Ramsey Numbers $r(C_5, G)$ for all Graphs G of Order Six

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ABSTRACT. The Ramsey numbers $r(C_5, G)$ are determined for all graphs G of order six.

1 Introduction

Ramsey numbers for small graphs have been studied assiduously since the earliest work on this subject by Chvátal and Harary [4]. For a constantly updated compilation of known results, the reader is referred to the useful electronic survey prepared by Radziszowski [14]. Various contributions have involved creating complete catalogues for limited families of graphs. An early effort in this direction was that of Clancy, who gave all but five Ramsey numbers $r(F, G)$ with $|V(F)| \leq 4$ and $|V(G)| \leq 5$ [6]. Additional diagonal Ramsey numbers for graphs of order five were found by Harborth and Mengersen [9]. Hendry extended Clancy's catalogue to cover, with six exceptions, all pairs where both F and G are of order at most five [10]. Another approach involves finding for some fixed graph F all Ramsey numbers $r(F, G)$ for graphs G of limited order. All triangle-graph Ramsey numbers for connected graphs of order six were found in [7]. By standard methods, Schelten and Schiermeyer found $r(K_3, G)$ for all but 39 of the 853 connected graphs G of order seven [17]. Using a computer, Brinkmann independently determined $r(K_3, G)$ for connected graphs of order seven, and he extended the calculations to cover connected graphs of order eight [3]. Brandt, Brinkmann, and Harmuth have now determined $r(K_3, G)$ for all connected graphs of order nine [2]. The authors established that

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$r(C_4, K_6) = 18$ [15] and then found $r(C_4, G)$ for all graphs of order six [12]. In this paper, we address the corresponding problem for C_5. An essential step in this program is taken in [16] where we show that $r(C_5, K_6) = 21$. In determining $r(C_5, G)$ for disconnected graphs G of order six, the results of [4], [6], [10], and [11] can be used. Most of the effort here is devoted to determining $r(C_5, G)$ where G is connected.

We consider only finite, undirected graphs without loops or multiple edges. Given graphs F and G with no isolates, the relation $K_p \to (F, G)$ holds if for every coloring of the edges of K_p using two colors, red and blue, the resulting red graph R contains (a subgraph isomorphic to) F or the blue graph B contains G. The Ramsey number $r(F, G)$ is the smallest positive integer p such that $K_p \to (F, G)$. For $p < r(F, G)$ a two-coloring of $E(K_p)$ with no red F and no blue G is called *good*. For $p = r(F, G) - 1$ such a good coloring is called *critical*. Given graphs G_1 and G_2, the graph $G_1 + G_2$ is obtained from disjoint copies of G_1 and G_2 by joining each vertex in $V(G_1)$ to every vertex in $V(G_2)$. Standard symbols will be used for cycles, complete graphs, and complete bipartite or multipartite graphs. The *book with m pages* is the graph $B_m = K_1 + K_{1,m}$.

2 Results

Theorem 1 *Let G be a graph of order six with no isolates. In case G is disconnected, $P_4 \cup K_2$, $K_{13} \cup K_2$, $2P_3$, $3K_2$, $(K_{1,3} + e) \cup K_2$, $K_3 \cup P_3$,*

$$r(C_5, G) = \begin{cases} 8, & G \cong C_4 \cup K_2, \\ 9, & G \cong B_2 \cup K_2, \\ 10, & G \cong 2K_3, \\ 13, & G \cong K_4 \cup K_2. \end{cases}$$

In case G is connected,

$$r(C_5, G) = \begin{cases} 21 & \text{if } G \cong K_6, \\ 17 & \text{if } K_5 \subset G \text{ and } G \not\cong K_6, \\ 13 & \text{if } K_4 \subset G \text{ and } K_5 \not\subset G, \\ 13 & \text{if } G \cong W_5 = K_1 + C_5, \\ 11 & \text{otherwise.} \end{cases}$$

The proof of the theorem is given in a sequence of lemmas. The first lemma takes care of the cases in which G is disconnected.

Lemma 1 *$P_4 \cup K_2$, $2P_3$, $3K_2$, $(K_{1,3}+e) \cup K_2$, $K_3 \cup P_3$,*

$$r(C_5, G) = \begin{cases} 8, & G \cong C_4 \cup K_2, \\ 9, & G \cong B_2 \cup K_2, \\ 10, & G \cong 2K_3, \\ 13, & G \cong K_4 \cup K_2. \end{cases}$$

Proof: The two-coloring of $E(K_7)$ in which $R \cong B_5$ shows that $r(C_5, C_4 \cup K_2) \geq 8$. Let (R, B) be any two-coloring of $E(K_8)$. Since $r(C_5, C_4) = 7$ we may assume that $C_4 \subset B$. Let $X = \{v_1, v_2, v_3, v_4\}$ be the vertex set of such a C_4. Then $C_4 \cup K_2 \subset B$ unless $Y = \{v_5, v_6, v_7, v_8\}$ spans a red K_4. If any vertex in X is adjacent in R to two vertices in Y there is a red C_5. Otherwise, in B any two vertices in X have two common neighbors in Y; this gives $C_4 \cup K_2 \subset B$.

Remark. If $r(C_5, G) \geq |V(G)| + 5$, then $r(C_5, G \cup K_2) = r(C_5, G \cup P_3) = r(C_5, G)$. Since $r(C_5, B_2) = 9$ we have $r(C_5, B_2 \cup K_2) = 9$.

The two-coloring of $E(K_9)$ with $B \cong K_{1,4,4}$ shows that $r(C_5, 2K_3) \geq 10$. In [10] we find that $r(C_5, W_4) = 9$ where $W_4 = K_1 + C_4$ is the four-spoked wheel. Let (R, B) be a two-coloring of $E(K_{10})$ with $C_5 \not\subset R$. If view of the result just mentioned, $W_4 \subset B$. Delete the hub vertex of such a blue copy of W_4 and apply the result again to find a second W_4 with a different hub. Now clearly we can pick four distinct vertices, two from the first wheel and two from the second (even if the rims overlap completely) to yield two disjoint triangles; hence $2K_3 \subset B$.

In view of the earlier remark and $r(C_5, K_4) = 13$, we have $r(C_5, K_4 \cup K_2) = 13$. The other values follow directly from the above remark and [6]. □

It is easy to see that $r(C_5, G) \geq 11$ for every connected graph G of order six. With the exception of the five-spoked wheel $W_5 = K_1 + C_5$, every K_4-free graph of order six is 3-colorable. Thus, except for W_5 and $B_4 = K_{1,1,4}$, the K_4-free graphs of order six are subgraphs of either $K_{2,2,2}$ or $K_{1,2,3}$. Our first goal, achieved in Lemma 3, is to show that if either $G \subset K_{2,2,2}$ or $G \subset K_{1,2,3}$ then $r(C_5, G) = 11$. Since $r(C_5, B_4) = 11$ [8], it is thus true that $r(C_5, G) = 11$ for every connected graph $G \ncong W_5$ of order six and satisfying $K_4 \not\subset G$.

Lemma 2 *If (R, B) is a two-coloring of $E(K_{11})$ such that $\delta(R) \geq 3$ and $C_5 \not\subset R$ then B contains both $K_{2,2,2}$ and $K_{1,2,3}$.*

Proof: Suppose that (R, B) is such a two-coloring of $E(K_{11})$. Let $V = \{v_1, v_2, \ldots, v_{11}\}$ denote the vertex set of the K_{11}.

Case 1. $K_4 \subset R$. Suppose $K = \{v_1, v_2, v_3, v_4\}$ is the vertex set of a red K_4. In view of the known results $r(C_5, K_{2,2}) = r(C_5, K_{1,3}) = 7$ [10], we may assume that the blue graph spanned by $V \setminus K$ contains both $K_{2,2}$ and $K_{1,3}$. If two vertices of K are adjacent in B to every member of $V \setminus K$ then B contains $K_{2,2,2}$ and $K_{1,2,3}$. Thus at most one vertex of K is adjacent in B to every member of $V \setminus K$. Take v_4 to hold this possibility. Since there is no red C_5, no two vertices in K are adjacent in R to the same vertex in $V \setminus K$. Thus there are distinct vertices $v_5, v_6, v_7 \in V \setminus K$ such that $\{v_1v_5, v_2v_6, v_3v_7\} \subset R$. Also $\{v_5, v_6, v_7\}$ is an independent set in R, and each vertex therein is adjacent in R to at least two vertices in $\{v_8, v_9, v_{10}, v_{11}\}$ since $\delta(R) \geq 3$. There must be a common adjacency and this gives $C_5 \subset R$, a contradiction.

Case 2. $K_4 - e \subset R$ and $K_4 \not\subset R$. Suppose $K = \{v_1, v_2, v_3, v_4\}$ induces a red $K_4 - e$; specifically, assume that v_3v_4 is the sole blue edge. As in case 1, the blue graph spanned by $V \setminus K$ contains both $K_{2,2}$ and $K_{1,3}$. Note that since $\delta(R) \geq 3$ neither v_3 nor v_4 can be adjacent in B to every member of $V \setminus K$. Also at most one of v_1, v_2 can be adjacent in B to every member of $V \setminus K$. Since there is no red C_5, we may therefore assume that there are distinct vertices $v_5, v_5, v_7 \in V \setminus K$ such that $\{v_1v_5, v_3v_6, v_4v_7\} \subset R$. Let $X = \{v_8, v_9, v_{10}, v_{11}\}$ and let N denote the neighborhood of v_5 in R. There are two subcases.

(a) $\{v_8, v_9\} \subset N \cap X$. In this case, the absence of a red C_5 implies that there is a blue $K_{2,2,2}$ with parts $\{v_3, v_6\}$, $\{v_4, v_7\}$, $\{v_8, v_9\}$ and also a blue $K_{1,2,3}$ with parts $\{v_6\}$, $\{v_8, v_9\}$, $\{v_2, v_4, v_7\}$.

(b) $N \cap X = \{v_8\}$. Then in order to fulfill $\delta(R) \geq 3$, we must have $v_2v_5 \in R$. The absence of a red C_5 implies a blue $K_{2,2,2}$ with parts $\{v_3, v_6\}, \{v_4, v_7\}, \{v_5, v_8\}$. Note that $v_iv_7 \in B$ for $i = 1, 2, 3, 5, 6, 8$; otherwise $C_5 \subset R$. Hence we may assume that $\{v_7v_9, v_7v_{10}\} \subset R$. If $v_8v_9 \in B$ then there is a blue $K_{1,2,3}$ with parts $\{v_8\}$, $\{v_7, v_9\}$, $\{v_1, v_2, v_3\}$; hence we may assume $v_8v_9 \in R$. Also $v_6v_9 \in B$ yields a blue $K_{1,2,3}$ with parts $\{v_6\}$, $\{v_7, v_9\}$, $\{v_1, v_2, v_5\}$, so $v_6v_9 \in R$. By symmetry, $v_8v_{10} \in R$ and $v_6v_{10} \in R$. Hence R contains the graph shown in Figure 1. The only additional edge that is possible is v_9v_{10}.

If $v_1v_{11} \in R$ then, in order to avoid a red C_5, we must have $v_iv_{11} \in B$ for $i = 3, 4, 5, 9, 10$, yielding a blue $K_{1,2,3}$ with parts $\{v_{11}\}$, $\{v_9, v_{10}\}$, $\{v_3, v_4, v_5\}$. Hence $v_1v_{11} \in B$. (Note that this argument does not use the fact that $v_1v_2 \in R$.) By symmetry then, $v_iv_{11} \in B$ for $i = 1, 2, 9, 10$. To fulfill the condition $\delta(R) \geq 3$, v_{11} must be adjacent in R to three vertices in $\{v_3, v_4, v_5, v_6, v_7, v_8\}$. Inspection shows that a red C_5 is produced unless $\{v_3v_{11}, v_4v_{11}, v_5v_{11}\} \subset B$ and $\{v_6v_{11}, v_7v_{11}, v_8v_{11}\} \subset R$. But this gives $K_{1,2,3} \subset B$ with parts $\{v_{11}\}$, $\{v_9, v_{10}\}$, $\{v_3, v_4, v_5\}$.

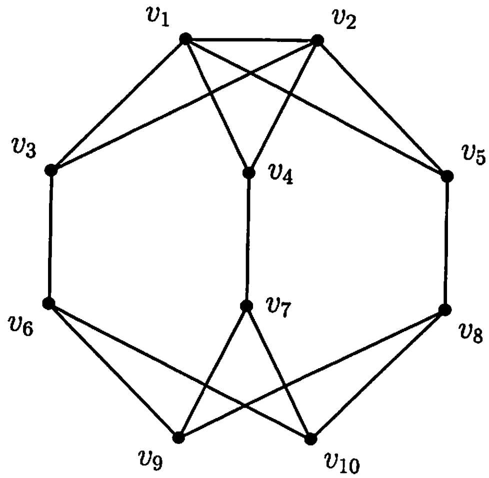


Figure 1

Case 3. $K_3 \subset R$ *and* $K_4 - e \not\subset R$. Suppose $K = \{v_1, v_2, v_3\}$ is the vertex set of a red K_3. Since $\delta(R) \geq 3$ and there is neither a red C_5 nor a red $K_4 - e$, there must be distinct vertices v_4, v_5, v_6 such that $\{v_1v_4, v_2v_5, v_3v_6\} \subset R$. Since there is no red $K_4 - e$, each of v_4, v_5, v_6 must be adjacent to at least two vertices in $\{v_7, v_8, \ldots, v_{11}\}$. Thus there must be a common adjacency. This gives $C_5 \subset R$, a contradiction.

Case 4. $K_3 \not\subset R$. Let v_1 be a vertex of maximum degree in R. Since $\delta(R) \geq 3$ and the number of vertices is odd, $\Delta(R) \geq 4$. For each vertex v_i, $i \geq 2$, let $d(v_1, v_i)$ denote the distance from v_1 to v_i. Accordingly, we define

$$N_1 = \{v_i|\ d(v_1, v_i) = 1\},\ N_2 = \{v_i|\ d(v_1, v_i) = 2\},\ N_{>2} = \{v_i|\ d(v_1, v_i) > 2\}.$$

Then $|N_1| \geq 4$, and the blue graph spanned by N_1 is complete. Since R contains neither C_3 nor C_5, the blue graph spanned by $\{v_1\} \cup N_2$ is complete. Hence we may assume $|N_2| \leq 4$; otherwise B contains K_6 and thus both $K_{2,2,2}$ and $K_{1,2,3}$. It follows that $|N_1 \cup N_{>2}| \geq 6$, so the blue graph spanned by $N_1 \cup N_{>2}$ contains $K_6 - e$ and thus both $K_{2,2,2}$ and $K_{1,2,3}$. □

Remark. A C_5-free graph G of order eight whose complement does not contain $K_{2,3}$ is isomorphic to one of the graphs $K_{4,4}$, $K_{4,4} - P_3$ or else contains $K_{4,4} - 4K_2$.

Proof of the remark: The proof is similar to the case analysis proof of the Lemma 7, so we simply give a summary of the possible extensions corresponding to the different cases of the minimum degree, as shown in Table 1. Let v be a vertex of minimum degree and let H be the graph spanned by the set of vertices disjoint from v and its neighborhood. By a well-known fact from extremal graph theory, if $\delta(G) \geq 4$ then G is pancyclic except when $G = K_{4,4}$. □

$\delta(G)$	$\|H\|$	H	Extension
1	6	$K_{3,3}$ (Lemma 5)	-
2	5	$K_{2,3}$, B_3 or $K_1 + 2K_2$ (Lemma 5)	$K_{4,4} - P_3$
3	4	Any Graph on 4 vertices without isolated vertices	Contains $K_{4,4} - 4K_2$

Table 1

Lemma 3 *If G is a connected graph of order six that is a subgraph of either $K_{1,2,3}$ or $K_{2,2,2}$, then $r(C_5, G) = 11$.*

Proof: The coloring of K_{10} in which $R \cong K_{5,5}$ shows that $r(C_5, G) \geq 11$ for any connected graph of order six. To complete the proof, it suffices to show that $K_{11} \to (C_5, K_{1,2,3})$ and $K_{11} \to (C_5, K_{2,2,2})$. By Lemma 2 and $r(C_5, K_{2,3}) = r(C_5, K_{1,2,2}) = 9$ [10], we may assume that in any counterexample $\delta(R) = 2$. Suppose that v_1 is a vertex of degree two in R with neighbors v_2 and v_3. Let us prove $K_{11} \to (C_5, K_{1,2,3})$. By the above remark, the red graph spanned by $\{v_4, v_5, \ldots, v_{11}\}$ will be a bipartite graph of order 8 that is isomorphic one of the graphs $K_{4,4}$, $K_{4,4} - P_3$ or else contains $K_{4,4} - 4K_2$. Specifically, we may assume that $\{v_4, v_5, v_6, v_7\}$ and $\{v_8, v_9, v_{10}, v_{11}\}$ are the color classes of an appropriate bipartite graph. Note that for each of the possibilities for the red bipartite graph spanned by $\{v_4, v_5, \ldots, v_{11}\}$, any two vertices in different color classes are joined by a path of length three. Since there is no red C_5, we may thus assume that v_2 is adjacent in B to each of the vertices v_4, v_5, v_6, v_7. Then $\{v_1, v_2, \ldots, v_7\}$ spans $K_6 - e$ in B, a contradiction. The proof for $K_{2,2,2}$ is similar. Details are left to the reader. □

We have shown that, except for W_5, every connected graph G of order six that does not contain K_4 satisfies $r(C_5, G) = 11$. In [5] Chvátal and Schwenk found that $r(C_5, W_5) = 13$. Our next goal is to show that if G contains K_4 but not K_5 then $r(C_5, G) = 13$. This is achieved in Lemmas 5 and 8.

Lemma 4 *A C_5-free graph G of order thirteen satisfying $\delta(G) \geq 4$ is bipartite.*

Proof: See the proof of Theorem 1 in [5]. □

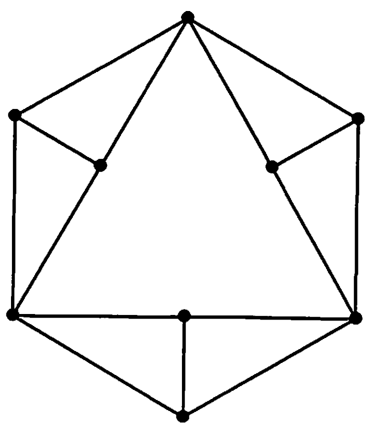
Lemma 5 *If $G \not\cong K_6 - K_3$ is a connected graph of order six that contains a K_4 but no K_5 then $r(C_5, G) = 13$.*

Proof: It is known that $r(C_5, K_4) = 13$ [6], so $r(C_5, G) \geq 13$. By exhaustive consideration of the possible graphs, it is seen that $G \subseteq K_6 - 2K_2$ for all graphs G under consideration. Thus it remains to show that $r(C_5, K_6 - 2K_2) \leq 13$. Let (R, B) be a two-coloring of $E(K_{13})$ with no red C_5 and no blue $K_6 - 2K_2$. In view of Lemma 4, we may assume that $\delta(R) \leq 3$ so $\Delta(B) \geq 9$. Consider a vertex of degree $\Delta(B)$ and its neighborhood in B. It is known that $r(C_5, K_5 - 2K_2) = 9$ [10]. Thus we find either $C_5 \subset R$ or else $K_6 - 2K_2 \subset B$, a contradiction. □

Lemma 6 *A C_5-free graph of order six with minimum degree at least 3 is isomorphic to $K_{3,3}$. A C_5-free graph of order five with minimum degree at least 2 is isomorphic to $K_{2,3}$, B_3, or $K_1 + 2K_2$ (a* bow tie*).*

Proof: The first fact is well known from extremal graph theory. The second can be found by direct inspection of all graphs of order five. □

Lemma 7 *The Ramsey number $r(C_5, B_3)$ is 10, and in any critical coloring of $E(K_9)$ the red graph is isomorphic to the graph $\mathcal{G}$ shown below.*



$\mathcal{G}$

Figure 2

Proof: See [8] for $r(C_5, B_3) = 10$, including a critical coloring of $E(K_9)$. Our goal is to prove uniqueness: if (R, B) is a critical coloring of $E(K_9)$ then $R \cong \mathcal{G}$. It is easily checked that the addition of any edge to $\mathcal{G}$ produces a C_5. Hence it suffices to prove that $\mathcal{G} \subset R$. We argue by cases depending on the minimum degree. In each case, take v_1 to be a vertex of minimum degree in R.

Case 1. $\delta(R) < 2$. Then the neighborhood of v_1 in B has at least seven vertices. Since $r(C_5, K_{1,3}) = 7$ we thus find either $C_5 \subset R$ or $K_1 + K_{1,3} = B_3 \subset B$, a contradiction, so there is no such critical coloring.

Case 2. $\delta(R) = 2$. Suppose v_1 has neighborhood $\{v_2, v_3\}$ in R. Since there is no blue B_3, the red graph spanned by $\{v_4, v_5, \ldots, v_9\}$ has minimum degree at least 3, so it is isomorphic to $K_{3,3}$ by Lemma 5. Take the color classes to be $\{v_4, v_5, v_6\}$ and $\{v_7, v_8, v_9\}$. If v_2 is adjacent in B to all three vertices in one of the color classes then $K_5 - e \subset B$. Otherwise, v_2 is adjacent in R to at least one vertex in each color class and then there is a red C_5. Again there is no such critical coloring.

Case 3. $\delta(R) = 3$. Suppose v_1 has neighborhood $\{v_2, v_3, v_4\}$ in R. Let $X = \{v_5, v_6, \ldots, v_9\}$ and let $\langle X \rangle_R$ denote the red graph spanned by this set. Note that $\langle X \rangle_R$ has minimum degree at least 2 (otherwise there is a blue B_3), so we can apply Lemma 6 to find that this graph contains $K_{2,3}$ or is isomorphic to the bow tie $K_1 + 2K_2$. We consider the corresponding subcases in turn.

(a) $K_{2,3} \subset \langle X \rangle_R$. Take the color classes to be $\{v_5, v_7, v_7\}$ and $\{v_8, v_9\}$. Since there is no red C_5 we have $\{v_5v_6, v_5v_7, v_6v_7\} \subset B$. If, for example, $\{v_2v_6, v_2v_7\} \subset B$, then the blue graph spanned by $\{v_1, v_2, v_5, v_6, v_7\}$ contains B_3; hence we may assume that $\{v_2v_5, v_2v_6\} \subset R$. Then there is a red C_5 unless $\{v_iv_j \mid 2 \le i \le 4, 8 \le j \le 9\} \subset B$. If $v_8v_9 \in R$ then $(v_2, v_5, v_8, v_9, v_6, v_2)$ is a red C_5, and if $v_8v_9 \in B$ there is a blue B_4.

(b) $\langle X \rangle_R \cong K_1 + 2K_2$. Take the red bow tie to consist of triangles $\{v_5, v_6, v_9\}$ and $\{v_7, v_8, v_9\}$. Consider vertices v_5 and v_6. In order to satisfy $\delta(R) \ge 3$, each of these vertices must be adjacent in R to some vertex in $\{v_2, v_3, v_4\}$. However, $v_iv_5 \in R$ and $v_jv_6 \in R$ where $2 \le i, j \le 4$ and $i \ne j$ gives the red cycle $(v_1, v_i, v_5, v_6, v_j, v_1)$, so we must have $i = j$. The same observation holds for the pair v_7, v_8, so there is no loss of generality in assuming that $\{v_2, v_5, v_6\}$ and $\{v_4, v_7, v_8\}$ induce red triangles. Now if v_3 is adjacent in R to any vertex v_i with $5 \le i \le 9$ there is a red C_5. Hence $\delta(R) \ge 3$ requires $\{v_3v_2, v_3v_4\} \subset R$, and therefore $\mathcal{G} \subset R$.

Case 4. $\delta(R) \ge 4$. The following result is left as an exercise for the reader: a C_5-free graph G of order 9 satisfying $\delta(G) \ge 4$ is bipartite. (The smallest odd cycle, if there is one, is clearly a C_3. Consider the following possibilities: (i) $K_3 \subset G$ and $K_4 - e \not\subset G$. (ii) $K_4 - e \subset G$ and $K_4 \not\subset G$, (iii) $K_4 \subset G$.) If R is such a graph, then $B_3 \subset K_5 \subset B$. □

Lemma 8. $r(C_5, K_6 - K_3) = 13$

Proof: Suppose (R, B) is good coloring of $E(K_{13})$. Since $r(C_5, B_3) = 10$ and $K_6 - K_3 = K_1 + B_3$, we have $\delta(R) \ge 3$.

Case 1. $\delta(R) = 3$. Let v_1 be a vertex whose neighborhood in R is $N = \{v_2, v_3, v_4\}$. Let $X = \{v_5, v_6, \ldots, v_{13}\}$; by assumption $B_3 \not\subset \langle X \rangle_B$.

In view of Lemma 7 we have $\langle X\rangle_R \cong \mathcal{G}$. Let P denote the set of vertices in X that are of degree 3 in the red copy of $\mathcal{G}$, and let Q denote the set of vertices of degree 4 in this copy. Observe that no vertex in N can be adjacent in the red graph to vertices in both P and Q; otherwise there is a red C_5. Hence we may assume that all edges from $\{v_1, v_2, v_3\}$ to P are blue or all edges from $\{v_1, v_2, v_3\}$ to Q are blue. Since $\langle P\rangle_B \cong K_6 - 3K_2$ and $\langle Q\rangle_B \cong K_3$ each contain K_3, in either either case we find $K_6 - K_3 \subset B$, a contradiction.

Case 2. $\delta(R) \geq 4$. Then R is bipartite by Lemma 3, and therefore $K_6 - K_3 \subset K_7 \subset R$, a contradiction. □

In [16] it is proved that the only C_5-free graphs of order 12 with independence number three are certain supergraphs of $3K_4$. In particular, we have the following fact.

Lemma 9. *A graph of order twelve must contain either C_5, K_4, or an independent set of four vertices.*

Lemma 10. *If G is a connected graph of order six, $K_5 \subset G$ and $G \ncong K_6$ then $r(C_5, G) = 17$.*

Proof: The two-coloring of $E(K_{16})$ with $R \cong 4K_4$ has neither a red C_5 nor a blue K_5. In view of this example, it suffices to show $r(C_5, K_6 - e) \leq 17$. Let $V = \{v_1, v_2, \ldots, v_{17}\}$ and let (R, B) be a two-coloring of edges of the complete graph on this set such that $C_5 \not\subset R$ and $K_6 - e \not\subset B$. From [7] we know that $r(K_3, K_6 - e) = 17$; hence $K_3 \subset R$. In view of $r(C_5, K_5 - e) = 13$ we have $\delta(R) \geq 4$.

First we prove that $K_4 \not\subset R$. Suppose $\{v_1, v_2, v_3, v_4\}$ is the vertex set of a red K_4. Since $\delta(R) \geq 4$ and $C_5 \not\subset R$, we may assume four additional vertices v_5, v_6, v_7, v_8 such that $\{v_1v_5, v_2v_6, v_3v_7, v_4v_8\} \subset R$. Now each of the vertices v_5, v_6, v_7, v_8 must be adjacent in R to three new vertices, and this is clearly impossible.

We know that $K_3 = B_1 \subset R$. Thus there is some $m \geq 1$ so that B_m is the largest book contained in R. Now we consider four cases. In each case, the given B_m has vertex set $\{v_1, v_2, \ldots, v_{m+2}\}$ where v_1v_2 is the spine of the book. Since there is no red K_4, the blue graph spanned by $\{v_3, v_4, \ldots, v_{m+2}\}$ is complete.

Case 1. $B_4 \subset R$. We may assume that $v_iv_{2i+1}, v_iv_{2i+2} \in R$ for $i = 3, 4, 5, 6$. Then either $C_5 \subset R$ or else we find that $\{v_1, v_2, v_7, v_9, v_{11}, v_{13}\}$ spans a blue $K_6 - e$, a contradiction.

Case 2. $B_3 \subset R$ and $B_4 \not\subset R$. There are two subcases.

(a) $\deg_R(v_1) \geq S$. Suppose $v_1v_6 \in R$. Then $v_2v_6 \in B$, and we have a configuration similar to that of case 1 except that $v_1v_6 \in B$, so we may assume $v_6v_{15} \in R$. Note that $v_2v_{13}, v_2v_{14}, v_2v_{15} \in B$; otherwise $C_5 \subset R$. If, for example, $v_1v_{13} \in B$ then $\{v_1, v_2, v_7, v_9, v_{11}, v_{13}\}$ spans a blue $K_6 - e$ as in

case 1. Hence we may assume that $v_2v_{13}, v_2v_{14}, v_2v_{15} \in R$ so $\{v_{13}, v_{14}, v_{15}\}$ spans a K_3 in blue. In this case $\{v_3, v_4, v_5, v_{13}, v_{14}, v_{15}\}$ spans a K_6 in B. Clearly the same argument applies in case $\deg_R(v_2) \geq 5$

(b) $\deg_R(v_1) = \deg_R(v_2) = 4$. By Lemma 9 and the fact that there is neither a red C_4 nor a red K_4, the blue graph spanned by $\{v_6, v_7, \ldots, v_{17}\}$ contains a K_4. Together with v_1 and v_2 this gives a blue $K_6 - e$.

Case 3. $B_2 \subset R$ and $B_3 \not\subset R$. Since v_1 and v_2 are of degree at least 4 in R, we may assume $v_1v_5 \in R$, $v_2v_5 \in B$, $v_2v_6 \in R$, $v_1v_6 \in B$. Since $\delta(R) \geq 4$ and $C_5 \not\subset R$ we may assume v_5 is adjacent to v_7, v_8, and $v_9 \in R$ and similarly v_6 is adjacent to v_{10}, v_{11}, and $v_{12} \in R$. Also $v_3v_{13} \in R$ and $v_4v_{14} \in R$. Observe that v_{13} cannot be adjacent in R to any vertex v_i with $i \leq 14$ other than v_3; otherwise there is a red C_5. Thus v_{13} is adjacent to v_{15}, v_{16}, and v_{17} in R. Now we have accounted for all of the vertices. Consider v_4. This vertex cannot be adjacent in R to any vertex other than v_1, v_2, and v_{14} without creating a red C_5. Since v_4 has degree at least 4 in R, we have reached a contradiction.

Case 4. $B_1 \subset R$ and $B_2 \not\subset R$. In this case $\{v_1, v_2, v_3\}$ spans a red K_3 and we must have $v_iv_{2i+2} \in R$ and $v_iv_{2i+3} \in R$ for $i = 1, 2, 3$ since there is no red B_2. Now each pair of vertices $\{v_4, v_5\}$, $\{v_6, v_7\}$, $\{v_8, v_9\}$ must account for at least three additional vertices. (For example, if $v_4v_5 \in B$ then we may assume that v_4 – and possibly v_5 as well – is adjacent in R to v_{10}, v_{11}, v_{12}, and if $v_4v_5 \in R$ we may assume that v_4 iS adjacent in R to v_{10} and v_{11} and v_5 is adjacent to v_{12} and v_{13}.) Thus we account for 18 vertices, a contradiction. □

By the results of Lemma 1 through Lemma 10, together with the special cases $r(C_5, B_4) = 11$ and $r(C_5, W_5) = 13$ found in [8] and [5], respectively, we have completed the proof of Theorem 1.

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