Generalized Steiner Triple Systems with Group Size g=7,8 *

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Abstract

Generalized Steiner triple systems, $GS(2,3,n,g)$ are used to construct maximum constant weight codes over an alphabet of size $g+1$ with distance 3 and weight 3 in which each codeword has length n. The existence of $GS(2,3,n,g)$ has been solved for $g = 2,3,4,5,6,9$. The necessary conditions for the existence of a $GS(2,3,n,g)$ are $(n-1)g \equiv 0 \pmod 2$, $n(n-1)g^2 \equiv 0 \pmod 6$, and $n \geq g+2$. In this paper, the existence of a $GS(2,3,n,g)$ for any given $g \geq 7$ is investigated. It is proved that if there exists a $GS(2,3,n,g)$ for all n, $g+2 \leq n \leq 9g+158$, satisfying the two congruences, then the necessary conditions are also sufficient. As an application it is proved that the necessary conditions for the existence of a $GS(2,3,n,g)$ are also sufficient for $g = 7,8$.

1 Introduction

A $(g+1)$-ary *constant weight code* (n,w,d) is a code $C \subseteq (Z_{g+1})^n$ of length n and minimum distance d, such that every $c \in C$ has Hamming weight w. To construct a constant weight code (n,w,d) with $w=3$ a *group divisible design* (GDD) will be used. A K−GDD is an ordered triple $(\mathcal{V},\mathcal{G},\mathcal{B})$ where $\mathcal{V}$ is a set of n elements, $\mathcal{G}$ is a collection of subsets of $\mathcal{V}$ called *groups* which partition $\mathcal{V}$ and $\mathcal{B}$ is a set of some subsets of $\mathcal{V}$ called *blocks*, such that each block intersects each group in at most one element and that each pair of elements from distinct groups occurs together in exactly one block in $\mathcal{B}$, where $|B| \in K$ for any $B \in \mathcal{B}$. The group type is the multiset

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$\{|G| : G \in \mathcal{G}\}$. A k-GDD(g^n) denotes a K-GDD with n groups of size g and $K = \{k\}$. If all blocks of a GDD can be partitioned into parallel classes, then the GDD is called *resolvable* GDD and denoted by RGDD, where a parallel class is a set of blocks partitioning the element set $\mathcal{V}$. In a 3-GDD(g^n), let $\mathcal{V} = (Z_{g+1} \setminus \{0\}) \times (Z_{n+1} \setminus \{0\})$ with n groups $G_i \in \mathcal{G}$, $G_i = (Z_{g+1} \setminus \{0\}) \times \{i\}$, $1 \leq i \leq n$ and blocks $\{(a,i),(b,j),(c,k)\} \in \mathcal{B}$. One can construct a constant weight code $(n,3,d)$ as stated in [7], [10]. From each block we form a codeword of length n by putting an a, b and c in positions i, j and k respectively and zeros elsewhere. This gives a constant weight code over Z_{g+1} with minimum distance 2 or 3. If the minimum distance is 3, then the code is a $(g+1)$-ary *maximum constant weight code* (MCWC) $(n,3,3)$ and the 3-GDD(g^n) is called *generalized Steiner triple system*, denoted by GS$(2,3,n,g)$. It is easy to see that a 3-GDD(g^n) is a GS$(2,3,n,g)$ iff any two intersecting blocks intersect at most two common groups of the GDD. The following result is known.

Lemma 1.1 ([7], [10]) *The following are the necessary conditions fr the existence of a GS$(2,3,n,g)$:*
(1) $(n-1)g \equiv 0 \ (mod\ 2)$;
(2) $n(n-1)g^2 \equiv 0 \ (mod\ 6)$;
(3) $n \geq g+2$.

The necessary conditions are shown to be sufficient for $g = 2,3$ with one exception by Etzion [7], for $g = 4,9$ by Phelps and Yin [9], [10], for $g = 5,6$ by Chen, Ge and Zhu [4], [5].

Lemma 1.2 ([7], [9], [10], [4], [5]) *The necessary conditions for the existence of a GS(2, 3, n, g) are also sufficient for g = 2, 3, 4, 5, 6, 9 with one exception of* $(g,n) = (2,6)$.

Recently, Blake-Wilson and Phelps [3] proved that the necessary conditions for the existence of a GS$(2,3,n,g)$ are also asymptotically sufficient for any g.

Since the existence of GS$(2,3,n,g)$ has been solved for $g \leq 6$, we need only to consider the case $g \geq 7$. For $g \geq 7$, let $T_g = \{n$: there exists a GS$(2,3,n,g)\}$, $B_g = \{n$: n satisfying the necessary conditions listed in Lemma 1.1 $\}$, $M_g = \{n$: $n \in B_g, n \leq 9g+158$ $\}$. In this paper, the following results are obtained.

Theorem 1.3 *For any $g \geq 7$, if $M_g \subset T_g$, then $B_g = T_g$. That is the necessary conditions for the existence of a GS$(2,3,n,g)$ are also sufficient.*

Theorem 1.4 *$B_7 = T_7$, that is the necessary conditions for the existence of a GS$(2,3,n,g)$ are also sufficient for $g = 7$.*

Theorem 1.5 *$B_8 = T_8$, that is the necessary conditions for the existence of a $GS(2,3,n,g)$ are also sufficient for $g = 8$.*

Combining Lemma 1.2, Theorem 1.4 and Theorem 1.5, it is known that the existence of a GS$(2,3,n,g)$ is completely determined for any $g \leq 9$.

2 Product Constructions

In product constructions, we will need the concept of both *holey generalized Steiner triple systems* and *disjoint incomplete Latin squares.*

A *holey group divisible design, $K - HGDD$*, is a fourtuple $(\mathcal{V}, \mathcal{G}, \mathcal{H}, \mathcal{B})$, where $\mathcal{V}$ is a set of points, $\mathcal{G}$ is a partition of $\mathcal{V}$ into subsets called *groups*, $\mathcal{H} \subset \mathcal{G}$, $\mathcal{B}$ is a set of *blocks* such that a group and a block contain at most one common point and every pair of points from distinct groups, not both in $\mathcal{H}$, occurs in a unique block in $\mathcal{B}$, where $|B| \in K$ for any $B \in \mathcal{B}$. A k-HGDD$(g^{(n,u)})$ denotes a K-HGDD with n groups of size g in $\mathcal{G}$, u groups in $\mathcal{H}$ and $K = \{k\}$. A *holey generalized Steiner triple system*, HGS$(2,3,(n,u),g)$, is a 3-HGDD$(g^{(n,u)})$ with the property that any two intersecting blocks intersect at most two common groups.

It is easy to see that if $u = 0$ or $u = 1$, then a HGS$(2,3,(n+u,u),g)$ is just a GS$(2,3,n,g)$ or a GS$(2,3,n+1,g)$ respectively.

A *Latin square* of side n, LS(n), is an $n \times n$ array based on some set S of n symbols with the property that every row and every column contains every symbol exactly once. An *incomplete Latin square*, ILS$(n+a,a)$, denotes a LS$(n+a)$ "missing" a sub LS(a). Without loss of generality, we may assume that the missing subsquare, or *hole*, is at the lower right corner. We say $(i,j,s) \in$ ILS$(n+a,a)$ if the entry in the cell (i,j) is s. Let A_1, A_2 be two ILS$(n+a,a)$s on the same symbol set. If $(i,j,s_1) \neq (i,j,s_2)$ for any $(i,j,s_1) \in A_1$, $(i,j,s_2) \in A_2$, then we say that A_1 and A_2 are *disjoint.* r DILS$(n+a,a)$ denotes r pairewise disjoint ILS$(n+a,a)$s and r DLS(n) denotes r pairewise disjoint LS(n)s.

The following singular indirect product construction for GS$(2,3,n,g)$s is first stated in [4].

Lemma 2.1 (Singular Indirect Product (SIP)) *Let m, n, t, u and a be integes such that $0 \leq a \leq u < n$. Suppose the following designs exist:*
(1) t DILS$(n+a,a)$;
(2) a $3 - GDD(g^m)$ with the propery that all blocks of the design can be partitioned into t sets $S_0, S_1, \cdots, S_{t-1}$, such that the minimum distance in $S_r, 0 \leq r \leq t-1$, is 3;
(3) a HGS$(2,3,(n+u,u),g)$.
Then there exists a HGS$(2,3,(c,d),g)$, where $c = m(n+a)+u-a, d = ma+u-a$. Further, if there exists

(4) a $GS(2,3,ma+u-a,g)$,
then there exists a $GS(2,3,m(n+a)+u-a,g)$.

Lemma 2.2 ([4]) *There exist $\delta(a)$ $DILS(n+a,a)$, where $\delta(0)=n$ and $\delta(a)=a$ for $1 \le a \le n$.*

Taking $a=0$ in Lemma 2.1, the singular direct product is then obtained, which is first appeared in [9]. From Lemma 2.2, t DLS(n) exist when $t \le n$.

Lemma 2.3 (Singular Direct Product (SDP)) *Let m, n, t, and u be integers such that $0 \le u < n$. Suppose $t \le n$ and the following designs exist:*
(1) a $3-GDD(g^m)$ with the propery that all blocks of the design can be partitioned into t sets $S_0, S_1, \cdots, S_{t-1}$, such that the minimum distance in $S_r, 0 \le r \le t-1$, is 3;
(2) a $HGS(2,3,(n+u,u),g)$.
Then there exists a $HGS(2,3,(mn+u,u),g)$. Further, if there exists a $GS(2,3,u,g)$, then there exists a $GS(2,3,mn+u,g)$.

Taking $u=1$ in Lemma 2.3, one gets the Construction D in Etzion [7]

Lemma 2.4 ([7]) *Let $(\mathcal{V},\mathcal{G},\mathcal{B})$ be a 3-$GDD(g^m)$, and suppose there exists a $GS(2,3,n,g)$. Then there exists a $GS(2,3,m(n-1)+1,g)$ if $\mathcal{B}$ can be partitioned into t sets $S_0, \cdots, S_{t-1}$, such that the minimum distance in $S_r, 0 \le r \le t-1$, is 3 and $t \le n-1$.*

Taking $u=0$ in Lemma 2.3, one gets the Construction C in Etzion [7]

Lemma 2.5 (Direct Product (DP)) *Let $(\mathcal{V},\mathcal{G},\mathcal{B})$ be a 3-$GDD(g^m)$, and suppose there exists a $GS(2,3,n,g)$. Then there exists a $GS(2,3,mn,g)$ if $\mathcal{B}$ can be partitioned into t sets $S_0, \cdots, S_{t-1}$, such that the minimum distance in $S_r, 0 \le r \le t-1$, is 3 and $t \le n$.*

Notice that the derived generalized Steiner triple system in Lemma 2.4 and Lemma 2.5 has a sub GS$(2,3,n,g)$, we state the fact in the following.

Lemma 2.6 *Let $(\mathcal{V},\mathcal{G},\mathcal{B})$ be a 3-$GDD(g^m)$. Suppose there exists a $GS(2,3,n,g)$. Then there exists a $HGS(2,3,(mn,n),g)$ or a $HGS(2,3,(m(n-1)+1,n),g)$ if $\mathcal{B}$ can be partitioned into t sets $S_0, \cdots, S_{t-1}$, such that the minimum distance in $S_r, 0 \le r \le t-1$, is 3 and $t \le n$ or $t \le n-1$ respectively.*

If one uses a 3-RGDD(g^m) in the constructions, then each parallel class becomes an S_r and there are $t=\frac{g(m-1)}{2}$ such classes. The following is stated in [4].

Lemma 2.7 *If there exists a GS$(2,3,n,g)$ and a 3-RGDD(g^m) with $t = \frac{g(m-1)}{2} \le n$ or $n-1$, then there exists a GS$(2,3,mn,g)$ or a GS$(2,3,m(n-1)+1,g)$ respectively.*

For the existence of a 3-RGDD(g^m), we have the following.

Lemma 2.8 ([1]) *A 3-RGDD(g^m) exists iff $(m-1)g \equiv 0 \pmod 2$, $mg \equiv 0 \pmod 3$ and $g^m \ne 2^3, 2^6$ and 6^3.*

Lemma 2.9 *For any $g \ge 7$, if there exists a GS$(2,3,n,g)$, then there exists a GS$(2,3,3n,g)$ and a GS$(2,3,3(n-1)+1,g)$. Consequently, there exists a HGS$(2,3,(3n,n),g)$ and a HGS$(2,3,(3(n-1)+1,n),g)$.*

In the remainder of this section, we shall discuss a new construction to obtain t DILS from some difference matrices. Let G be an Abelian group, $|G| = n$. An $(n,k;\lambda)$-*difference matrix* is a $k \times n\lambda$ matrix $D = (d_{ij})$ with entries from G, so that for each $1 \le i < j \le k$, the set $\{d_{il} - d_{jl} : 1 \le l \le n\lambda\}$ contains every element of G λ times. Let $(n,k;\lambda)$-DM denote an $(n,k;\lambda)$-difference matrix.

Theorem 2.10 *If there exists an $(n,4;1)$-DM, then there exist n DILS$(n+a,a)$ for any a, $0 \le a \le n$.*

Proof. If $a = 0$, the conclusion follows from Lemma 2.2, we need only to consider the case $1 \le a \le n$. Let $G = \{a_0 = 0, a_1, \cdots, a_{n-1}\}$ be an Abelian group. By the assumption, we have two mutually orthogonal Latin squares $L_1 = (c_{ij})$, $L_2 = (d_{ij})$, which are generated from the $(n,4;1)$-DM, $M = (m_{ij})$ as follows: For any $a_h \in G$ and $1 \le t \le n$, if $m_{1t} + a_h = a_i$, and $m_{2t} + a_h = a_j$, we take $c_{ij} = m_{3t} + a_h$ and $d_{ij} = m_{4t} + a_h$.

Now we construct an ILS$(n+a,a)$, denoted by A_0, based on $G \cup \{\infty_0, \cdots, \infty_{a-1}\}$ as follows. For $0 \le k \le a-1$, if $(a_i, a_j, s) \in L_1$ and $(a_i, a_j, a_k) \in L_2$, then $(a_i, a_j, \infty_k) \in A_0$, $(a_i, \infty_k, s) \in A_0$ and $(\infty_k, a_j, s) \in A_0$; for $a \le k \le n-1$, if $(a_i, a_j, s) \in L_1$ and $(a_i, a_j, a_k) \in L_2$, then $(a_i, a_j, s) \in A_0$. Let $\pi_h (0 \le h \le n-1)$ be a permutation on $G \cup \{\infty_0, \cdots, \infty_{a-1}\}$ given by

$$\pi_h(x) = \begin{cases} x + a_h, & \text{for } x \in G, \\ x, & \text{for } x \in \{\infty_1, \cdots, \infty_{a-1}\}. \end{cases}$$

From A_0, we can construct $n-1$ ILS$(n+a,a)$s, $A_1, \cdots, A_{n-1}$, whose entries are defined as follows. For $1 \le h \le n-1$, define

$$(u, v, w_0) \in A_h \text{ if } (\pi_h(u), \pi_h(v), w_0) \in A_0.$$

Notice that for any given $u, v \in G \cup \{\infty_0, \cdots, \infty_{a-1}\}$, not both in $\{\infty_0, \cdots, \infty_{a-1}\}$, the entries of A_0 in the cells $(\pi_h(u), \pi_h(v))$, $0 \le h \le n-1$, are distinct. So, $A_0, A_1, \cdots, A_{n-1}$ are pairwise disjoint. □

The following Lemma is known.

Lemma 2.11 *([6]) There exists a $(q, q; 1)$-DM for any prime power q.*

Lemma 2.12 *If there exists an $(m_1, k; 1)$-DM and an $(m_2, k; 1)$-DM, then there exists an $(m_1m_2, k; 1)$-DM.*

Proof. Suppose D_i is an $(m_i, k; 1)$-DM based on G_i, where $|G_i| = m_i, 1 \leq i \leq 2$, and $D_1 = (a_{ij})_{k\times m_1}$, $D_2 = (b_{ij})_{k\times m_2}$. Let

$$D = \begin{pmatrix} (a_{11}, b_{11}) & \cdots & (a_{11}, b_{1m_2}) & \cdots & (a_{1m_1}, b_{11}) & \cdots & (a_{1m_1}, b_{1m_2}) \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ (a_{k1}, b_{k1}) & \cdots & (a_{k1}, b_{km_2}) & \cdots & (a_{km_1}, b_{k1}) & \cdots & (a_{km_1}, b_{km_2}) \end{pmatrix}.$$

It is easy to see that D is an $(m_1m_2, k; 1)$-DM based on $G_1 \times G_2$ □

Lemma 2.13 *Suppose n is a positive integer, $4|n$, then there exists an $(n, 4; 1)$-DM.*

Proof. We can write $n = 4 \cdot 2^\alpha \cdot 3^\beta \cdot n_1$, such that the prime factor of n_1 is no less than 5. From Lemma 2.11 and Lemma 2.12, there exists an $(n_1, 4; 1)$-DM, we need only to prove that there exists an $(n_2, 4; 1)$-DM for $n_2 = 4 \cdot 2^\alpha \cdot 3^\beta$. We distinguish two cases
Case 1 $\beta \neq 1$. If $\beta = 0$, then $n_2 = 2^{2+\alpha}$. From Lemma 2.11, there exists an $(n_2, 4; 1)$-DM. Otherwise $\beta \geq 2$, from Lemma 2.11 and Lemma 2.12, there exists an $(n_2, 4; 1)$-DM;
Case 2 $\beta = 1$. $n_2 = 12 \cdot 2^\alpha$. If $\alpha < 2$, then $n_2 = 12$ or $n_2 = 24$, from [6, II Theorem 2.35, Theorem 2.43], there exists an $(n_2, 4; 1)$-DM. Otherwise $\alpha \geq 2$, from Lemma 2.11 and Lemma 2.12, there exists an $(n_2, 4; 1)$-DM.
□

As a corollary we have the following lemma which will be used very often.

Lemma 2.14 *Suppose n is a positive integer, $4|n$, then there exists n DILS$(n + a, a)$ for $0 \leq a \leq n$.*

Note added (May, 2000): New results on n DILS$(n + a, a)$ can be found in [11] and [8].

3 Proof of Theorem 1.3

For $g \geq 7$, let

$$f(g) = \begin{cases} 1, \ 3 & \text{if } g \equiv 1,5 \ (mod\ 6) \\ 0, \ 1, \ 3, \ 4 & \text{if } g \equiv 2,4 \ (mod\ 6) \\ 1, \ 3, \ 5 & \text{if } g \equiv 3 \ (mod\ 6) \\ 0, \ 1, \ 2, \ 3, \ 4, \ 5 & \text{if } g \equiv 0 \ (mod\ 6) \end{cases}$$

$$\delta(k) = \begin{cases} 0, & \text{if } k \text{ is even,} \\ 1, & \text{if } k \text{ is odd.} \end{cases}$$

From Lemma 1.1, it is easy to see that the necessary conditions for the existence of a GS$(2,3,n,g)$ are $n \equiv f(g)\ (mod\ 6)$, and $n \geq g+2$.

Lemma 3.1 *For $g \geq 7$, suppose $v = 54p + 6j + f(g)$, $0 \leq j \leq 8$. If $6p + 6 + \delta(f(g)) \in T_g$, $18p + 6j + f(g) - 36 \in T_g$, and $p \geq \lceil \frac{7-j}{2} \rceil$, then $v \in T_g$.*

Proof. Apply Lemma 2.1 with $m = 3, n = 12p + 12, t = g, u = 6p + 6 + \delta(f(g)), a = 6p + 3j - 21 + \lfloor \frac{f(g)}{2} \rfloor$. It is easy to check that $a \leq u < n$. Since $\lfloor \frac{f(g)}{2} \rfloor \geq 0$ and $p \geq \lceil \frac{7-j}{2} \rceil = \frac{7-j+\delta(7-j)}{2}$, it is easy to see that $a \geq 0$. From Lemma 2.14, there exist n DILS$(n+a,a)$ for $0 \leq a \leq n$. We further have t DILS$(n+a,a)$ since $t \leq u-2 < n$. Thus the condition (1) of Lemma 2.1 is satisfied. For $g \geq 7$, a 3-RGDD(g^3) always exists by Lemma 2.8, which has g parallel classes. So, condition (2) is also satisfied. From $u \in T_g$ we apply Lemma 2.9 to obtain a HGS$(2,3,(n+u,u),g)$, providing the design in condition (3). Finally, since $2\lfloor \frac{f(g)}{2} \rfloor + \delta(f(g)) = f(g)$, we know that $ma + u - a = 18p + 6j + f(g) - 36 \in T_g$, the condition (4) is satisfied. Therefore, we have the conclusion that $v = m(n+a) + u - a \in T_g$. This completes the proof. □

Now, we are in a position to prove Theorem 1.3.

Proof of Theorem 1.3. We need to show that $M_g \subset T_g$ implies that $B_g \subset T_g$. Suppose $n \in B_g$. If $n \in M_g$, then $n \in T_g$. Otherwise, $n = 54p + 6j + f(g) > 9g + 158$, where $0 \leq j \leq 8$. We first claim that $p \geq \lceil \frac{7-j}{2} \rceil$. If not so, then $p < \lceil \frac{7-j}{2} \rceil = \frac{7-j+\delta(7-j)}{2}$. Thus $n < 189 - 21j + 27\delta(7-j) + f(g)$. Since $0 \leq j \leq 8, \delta(7-j) \leq 1, f(g) \leq 5$ and $g \geq 7$, we have $n < 221 \leq 9g + 158$, a contradiction.

Next, it is easy to see that $n > 9g+158$ implies that $6p \geq g+11$. Then, it is easily checked that $\alpha = 6p+6+\delta(f(g)) \geq g+2, \beta = 18p+6j+f(g)-36 \geq g+2$. Since $\beta \equiv f(g)\ (mod\ 6)$, we see that $\beta \in B_g$. It is also easily verified that $\alpha \in B_g$. If we have both $\alpha \in M_g$ and $\beta \in M_g$, then Lemma 3.1 guarantees that $n \in T_g$ and the proof is completed. If at least one of α and

β is not in M_g, then we can repeat the above process taking it as n' and using new α' and β'.

After certain steps α' and β' will be small enough so that both α' and β' are in M_g. This makes both $\alpha \in T_g$ and $\beta \in T_g$, thus $n \in T_g$. This completes the proof. □

4 Proof of Theorem 1.4

For $g = 7$, the necessary conditions for the existence of a GS$(2,3,n,g)$ become $n \equiv 1,3 \pmod 6$ and $n \geq 9$. It is known that there exists a GS$(2,3,q+1,q-1)$ for any prime power q in [7, Section 4]. Taking $q = 8,9$, we get a GS$(2,3,9,7)$ and a GS$(2,3,10,8)$.

For $n \equiv 3 \pmod 6$, to construct a GS$(2,3,n,7)$ in Z_{7n}, it suffices to find a set of generalized base blocks, $\mathcal{A} = \{B_1, \cdots, B_s\}$, $s = \frac{7(n-1)}{2}$, such that $(\mathcal{V}, \mathcal{G}, \mathcal{B})$ forms a GS$(2,3,n,7)$, where $\mathcal{V} = Z_{7n}$, $G = \{G_0, G_1, \cdots, G_{n-1}\}$, $G_i = \{i + nj : 0 \leq j \leq 6\}, 0 \leq i \leq n-1$, and $\mathcal{B} = \{B + 3j : B \in \mathcal{A}, 0 \leq j \leq \frac{7n}{3} - 1\}$. For convenience, we write $\mathcal{A} = \bigcup_{i=0}^{2} \{\{i,x,y\} : \{x,y\} \in S_i\}$. So, for each $\mathcal{A}$ we need only display the corresponding $S_i, 0 \leq i \leq 2$.

Lemma 4.1 *There exists a GS$(2,3,n,7)$ for $n \in F_1$, where $F_1 = \{9, 15, 21, 33, 51\}$*

Proof. For $n = 9$, as mentioned above, there exists a GS$(2,3,n,7)$. For other values $n \in F_1$, with the aid of a computer, we have found a set of generalized base blocks of a GS$(2,3,n,7)$. Here, we only list the $S_i, 0 \leq i \leq 2$ for $n = 15$. For the remaining values n, the corresponding $S_i, 0 \leq i \leq 2$ are listed in Appendix A.

$n = 15$, $\mathcal{A} = \bigcup_{i=0}^{2} \{\{i,x,y\} : \{x,y\} \in S_i\}$,

$S_0 = \{\{79,101\}, \{41,97\}, \{54,55\}, \{3,7\}, \{10,28\}, \{9,56\}, \{13,69\}, \{39,59\}, \{52,84\}, \{83,85\}, \{18,58\}, \{57,63\}, \{62,71\}, \{61,80\}, \{81,100\}, \{2,12\}, \{35,77\}\}$;

$S_1 = \{\{24,32\}, \{5,77\}, \{89,94\}, \{34,63\}, \{29,83\}, \{18,35\}, \{69,103\}, \{3,17\}, \{56,70\}, \{41,52\}, \{14,75\}, \{7,90\}, \{15,79\}, \{25,62\}, \{43,44\}, \{53,85\}, \{58,67\}, \{11,59\}, \{26,81\}, \{12,65\}, \{47,60\}\}$;

$S_2 = \{\{40,75\}, \{78,83\}, \{23,39\}, \{8,28\}, \{29,37\}, \{42,69\}, \{9,81\}, \{20,21\}, \{84,95\}, \{33,100\}, \{68,104\}\}$. □

Lemma 4.2 *There exists a GS$(2,3,n,7)$ for $n \in F_2 = \{13, 19, 31\}$*

Proof. With the aid of a computer, we have found a set of base blocks $\mathcal{A}$ of a GS$(2,3,n,7)$ for $n \in F_2$.

For convenience, we write $\mathcal{A} = \{\{0, x, y\} : \{x, y\} \in S\}$. So, for each $\mathcal{A}$ we need only display the corresponding S.

$n = 13$, $S = \{\{15, 34\}, \{20, 41\}, \{22, 45\}, \{24, 55\}, \{25, 53\}, \{1, 33\}, \{2, 37\}, \{3, 43\}, \{4, 86\}, \{6, 83\}, \{10, 84\}, \{11, 75\}, \{12, 42\}, \{18, 47\}\}$.

$n = 19$, $S = \{\{62, 93\}, \{87, 89\}, \{15, 49\}, \{88, 116\}, \{58, 108\}, \{100, 132\}, \{124, 129\}, \{30, 122\}, \{56, 59\}, \{26, 90\}, \{42, 97\}, \{8, 22\}, \{67, 96\}, \{21, 106\}, \{20, 80\}, \{54, 126\}, \{16, 68\}, \{10, 120\}, \{47, 98\}, \{24, 63\}, \{6, 18\}\}$.

$n = 31$, $S = \{\{82, 182\}, \{95, 208\}, \{67, 190\}, \{40, 203\}, \{51, 136\}, \{97, 108\}, \{78, 103\}, \{16, 106\}, \{107, 128\}, \{134, 164\}, \{70, 129\}, \{5, 131\}, \{12, 22\}, \{2, 34\}, \{66, 140\}, \{197, 214\}, \{57, 72\}, \{50, 137\}, \{84, 153\}, \{18, 198\}, \{118, 179\}, \{24, 76\}, \{48, 116\}, \{115, 173\}, \{46, 79\}, \{144, 157\}, \{23, 49\}, \{28, 36\}, \{47, 112\}, \{43, 98\}, \{39, 45\}, \{1, 42\}, \{56, 63\}, \{142, 213\}, \{29, 121\}\}$. □

Lemma 4.3 *There exists a $GS(2, 3, q, 7)$ for any prime power $q, q \equiv 1$ (mod 6), $q \geq 43$.*

Proof. We apply Theorem 2 in [3] to obtain the result. There exists an STS(7) , which can be split into 7 partial parallel classes. Let $q = 6s + 1$, since $q \geq 43$, we have $s \geq 7$. The desired idempotent Latin squares needed in the Theorem comes from Lemma 7 in [3] □

Lemma 4.4 *There exists a $GS(2, 3, v, 7)$ for all $v \in F_3 = \{e : e \in B_7, e \leq 73\}$.*

Proof. For $v \in F_1 \cup F_2$, the conclusion comes from Lemma 4.1 and Lemm 4.2. Since $25 = 3 \cdot 8 + 1$ and there exists a GS(2, 3, 9, 7), there exists a GS(2, 3, 25, 7) and a HGS(2, 3, (25, 9), 7) by Lemma 2.9. For $v = 43, 49, 61, 67, 73$, the conclusion follows from Lemm 4.3. For $v = 69$, there exist 16 DILS(16 + 6, 6) and a GS(2, 3, 21, 7) by Lemma 2.14 and Lemma 4.1. Apply Lemma 2.1 with $m = 3, n = 16, t = 7, u = 9, a = 6$, we get a GS(2, 3, 69, 7). For the remaining values v, we write $v = 3n$ or $v = 3(n - 1) + 1$ for $n \in [9, 25]$. By Lemma 2.9, Lemma 4.1 and Lemma 4.2, there exists a GS(2, 3, v, 7). Here, we list the pairs (v, n) in Table 4.1. □

v	n	v	n	v	n
$27 = 3 \cdot 9$	9	$37 = 3 \cdot 12 + 1$	13	$39 = 3 \cdot 13$	13
$45 = 3 \cdot 15$	15	$55 = 3 \cdot 18 + 1$	19	$57 = 3 \cdot 19$	19
$63 = 3 \cdot 21$	21				

Table 4.1 pairs (v, n) for $v \in F_3 \setminus (F_1 \cup F_2 \cup \{25, 43, 49, 61, 67, 69.73\})$

Lemma 4.5 *There exists a* $GS(2,3,v,7)$ *for all* $v \in F_4 = \{e : e \equiv 1,3,7, 9 \ (mod\ 18), 9 \le v \le 219\}$.

Proof. For $v \equiv 1,3\ (mod\ 18)$, write $v = 18t + k, k = 1,3$, where $t \le 12$ since $v \le 219$. Let $n = 6t + 1$, then $n \le 73$ and a GS$(2,3,n,7)$ exists from Lemma 4.4. Since $18t + 1 = 3(n-1) + 1$ and $18t + 3 = 3n$, a GS$(2,3,v,7)$ exists from Lemma 2.9;

For $v \equiv 7,9\ (mod\ 18)$, write $v = 18t+k$, where $k = 7,9$ and $t \le 11$. Let $n = 6t + 3$, then $n \le 69$ and a GS$(2,3,n,7)$ exists from Lemma 4.4. Since $v = 3(n-1) + 1$ or $3n$, a GS$(2,3,v,7)$ exists from Lemma 2.9. □

Lemma 4.6 *There exists a* $GS(2,3,v,7)$ *for all* $v \in F_5 = \{e : e \equiv 13, 15\ (mod\ 18), 9 \le v \le 213\}$.

Proof. We can write $v = 54h + k, k = 13,15,31,33,49,51$. Since $v \le 213$, we have $h \le 3$.

For $k = 13$, if $h = 0$, from Lemma 4.4 there exists a GS$(2,3,v,7)$. Otherwise $1 \le h \le 3$. Since $6h + 3 \le 21$ and $18h + 1 \le 55$, from Lemma 4.4 and Lemma 2.9 there exists a HGS$(2,3,(18h+7, 6h+3),7)$, and a GS$(2,3,18h+1,7)$. From Lemma 2.14 there exist $12h+4$ DILS$(12h+4+6h-1, 6h-1)$. Taking $m = 3, n = 12h+4, t = 7, u = 6h+3, a = 6h-1$ in Lemma 2.1, we get a GS$(2,3,v,7)$.

For $k = 15,31,33,49,51$, if $h = 0$, then there exists a GS$(2,3,v,7)$ from Lemma 4.4. Otherwise, $1 \le h \le 3$. The discussion is similar to the case $k = 13$. We list the parameters needed in Lemma 2.1. Taking $m = 3, t = 7$. n, u and a are taken according to different k. We list the fourtuple (k,n,u,a) in Table 4.2. □

k	n	u	a	k	n	u	a
15	12h+4	6h+3	6h	31	12h+12	6h+7	6h-6
33	12h+12	6h+7	6h-5	49	12h+12	6h+7	6h+3
51	12h+12	6h+7	6h+4				

Table 4.2 fourtuples (k,n,u,a) needed in Lemma 2.1

Now, we are in a position to prove Theorem 1.4.

Proof of Theorem 1.4: From Theorem 1.3, we need only to consider the values v, such that $v \in B_7, v \le 219$. The result comes from Lemma 4.5 and Lemma 4.6. □

5 Proof of Theorem 1.5

For $g = 8$, the necessary conditions for the existence of a GS$(2,3,n,g)$ become $n \equiv 0,1 \pmod 3$ and $n \geq 10$. In [5], by introducing a $K -^* GDD$, Wilson's Fundamental construction can be used to construct generalized Steiner triple systems.

Definition 5.1 *A $K - GDD$ is said to have "star" property and denoted by $K -^* GDD$ if any two intersecting blocks intersect at most two common groups.*

With this definition a GS$(2,3,n,g)$ is just the same as a $3 -^* GDD(g^n)$. Using a $K -^* GDD$ as a master GDD, the well known Wilson's Fundamental Construction can be used to construct GS$(2,3,n,g)$s, which we state below.

Lemma 5.2 *(Weighting) Let $(\mathcal{V},\mathcal{G},\mathcal{B})$ be a $K -^* GDD$ (the master GDD) with groups G_1, G_2, $\cdots$, G_t. Suppose there exists a function w: $\mathcal{V} \longrightarrow Z^+ \bigcup \{0\}$ (a weighting function) which has the property that for each block $B = \{x_1, x_2, \cdots, x_k\} \in \mathcal{B}$ there exists a $3 -^* GDD$ of group type $(w(x_1), w(x_2), \cdots, w(x_k))$ (such a GDD is an "ingredient" GDD). Then there exists a $3 -^* GDD$ of group type $(\sum_{x \in G_1} w(x), \sum_{x \in G_2} w(x), \cdots, \sum_{x \in G_t} w(x))$.*

Lemma 5.3 *([5])If there exists a $K -^* GDD(g^n)$, and there exists a GS$(2, 3, k, h)$ for any $k \in K$, then there exists a GS$(2,3,n,gh)$.*

Lemma 5.4 *There exists a $4 -^* GDD(4^n)$ for $n \in E_1$, where $E_1 = \{13, 16, 19, 22, 25, 31, 58\}$.*

Proof. For each $n \in E_1$, with the aid of a computer, we have found a set of base blocks $\mathcal{A}$ for such a 4-*GDD(4^n) in Z_{4n} with the groups $G_i = \{i + jn : 0 \leq j \leq 3\}, 0 \leq i \leq n-1$, which is listed as follows.

$n = 13$, $\mathcal{A} = \{\{0,1,3,11\}, \{0,4,16,25\}, \{0,5,19,37\}, \{0,6,23,30\}\}$.

$n = 16$, $\mathcal{A} = \{\{0,1,3,7\}, \{0,5,18,39\}, \{0,8,17,44\}, \{0,10,33,52\}, \{0,11,26,40\}\}$.

$n = 19$, $\mathcal{A} = \{\{0,1,3,7\}, \{0,5,13,36\}, \{0,9,24,42\}, \{0,10,26,54\}, \{0,11,41,62\}, \{0,12,29,49\}\}$.

$n = 22$, $\mathcal{A} = \{\{0,1,9,55\}, \{0,2,19,72\}, \{0,3,26,41\}, \{0,4,10,68\}, \{0,5,45,56\}, \{0,7,36,67\}, \{0,12,25,39\}\}$.

$n = 25$, $\mathcal{A} = \{\{0,1,32,60\}, \{0,2,76,90\}, \{0,3,61,80\}, \{0,4,17,66\}, \{0,5,57,84\}, \{0,6,35,91\}, \{0,7,53,89\}, \{0,8,30,63\}\}$.

$n = 31$, $\mathcal{A} = \{\{0,1,78,106\}, \{0,2,39,56\}, \{0,3,67,89\}, \{0,4,73,83\}, \{0,5,32,66\}, \{0,6,59,80\}, \{0,8,84,117\}, \{0,9,81,104\}, \{0,11,24,99\}, \{0,12,26,42\}\}$.

$n = 58$, $\mathcal{A} = \{\{0,1,41,118\}, \{0,2,65,86\}, \{0,3,52,125\}, \{0,4,54,101\},$

$\{0,5,95,134\},\{0,6,25,89\},\{0,7,127,205\},\{0,8,31,204\},$
$\{0,9,70,132\},\{0,10,45,82\},\{0,11,85,177\},\{0,12,79,212\},$
$\{0,13,46,176\},\{0,14,152,203\},\{0,15,96,126\},\{0,16,87,144\},$
$\{0,17,93,141\},\{0,22,60,179\},\{0,24,188,214\}\}.$ □

As mentioned in Section 4, there exists a GS(2, 3, 10, 8), by Lemma 1.2 we know that there exists a GS(2, 3, 4, 2). So, by Lemma 5.3 and Lemma 5.4 we have the following.

Lemma 5.5 *There exists a $GS(2,3,n,8)$ for $n \in E_1 \cup \{10\}$.*

To construct a GS(2, 3, n, 8) in Z_{8n} for some $n \equiv 0$ $(mod\ 3)$, it suffices to find a set of generalized base blocks, $\mathcal{A} = \{B_1, B_2, \cdots, B_s\}$, $s = 4(n-1)$, such that $(\mathcal{V},\mathcal{G},\mathcal{B})$ forms a GS(2, 3, n, g), where $\mathcal{V}$=Z_{8n}, $\mathcal{G}$=$\{G_i : 0 \le i \le n-1\}$, G_i=$\{i + jn : 0 \le j \le 7\}$, $0 \le i \le n-1$, and $\mathcal{B}$=$\{B + 3i : B \in \mathcal{A}, 0 \le i \le \frac{8n}{3} - 1\}$.

Lemma 5.6 *There exists a $GS(2,3,n,g)$ for each $n \in E_2$, where E_2=$\{12$, 15, 18, 21, 24, 27, 33, 42, 51$\}$.*

Proof. For each $n \in E_2$, with the aid of a computer, we have found a set of generalized base blocks $\mathcal{A}$. For convenience, we can write $\mathcal{A} = \bigcup_{i=0}^{2}\{\{i,x,y\} : \{x,y\} \in S_i\}$. So, for each $\mathcal{A}$ we need only display the corresponding S_i, $0 \le i \le 2$. Here we only list the S_i, $0 \le i \le 2$ for $n = 12$, for other values, the corresponding S_i, $0 \le i \le 2$ are listed in Appendix B (In order to save space, we omit Appendix B, the interested reader may contact the authors for a copy).

$n = 12$, $\mathcal{A} = \bigcup_{i=0}^{2}\{\{i,x,y\} :\ \{x,y\} \in S_i\}$,

$S_0 = \{\{18,53\},\{69,85\},\{2,82\},\{49,64\},\{22,80\},\{5,37\},\{3,50\},\{40,71\},$
$\{21,52\},\{41,62\},\{1,90\},\{13,38\},\{33,87\},\{26,79\},\{17,34\}\};$

$S_1 = \{\{50,69\},\{36,56\},\{64,93\},\{68,77\},\{67,86\},\{72,87\},\{2,58\},\{5,19\},$
$\{39,94\},\{7,9\},\{6,38\},\{14,51\},\{10,52\},\{8,47\},\{27,78\}\};$

$S_2 = \{\{4,56\},\{37,64\},\{33,76\},\{8,9\},\{30,69\},\{70,90\},\{24,68\},\{10,83\},$
$\{12,42\},\{15,88\},\{47,80\},\{71,75\},\{7,28\},\{84,95\}\}.$ □

Lemma 5.7 *There exists a $3-GDD(8^m)$ for $m = 3,4,6$ and 7 with the property that all blocks of the design can be partitioned into t sets S_0, S_1, $\cdots$, S_{t-1} such that $t \le 8$ for $m = 3,6$, $t \le 4$ for $m = 4,7$, and the minimum distance in S_r, $0 \le r \le t-1$, is 3*

Proof. For $m = 3$, from Lemm2.8, there exists a 3-RGDD(8^3), which has 8 parallel classes. By Lemma 1.2 there exists a $GS(2,3,4,2)$ and a

$GS(2,3,7,2)$. In [7], a $3-GDD(2^6)$ is presented, in which all blocks can be partitioned into 2 sets S_0, S_1, such that the minimum distance in S_0, S_1, is 3. Use these designs as master GDDs in Wilson's Fundamental construction and give weight four to each element, we get a $3-GDD(8^m)$ for $m = 4,7,6$ respectively. Since a $3-RGDD(4^3)$ exists, it is not difficult to see that the resultant designs are desired ones. □

Since the existence of a $GS(2,3,n,8)$ implies that $n \geq 10$, from Lemma 5.7, it is certain that for $g = 8$, the m in DP and SDP can be choosen to be 3, 4, 6 and 7. So by Lemma 2.3 and Lemma 2.4 we have the following.

Lemma 5.8 *If there exists a GS(2,3,n,8), then there exists a GS(2, 3, mn,8) and a GS(2, 3,* $m(n-1)+1$, *8), where* $m = 3,4,6$ *and* 7.

Lemma 5.9 *There exists a* $GS(2,3,v,g)$ *for all* $v \in E_3$, *where* $E_3 = \{e : e \equiv 0,1 \ (mod\ 3), e \leq 76\}$.

Proof. For $v \in E_1 \cup E_2 \cup \{10\}$, the conclusion comes from Lemma 5.5 and Lemm 5.6. For the remaining values v, we can write $v = mn$ or $v = m(n-1)+1$ for some $m \in \{3,4,6,7\}$ and $n \in E_1 \cup E_2 \cup \{10\}$. By Lemma 5.5, Lemma 5.6 and Lemma 5.8, there exists a $GS(2,3,v,8)$. Here, we list the triples (v,m,n) in Table 5.1. □

v	m	n	v	m	n	v	m	n
$28 = 3 \cdot 9 + 1$	3	10	$30 = 3 \cdot 10$	3	10	$34 = 3 \cdot 11 + 1$	3	12
$36 = 3 \cdot 12$	3	12	$37 = 3 \cdot 12 + 1$	3	13	$39 = 3 \cdot 13$	3	13
$40 = 4 \cdot 10$	4	10	$43 = 3 \cdot 14 + 1$	3	15	$45 = 3 \cdot 15$	3	15
$46 = 3 \cdot 15 + 1$	3	16	$48 = 3 \cdot 16$	3	16	$49 = 4 \cdot 12 + 1$	4	13
$52 = 3 \cdot 17 + 1$	3	18	$54 = 3 \cdot 18$	3	18	$55 = 3 \cdot 18 + 1$	3	19
$57 = 3 \cdot 19$	3	19	$60 = 4 \cdot 15$	4	15	$61 = 4 \cdot 15 + 1$	4	16
$63 = 3 \cdot 21$	3	21	$64 = 4 \cdot 16$	4	16	$66 = 3 \cdot 22$	3	22
$67 = 6 \cdot 11 + 1$	6	12	$69 = 4 \cdot 17 + 1$	4	18	$70 = 7 \cdot 10$	7	10
$72 = 3 \cdot 24$	3	24	$73 = 3 \cdot 24 + 1$	3	25	$75 = 3 \cdot 25$	3	25
$76 = 4 \cdot 18$	4	18						

Table 5.1 triples (v,m,n) for $v \in E_3 \setminus (E_1 \cup E_2 \cup \{10\})$

Lemma 5.10 *There exists a* $GS(2,3,v,8)$ *for all* $v \in E_4$, *where* $E_4 = \{e : e \equiv 0,1,\ 3,7 \ (mod\ 9),\ 10 \leq v \leq 228\}$.

Proof. For $v \equiv 0,1,3 \ (mod\ 9)$, write $v = 9t + k$, where $k = 0,1,3$. If $t \leq 3$, the result follows from Lemma 5.9. Otherwise, $t \geq 4$. Let $n = 3t$, then $v = 3n, 3n+1$ or $3(n+1)$. Since $v \leq 228$, we have $4 \leq t \leq 25$, hence $n \leq 75, n+1 \leq 76$. Notice that $n \in B_8$ and $n+1 \in B_8$, by Lemma 5.8 and Lemma 5.9, there exists a $GS(2,3,v,8)$.

For $v \equiv 7 \ (mod\ 9)$, write $v = 9t + 7$. If $t \leq 2$, the result follows from Lemma 5.9. Otherwise, $t \geq 3$. Let $n = 3t + 3$, then $v = 3(n-1) + 1$. Since $v \leq 228$, we have $t \leq 24$, hence $n \leq 75$. Notice that $n \in B_8$, by Lemma 5.8 and Lemma 5.9, there exists a GS$(2, 3, v, 8)$. □

Lemma 5.11 *There exists a* $GS(2,3,v,8)$ *for all* $v \in E_5$*, where* $E_5 = \{e: e \equiv 4, 6, 13, 24, 31, 33 \ (mod\ 36),\ 10 \leq v \leq 229\}$.

Proof. Write $v = 36t + k, k = 4, 6, 13, 24, 31, 33$. If $t \leq 1$, the result comes from Lemma 5.9. For $t \geq 2$, notice $v \leq 229$, we can write $v = mn$ or $v = mn + 1$ for some $m \in \{4, 6\}$ and $n \in B_8, n \leq 58$. From Lemma 5.8 and Lemma 5.9, there exists a GS$(2, 3, v, 8)$. here we list the fourtuples (k, v, m, n) in Table 5.2 . □

k	v	m	n	k	v	m	n
4	$v = 4 \cdot (9t+1)$	4	9t+1	6	$v = 6 \cdot (6t+1)$	6	6t+1
13	$v = 4 \cdot (9t+3) + 1$	4	9t+4	24	$v = 6 \cdot (6t+4)$	6	6t+4
31	$v = 6 \cdot (6t+5) + 1$	6	6t+6	33	$v = 4 \cdot (9t+8) + 1$	4	9t+9

Table 5.2 fourtuples (k, v, m, n) for Lemma 5.11

Lemma 5.12 *There exists a* $GS(2,3,v,8)$ *for all* $v \in E_6$*, where* $E_6 = \{e: e \equiv 15, 22 \ (mod\ 36),\ 10 \leq v \leq 229\}$.

Proof. For $v \equiv 15 \ (mod\ 36)$, write $v = 36e + 15$. If $e = 1$, then $v = 51$, from Lemma 5.9, there exists a GS$(2, 3, 51, 8)$. If $e = 2$, then $v = 87$. Apply Lemma 2.1 with $m = 3, n = 24, t = 8, u = 13, a = 1$. There exist 24 DILS$(24 + 1, 1)$ by Lemma 2.14, and there exist t DILS$(24 + 1, 1)$ too, condition (1) is satisfied. As mentioned before, condition (2) is also satisfied. Since there exists a GS$(2, 3, 13, 8)$ by Lemma 5.9, we get a HGS$(2, 3, (37, 13)^{*}, 8)$ by Lemma 2.9, thus codition (3) is satisfied. There exists a GS$(2, 3, 15, 8)$ by Lemma 5.9, this is the design desired in condition (4). So, we obtain a GS$(2, 3, 87, 8)$. For $e \geq 3$, $3e - 4 \geq 5$. Apply Lemma 2.1 with $m = 4, n = 6e + 6, t = 4, u = 3e + 3, a = 3e - 4$. From Lemma 2.2, there exist $3e - 4$ DILS$(n + a, a)$. Therefore, there exist t DILS$(n + a, a)$, this is the condition (1). Condition (2) is satisfied by Lemm 5.7. Since $v \leq 229$, we have $3 \leq e \leq 5$, hence $12 \leq u \leq 18$. From Lemma 5.9, there exists a GS$(2, 3, u, 8)$. So there exists a HGS$(2, 3, (n + u, u), 8)$, providing the design needed in condition (3). Since $27 \leq ma + u - a = 12e - 9 \leq 51$, by Lemma 5.9, there exists a GS$(2, 3, ma + u - a, 8)$. This is the design needed in condition (4). Thus, we obtain a GS$(2, 3, v, 8)$.

For $v \equiv 22 \ (mod\ 36)$, write $v = 36e+22$. If $e \leq 1$, the result follows from Lemma 5.9. Otherwise, $e \geq 2$. Just as we did in the case $v \equiv 15 \ (mod\ 36)$, apply Lemma 2.1 with $m = 4, n = 6e + 6, t = 4, u = 3e + 4, a = 3e - 2$, we obtain a GS$(2, 3, v, 8)$. □

Combining Lemma 5.11 and Lemma 5.12, we have the following.

Lemma 5.13 *There exists a* $GS(2,3,v,8)$ *for all* $v \in E_7$, *where* $E_7 = \{e: e \equiv 4, 6 \ (mod\ 9),\ 10 \le v \le 229\}$.

Now, we are in a position to prove Theorem 1.5.

Proof of Theorem 1.5: From Theorem 1.3, we need only to consider the values v, such that $v \in B_8, v \le 229$, the result comes from Lemma 5.10 and Lemma 5.13. □

Appendix A.

$n = 21,\ \mathcal{A} = \bigcup_{i=0}^{2} \{\{i,x,y\}:\ \{x,y\} \in S_i\}$,

$S_0 = \{\{70,110\},\{98,104\},\{51,119\},\{30,79\},\{107,120\},\{48,80\},\{41,75\},$
$\{55,141\},\{22,130\},\{5,36\},\{40,97\},\{108,122\},\{11,56\},\{53,54\},$
$\{90,114\},\{15,73\},\{9,124\},\{3,65\}\}$;

$S_1 = \{\{8,120\},\{12,121\},\{23,79\},\{56,113\},\{71,135\},\{31,114\},\{105,107\},$
$\{6,66\},\{9,34\},\{35,54\},\{38,52\},\{14,111\},\{37,119\},\{3,49\},$
$\{30,116\},\{82,143\},\{48,137\},\{11,13\},\{5,86\},\{63,81\},\{19,20\},$
$\{94,103\},\{60,129\},\{57,133\},\{45,61\},\{7,27\},\{15,25\}\}$;

$S_2 = \{\{61,137\},\{48,49\},\{29,129\},\{11,62\},\{41,85\},\{28,125\},\{17,55\},$
$\{121,124\},\{82,123\},\{126,133\},\{6,10\},\{76,111\},\{38,141\},\{75,106\},$
$\{24,116\},\{12,78\},\{18,70\},\{9,54\},\{19,77\},\{20,90\},\{95,118\},$
$\{22,97\},\{120,132\},\{32,101\},\{100,146\}\}$.

$n = 33,\ \mathcal{A} = \bigcup_{i=0}^{2} \{\{i,x,y\}:\ \{x,y\} \in S_i\}$,

$S_0 = \{\{69,181\},\{204,211\},\{39,177\},\{140,220\},\{59,73\},\{12,77\},\{46,124\},$
$\{8,31\},\{47,130\},\{176,196\},\{114,135\},\{29,152\},\{62,84\},\{144,205\},$
$\{23,223\},\{44,155\},\{45,203\},\{81,137\},\{9,175\},\{52,75\},\{2,188\},$
$\{60,228\},\{154,172\},\{94,207\},\{15,120\},\{195,214\},\{18,38\},\{86,157\},$
$\{90,103\},\{101,227\},\{78,95\},\{113,151\},\{26,42\},\{134,190\},\{6,49\},$
$\{149,226\},\{55,160\},\{48,217\}\}$;

$S_1 = \{\{164,181\},\{22,167\},\{118,190\},\{47,134\},\{14,172\},\{50,71\},\{113,222\},$
$\{35,99\},\{5,117\},\{26,174\},\{69,91\},\{206,216\},\{40,197\},\{136,203\},$
$\{89,123\},\{87,144\},\{65,126\},\{54,105\},\{124,125\},\{112,156\},\{56,107\},$
$\{148,229\},\{37,204\},\{162,192\},\{93,191\},\{95,96\},\{62,163\},\{29,178\},$
$\{111,145\},\{7,147\},\{132,185\},\{46,104\},\{150,217\},\{28,44\},\{170,195\},$
$\{13,76\},\{135,139\},\{20,25\},\{122,153\},\{30,98\},\{3,202\}\}$;

$S_2 = \{\{9,157\},\{44,90\},\{69,141\},\{13,61\},\{8,97\},\{94,196\},\{181,219\},$
$\{185,193\},\{32,222\},\{65,118\},\{39,143\},\{71,211\},\{104,218\},\{108,197\},$
$\{153,154\},\{54,162\},\{52,226\},\{11,115\},\{34,123\},\{29,228\},\{56,126\},$
$\{124,133\},\{60,95\},\{21,46\},\{4,51\},\{15,117\},\{149,161\},\{42,173\},$
$\{137,176\},\{20,105\},\{148,158\},\{80,83\},\{26,183\}\}$.

$n=51$, $\mathcal{A}=\bigcup_{i=0}^{2}\{\{i,x,y\}:\ \{x,y\}\in S_i\}$,

$S_0=\{\{250,261\},\{100,289\},\{224,262\},\{43,246\},\{194,241\},\{131,167\},\{209,336\},$
$\{237,331\},\{84,311\},\{149,348\},\{182,186\},\{268,354\},\{217,267\},\{218,221\},$
$\{44,343\},\{113,291\},\{276,317\},\{199,270\},\{314,339\},\{53,210\},\{76,150\},$
$\{216,322\},\{140,164\},\{52,247\},\{233,287\},\{27,67\},\{38,297\},\{20,48\},$
$\{169,232\},\{45,257\},\{65,144\},\{188,240\},\{126,324\},\{55,274\},\{39,127\},$
$\{101,266\},\{83,260\},\{128,248\},\{222,269\},\{28,103\},\{138,180\},\{104,123\},$
$\{166,342\},\{6,19\},\{152,326\},\{30,192\},\{46,133\},\{58,74\},\{183,319\},$
$\{163,275\},\{36,73\},\{31,347\},\{189,193\},\{302,335\},\{110,238\},\{328,333\}\}$;

$S_1=\{\{104,265\},\{183,313\},\{275,350\},\{114,186\},\{38,286\},\{18,175\},\{11,43\},$
$\{123,238\},\{248,257\},\{241,355\},\{224,254\},\{135,341\},\{21,191\},\{68,80\},$
$\{128,129\},\{242,335\},\{101,164\},\{78,132\},\{66,83\},\{181,237\},\{24,153\},$
$\{130,253\},\{138,206\},\{203,232\},\{7,98\},\{140,251\},\{239,356\},\{42,332\},$
$\{112,144\},\{37,182\},\{147,236\},\{193,348\},\{10,208\},\{89,214\},\{268,280\},$
$\{95,337\},\{131,304\},\{19,100\},\{287,294\},\{26,261\},\{331,333\},\{210,328\},$
$\{93,344\},\{16,213\},\{65,81\},\{171,249\},\{99,207\},\{155,302\},\{61,234\},$
$\{137,187\},\{40,109\},\{63,202\},\{32,47\},\{150,162\},\{62,148\},\{41,274\},$
$\{25,297\},\{301,309\},\{134,152\},\{290,351\},\{54,342\}\}$;

$S_2=\{\{197,300\},\{234,263\},\{145,173\},\{139,172\},\{22,191\},\{46,333\},\{288,351\},$
$\{314,353\},\{240,352\},\{114,357\},\{241,289\},\{37,303\},\{23,152\},\{117,278\},$
$\{60,94\},\{142,283\},\{87,340\},\{75,218\},\{64,221\},\{264,287\},\{238,282\},$
$\{186,279\},\{50,92\},\{151,286\},\{267,346\},\{254,348\},\{204,309\},\{184,316\},$
$\{39,236\},\{79,354\},\{120,345\},\{146,337\},\{76,243\},\{15,161\},\{168,225\},$
$\{78,100\},\{160,302\},\{86,166\},\{251,310\},\{116,181\},\{7,62\},\{211,307\},$
$\{148,327\},\{71,137\},\{273,274\},\{222,297\},\{97,144\},\{133,134\},\{101,332\},$
$\{158,355\},\{13,281\},\{183,325\},\{187,253\},\{51,252\},\{66,324\},\{55,156\},$
$\{89,123\},\{115,174\}\}$.

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