TWO PROPERTIES OF THE GENERALIZED SEQUENCE {Wn} RELEVANT TO RECURRING DECIMAL

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1. Introduction

In this paper, we deduce two properties of the generalized sequence $\{W_n\}$ relevant to recurring decimal , and further get some interesting properties ofspecial sequences $\{F_n\}$、$\{L_n\}$、$\{P_n\}$ and $\{Q_n\}$.

The generalized sequence $\{W_n\}$is defined for all integers n by the recurrencerelation (see[1])

$$W_{n+2} = pW_{n+1}+qW_n \tag{1}$$

in which

$$W_0 = a,\ W_1 = b \tag{2}$$

where,a,b,p,q are arbitrary integers.

Its general term formula is

$$W_n = \frac{(b-a\beta)\alpha^n-(b-a\alpha)\beta^n}{\alpha-\beta} \tag{3}$$

where

$$\alpha = \frac{p+\sqrt{p^2+4q}}{2}, \quad \beta = \frac{p-\sqrt{p^2+4q}}{2} \tag{4}$$

are the roots of the proper equation

$$x^2 - px - q = 0 \tag{5}$$

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Special cases of $\{W_n\}$ which interest us are

the Fibonacci sequence $\{F_n\}$: p=1,q=1,a=0,b=1

the Lucas sequence $\{L_n\}$: p=1,q=1,a=2,b=1

the Pell sequence $\{P_n\}$: p=2,q=1,a=0,b=1 (6)

the Pell-Lucas sequence $\{Q_n\}$: p=2,q=1,a=2,b=2

It is easy of to get the generating functions of the generalized sequence {Wn} as follows :

$$\sum_{i=0}^{\infty} W_i x^i = \frac{a+bx-apx}{1-px-qx^2} \tag{7}$$

2. Two Properties of {Wn} Relevant to Recurring Decimal

Theorem 1:

$$\sum_{i=0}^{\infty} \frac{W_i}{10^{i+1}} = \frac{10a+b-ap}{100-10p-q} \qquad (where\ p^2+q^2 \neq 0) \tag{8}$$

Proof: In equation (7) putting $x = \frac{1}{10}$, we get

$$\sum_{i=1}^{\infty} \frac{W_i}{10^i} = \frac{100a+10b-10ap}{100-10p-q} .$$

And dividing both sides of the equation by 10 we have

$$\sum_{i=1}^{\infty} \frac{W_i}{10^{i+1}} = \frac{10a+b-ap}{100-10p-q} .$$

Corollary 1 : When a=0,b=1,from theorem 1 we get

$$\sum_{i=0}^{\infty} \frac{W_i}{10^{i+1}} = \frac{1}{100-10p-q} . \tag{9}$$

(where $0 \leqslant p \leqslant 9$, $0 \leqslant q \leqslant 9$, $p^2+q^2 \neq 0$)

In fact, when a=0, b=1, $0 \leqslant p \leqslant 9$, $0 \leqslant q \leqslant 9$, $p^2+q^2 \neq 0$, $\sum_{i=0}^{\infty} \frac{W_i}{10^{i+1}}$ take the reciprocales of 1—99.

Corollary 2: When p=1, q=1, a=0 and b=1, from theorem 1 we get a property of the Fibonacci sequence $\{F_n\}$:

$$\sum_{i=0}^{\infty} \frac{F_i}{10^{i+1}} = \frac{1}{89} , \tag{10}$$

this is

```
   0
    1
     1
      2
       3
        5
         8
         13
          21.
            .
+)          .
-------------------------------
   0.011235951......—> 1/89
```

Corollary 3 : When p=1, q=1, a=2 and b=1, from theorem 1 we get a propertyof the Lucas sequence $\{L_n\}$:

$$\sum_{i=0}^{\infty} \frac{L_i}{10^{i+1}} = \frac{19}{89} , \tag{11}$$

that is

```
   2
    1
     3
      4
       7
       11
        18
         29
          47
           76.
             .
+)           .
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   0.2134831446 ......—> 19/89
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Corollary 4 : When p=2, q=1, a=0, and b=1, from theorem 1 we get a property of the Pell sequence $\{P_n\}$:

$$\sum_{i=0}^{\infty} \frac{P_i}{10^{i+1}} = \frac{1}{79} , \tag{12}$$

that is

$$\begin{array}{l} 0 \\ \ \ 1 \\ \ \ \ \ 2 \\ \ \ \ \ \ \ 5 \\ \ \ \ \ \ \ 12 \\ \ \ \ \ \ \ \ \ 29 \\ \ \ \ \ \ \ \ \ \ \ 70 \\ \ \ \ \ \ \ \ \ \ \ 169 \\ \ \ \ \ \ \ \ \ \ \ \ \ 408 \\ \ \ \ \ \ \ \ \ \ \ \ \ \ \ 985 \\ \ \ \ \ \ \ \ \ \ \ \ \ \ 2378. \\ \cdot \\ +) \cdot \\ \hline 0.01265822028\ldots\ldots \longrightarrow 1/79 \end{array}$$

Corollary 5: When p=2, q=1, a=2 and b=2, from theorem 1 we get a property of the Pell-Lucas sequence $\{Q_n\}$:

$$\sum_{i=0}^{\infty} \frac{Q_i}{10^{i+1}} = \frac{18}{79} , \tag{13}$$

that is

$$\begin{array}{l} 2 \\ \ \ 2 \\ \ \ \ \ 6 \\ \ \ \ \ 14 \\ \ \ \ \ \ \ 34 \\ \ \ \ \ \ \ \ \ 82 \\ \ \ \ \ \ \ \ \ 198 \\ \ \ \ \ \ \ \ \ \ \ 478 \\ \ \ \ \ \ \ \ \ \ \ 1154 \\ \ \ \ \ \ \ \ \ \ \ \ \ 2786. \\ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \cdot \\ +) \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \cdot \\ \hline 0.2278480126\ldots\ldots \longrightarrow 18/79 \end{array}$$

Theorem 2 :

$$\sum_{i=0}^{\infty}\frac{W_i}{10^{2i+1}} = \frac{100a+b-ap}{10000-100p-q} \qquad (where\ p^2+q^2\neq 0) \qquad (14)$$

Proof : In equation (7) putting $x=\frac{1}{100}$, we get

$$\sum_{i=1}^{\infty}\frac{W_i}{10^{2i}} = \frac{10000a+100b-100ap}{10000-100p-q} \ .$$

And dividing both sides of the equation by 100 we have

$$\sum_{i=1}^{\infty}\frac{W_i}{10^{2i+2}} = \frac{100a+b-ap}{10000-100p-q} \ .$$

Corollary 6 : When p=1, q=1, a=0 and b=1, from theorem 2 we get a property of the Fibonacci sequence $\{F_n\}$:

$$\sum_{i=0}^{\infty}\frac{F_i}{10^{2i+2}} = \frac{1}{9899} \ , \qquad (15)$$

that is

```
   00
     01
       01
         02
           03
             05.
                .
+)              .
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   0.000101020305...... —> 1/9899
```

Corollary 7 : When p=1, q=1, a=2 and b=1, from theorem 2 we get a property of the Lucas sequence $\{L_n\}$:

$$\sum_{i=0}^{\infty} \frac{L_i}{10^{2i+2}} = \frac{199}{9899} , \qquad (16)$$

that is

$$\begin{array}{l} 02 \\ \quad 01 \\ \qquad 03 \\ \qquad\quad 04 \\ \qquad\qquad 07 \\ \qquad\qquad\quad 11. \\ \qquad\qquad\qquad \cdot \\ +) \qquad\qquad\qquad \cdot \\ \hline 0.020103040711\ldots\ldots \longrightarrow 199/9899 \end{array}$$

Corollary 8 : When p=2, q=1, a=0 and b=1, from theorem 2 we get a property of the Pell sequence $\{P_n\}$:

$$\sum_{i=0}^{\infty} \frac{P_i}{10^{2i+2}} = \frac{1}{9799} , \qquad (17)$$

that is

$$\begin{array}{l} 00 \\ \quad 01 \\ \qquad 02 \\ \qquad\quad 05 \\ \qquad\qquad 12 \\ \qquad\qquad\quad 29. \\ \qquad\qquad\qquad \cdot \\ +) \qquad\qquad\qquad \cdot \\ \hline 0.000102051229\ldots\ldots \longrightarrow 1/9799 \end{array}$$

Corollary 9: When p=2, q=1, a=2 and b=2, from theorem 2 we get a property of the Pell-Lucas sequence $\{Q_n\}$:

$$\sum_{i=0}^{\infty} \frac{Q_i}{10^{2i+2}} = \frac{198}{9799} , \qquad (18)$$

that is

```
           02
             02
               06
                 14
                   34
                     82.
                        .
 +)                     .
 0.020206143482...... —> 198/ 9799
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References

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