The values $\sqrt{2q}$ and $\log_2 q$: their relationship with k-arcs

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Abstract

We show that the secants of an arc of size near to $\sqrt{2q}$ cover almost half plane; also, a random union of - about - $\log_2 q$ arcs of this size is such that its secants cover the plane.

The values $\sqrt{2q}$ and $\log 2q$ arise naturally in some propositions about the study of (complete) k-arcs and k-saturating sets (i.e. sets which cover by their secants the whole plane) of small order, in a projective plane of order q.

For example, let us recall

1) [Lunelli-Sce bound] If K is a complete k-arc , then

$$k \geq \frac{1}{2}\left(3+\sqrt{8q+1}\right) \approx \sqrt{2q}$$

2) [Ughi] The points of 3 lines of a Baer subplane form a saturating set where

$$k = 3\sqrt{q}$$

3) [Kovacs] There exists a saturating set K with

$$k \leq 6\sqrt{3q} \cdot \sqrt{\log_2 q}$$

4) [Kim-Vu] There exists a k-arc (in fact, a lot of them) for which k satisfies

$$k \leq \sqrt{q} \log^c q$$

where c is a constant.

In order to better understand the meaning of these values, in this paper we will prove that, roughly speaking, a k-arc - where $k \sim \sqrt{2q}$ - covers more than (approximately) half plane, and that it is possible to obtain a saturating set by collecting about $\log_2 q^2$ such arcs.

From now on, let K a k-arc in a proiective plane π of order q and cardinality q^2+q+1.

Write $K = \{P_1, ..., P_k\}$. Put

$$A_1 = \{P_1\}$$

$$A_i = \left\{ \begin{array}{l} P \in \pi | P \text{ is covered by a secant of } \{P_1, ..., P_i\} \\ \text{but is not covered by a secant of } \{P_1, ..., P_{i-1}\} \end{array} \right\} \text{ when } i \geq 2 \ .$$

The sets A_i are obviously disjoint, and

$$\# \text{ points covered by } K = \sum_{i=1,...,k} |A_i| \ .$$

Then
$|A_1| = 1$
$|A_2| = q$
$|A_3| = 1 + 2(q-1)$
$|A_4| = 1 + 3(q-1-1) = 3q - 5$.
Let us put

$$b_i = 1 + (i-1) \cdot \left[q - 1 - \binom{i-2}{2} \right] \ = \ qi - q - \frac{1}{2}i^3 + 3i^2 - \frac{13}{2}i + 5$$

$$c_i = 1 + (i-1) \cdot [q - 1 - (i-3)] \ = \ qi - q - i^2 + 3i - 1 \ .$$

In general, it is possible to prove

Proposition 1

$$b_i \leq |A_i|$$

Proposition 2

$$|A_i| \leq c_i$$

Proof.
Obvious for $i = 1$. Now let i be ≥ 2.
Let us consider the $(i-1)$ lines $r_1 = \overline{P_1 P_i}, ..., r_{i-1} = \overline{P_{i-1} P_i}$.
When we add P_i to $\{P_1, ..., P_{i-1}\}$ we obtain - as new covered points -
1) the point P_i
2) on each line r_j, all the points of r_j except those which were already covered by $\{P_1, ..., P_{i-1}\}$.
So we can write

$$A_i \ = \ P_i \cup \left\{ \bigcup_{j=1,...,i-1} (r_j \setminus P_i) \ \setminus \ A_{i-1} \right\} \ .$$

Let

N = # points intercepted on a fixed line r_j by the $\binom{i-2}{2}$ secants of

$$\left\{P_1, ..., \underset{\text{dropped out}}{\dot{P_j}}, ..., P_{i-1}\right\}.$$

Then obviously $N \leq \binom{i-2}{2}$; moreover $N \geq (i-3)$; we can suppose - without loss of generality - that $j = 1$ .Then the points

$$\overline{P_2P_3} \cap r_1 \ , ..., \ \overline{P_2P_{i-1}} \cap r_1$$

are in fact $(i-3)$ distinct points of r_1 already covered by $\{P_1, ..., P_{i-1}\}$.
The statements now follow.

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Proposition 3 *If* $k = \left[\sqrt{2q}\right] + 2$ *then*

$$\# \textit{ points covered by secants of } K =$$

$$= \sum_{i=1,\dots,k} |A_i| \geq \frac{1}{8}\left(4q^2 - 46q - 56\sqrt{2q} - 52\right) \approx \textit{ half plane} .$$

Proof.
Put

$$a = \left[\sqrt{2q} + 2\right] = \left[\sqrt{2q}\right] + 2 .$$

Observe that $b_i < 0$ when $i > a$, but still - obviously - $|A_i| \geq 0$, so that

$$\sum_{i=1,\dots,k} |A_i| \geq \sum_{i=1,\dots,a} b_i = \frac{1}{2} \sum_{i=1,\dots,a} \left(2qi - 2q - i^3 + 6i^2 - 13i + 10\right) =$$

$$= \frac{1}{2} \sum_{i=1,\dots,a} \left[-i^3 + 6i^2 + (2q-13)\, i + 10 - 2q\right] =$$

$$= \frac{1}{2}\left\{-\left(\frac{a(a+1)}{2}\right)^2 + 6\frac{a(a+1)(2a+1)}{6} + \frac{(2q-13)a(a+1)}{2} + (10-2q)a\right\} \geq$$

$$\geq \frac{1}{8}\left(4q^2 - 46q - 56\sqrt{2q} - 52\right) .$$

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Proposition 4 *If K is a complete k-arc then*

$$q^2 + q + 1 = \# \text{ covered points } =$$

$$= \sum_{i=1,\ldots,k} |A_i| \leq \frac{1}{6}\{-2k^3 + (6+3q)k^2 + (2-3q)k\} \ .$$

Proof.
It is a straightforwward computation, using the inequalities

$$A_i \leq c_i = qi - q - i^2 + 3i - 1 \ .$$

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Remark . This proposition seems qualitatively different from that of Lunelli-Sce, but the integer bounds (on k) they gives are essentially the same.

But, our proposition says not only that a complete k-arc is such that $k \gtrsim \sqrt{2q}$, but also that a complete k-arc for which k is near to $\sqrt{2q}$ is very "strange", in the sense that for every $i = 1, ..., k$, and every secant $\overline{P_i P_j}$ the points intersected on $\overline{P_i P_j}$ by the secants of $\left\{P_1, \ldots \dot{P_i}, \ldots, \dot{P_j}, \ldots, P_k\right\}$ are nearly as few as possible.

I do hope that this observation can be useful in a exaustive research of complete k-arcs (at least for small k), because it is possible now to exclude that that certain arcs (whose size is smaller than $\sqrt{2q}$) can be extended to complete arcs of order near to $\sqrt{2q}$.

From our arguments it is now possible to give a construction which -even if weaker than the previous quoted ones - seems enlightening. Let

$$X = \left\{K | K \text{ is a } k\text{-arc, where } k = \left[\sqrt{2q} + 2\right]\right\} \ .$$

Roughly speaking, the idea is as follows: choose random K_1 in X. Then you can cover about half plane. Now, choose -random again - K_2 in X. Then it seems that you should cover about half of the not previously covered points, and so on. So, the "right" number of choices seems to be near to $\log_2|\pi|$.

Proposition 5 *Let π be a plane of order q, such that $Aut(\pi)$ is 1-transitive over the points of the plane. Let us suppose $q \geq 41$. Then, in π there is a k-set W , obtained as union (not necessarily disjoint) of k-arcs with size $=\left[\sqrt{2q} + 2\right]$, such that*

$$|W| \leq 2 \ log_2 \left(q^2 + q + 1\right) \ \left[\sqrt{2q} + 2\right]$$

and W covers by its secants all the plane.

Proof.
Fix K in X, and choose Q in π. Then

$$\underset{\text{here the event is the choice of Q in } \pi}{\text{prob}(Q \text{ is not covered by secant of K})} \leq$$

$$1 - \frac{\frac{1}{8}\left(4q^2 - 46q - 56\sqrt{2q} - 52\right)}{q^2+q+1} = \frac{1}{2} + \varepsilon(q)$$

$$\text{where } \varepsilon(q) = \frac{50q + 56\sqrt{2q} + 56}{8(q^2+q+1)} .$$

Observe that $\lim_{q\to\infty} \varepsilon(q) = 0$, and that -for example - $\varepsilon(q) < \frac{1}{5}$ for $q \geq 41$.
Let $x = \underset{\text{here Q is fixed, and the event is the choice of K in X}}{\text{prob}(Q \text{ is not covered by secant of K})}$.
I will prove that $x \leq \frac{1}{2} + \varepsilon$.

First of all, we observe that x does not depend on Q, because of our assumptions on the group of collineations of the plane.

In fact, let Q' be another point, and φ a collineation sending Q in Q'. Then φ gives a bijection of the elements K of X , such that

$$K \text{ covers } Q \Leftrightarrow \varphi(K) \text{ covers } \varphi(Q) = Q' .$$

Let us consider now

$$\text{P} = \underset{\text{here the event is the choice of (Q,K) in } \pi\times X}{\text{prob}(Q \text{ is not covered by secant of K})} .$$

This number can be computed in the two different ways:

$$P = \sum_{Q\in\pi} \left[\text{prob}(Q) \cdot \underset{Q \text{ being fixed}}{\text{prob}\,(Q \text{ is not covered by secants of K})}\right] =$$

$$= \sum_{Q\in\pi} \left[\frac{1}{\#\pi} \cdot x\right] = \frac{\#\pi}{\#\pi} \cdot x = x$$

$$P = \sum_{K\in X} \left[\text{prob}(K) \cdot \underset{K \text{ being fixed}}{\text{prob}\,(Q \text{ is not covered by } K))}\right] \leq$$

$$\leq \sum_{K\in X} \left[\frac{1}{|X|} \cdot \left(\frac{1}{2} + \varepsilon\right)\right] = |X| \; (\frac{1}{2} + \varepsilon) = (\frac{1}{2} + \varepsilon)$$

so we can conclude that $x \leq (\frac{1}{2}+\varepsilon)$.

Now, let Q be a fixed point. I want to prove that

$$\underset{\text{here the event is the independent choice of } K_1,\dots,K_l \text{ in } X}{\text{prob}(Q \text{ is not covered by secant of } K_1 \cup \dots \cup K_l)} \leq (\frac{1}{2}+\varepsilon)^l \ .$$

In fact

Q is not covered by $K_1 \cup \dots \cup K_l \Rightarrow$

Q is not covered by each one among $K_1, \dots, K_l$

so that

prob(Q is not covered by secant of $K_1 \cup \dots \cup K_l$)$\leq$

$\leq \prod_{i=1,\dots,l}$ prob(Q is not covered by K_i) $\leq (\frac{1}{2}+\varepsilon)^l$.

Consider now

$$y = \underset{\text{where the event is the independent choice of } K_1,\dots,K_l \text{ in } X}{\text{prob}(\exists Q \in \prod \mid Q \text{ is not covered by } K = K_1 \cup \dots \cup K_l)} \ .$$

Then

$$y \leq \sum_{Q \in \pi} \text{prob}(Q \text{ is not covered by } K = K_1 \cup \dots \cup K_l) \leq |\pi| \ (\frac{1}{2}+\varepsilon)^l \ .$$

If we choose l such that the term on the right is < 1,then we are done, because there is a non zero probability of finding $K_1, \dots, K_l$ in such a way that $W = K_1 \cup \dots \cup K_l$ satisfies our thesis.

It is enough to choose

$$l = [\log_{\frac{1}{\frac{1}{2}+\epsilon}} |\pi|] + 2 = [\ \log_{\frac{2}{1+2\epsilon}} |\pi|] + 2 = [(\log_{\frac{2}{1+2\epsilon}} 2) \cdot \log_2 |\pi|] + 2$$

so that

$$l \ \leq (\log_{\frac{2}{1+2\epsilon}} 2) \cdot \log_2 |\pi| \ + 2 \underset{because\ q \geq 41}{<} 2 \log_2 |\pi| \ = \ 2 \log_2 \left(q^2+q+1\right) \ .$$

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References

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