss a several variable version of the polynomial.

# References


[1] B. Bollobás: *Modern Graph Theory*, Springer-Verlag, 1998.

[2] P. Orlik, H. and Terao: *Arrangements of Hyperplanes*, Springer-Verlag, 1992.



*E-mail address:* ffjjw@uaf.edu