KRASNOSEL'SKII NUMBERS AND NON SIMPLY CONNECTED ORTHOGONAL POLYGONS

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ABSTRACT. Let $S = T \sim (\cup\{A : A \text{ in } \mathcal{A}\})$, where T is a simply connected orthogonal polygon and $\mathcal{A}$ is a collection of n pairwise disjoint open rectangular regions contained in T. Point x belongs to the staircase kernel of S, Ker S, if and only if x belongs to Ker T and neither the horizontal nor the vertical line through x meets any A in $\mathcal{A}$. This produces a Krasnosel'skii-type theorem for S in turns of n. However, an example shows that, independent of n, no general Krasnosel'skii number exists for S.

1. Introduction. We begin with some definitions from [3]. Let S be a nonempty set in $\mathbb{R}^2$. Set S is called an *orthogonal polygon* or a *rectilinear polygon* if and only if S is a connected union of finitely many convex polygons (possibly degenerate) whose edges are parallel to the coordinate axes. We say that two vectors have the same direction if and only if one is a positive multiple of the other. Let λ be a simple polygonal path in $\mathbb{R}^2$ whose edges $[v_{i-1}, v_i]$, $1 \leq i \leq m$, are parallel to the coordinate axes. Path λ is called a *staircase path* if and only if the associated vectors determined by its edges alternate in direction. That is, for an appropriate labeling,for i odd the vectors $\overrightarrow{v_{i-1}v_i}$ have the same horizontal direction, and for i even the vectors $\overrightarrow{v_{i-1}v_i}$ have the same vertical direction. We say that point v_i is (directly) *north, south, east,* or *west* of v_{i-1} according to the direction of vector $\overrightarrow{v_{i-1}v_i}$. Similarly, we use the terms *northeast, northwest, southeast, southwest* to describe the relative position of points. For points x and y in set S, we say *x sees y* (*x is visible from y*) *via staircase paths* if and only if there is a staircase path in S which contains both x and y. Set S is *starshaped via staircase paths* if and only if for some point p in S, p sees each point of S via staircase paths, and the set of all such points p is the

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staircase kernel of S, Ker S. Set S is called *horizontally* convex if and only if for each x, y in S with $[x, y]$ horizontal, it follows that $[x, y] \subseteq S$. *Vertically convex* is defined analogously. Finally, set S is called an *orthogonally convex polygon* if and only if S is an orthogonal polygon which is both horizontally convex and vertically convex.

Many results in convexity involve the idea of visibility via straight line segments. (See [5], [6], [7], [8]). However, in work with orthogonal polygons ([4], [9], [10]), often it is useful to replace visibility via segments with the related concept of visibility via staircase paths. It turns out that, with this replacement, we can establish analogues of some familiar segment visibility results. For example, the Krasnosel'skii theorem [7] states that for S nonempty and compact in $\mathbb{R}^2$, S is starshaped (via segments) if and only if every 3 points of S are visible (via segments) from a common point. We may replace points of S with boundary points of S for a stronger result. Analogously, for S a nonempty simply connected orthogonal polygon in $\mathbb{R}^2$, S is starshaped via staircase paths if and only if every 2 boundary points of S are visible (via staircase paths) from a common point [1]. Moreover, Victor Chepoi [4] has extended the result, proving that for a finite union of boxes S in $\mathbb{R}^n$ whose intersection graph is a tree, S is starshaped (via staircase paths) if and only if every 2 boundary points of S are visible (via staircase paths) from a common point. A natural question to ask is whether or not a Krasnosel'skii- type theorem exists, even in $\mathbb{R}^2$, when S is an arbitrary union of boxes. As it turns out, when $\mathbb{R}^2 \sim S$ contains exactly n bounded components, each a rectangular region, then there is a Krasnosel'skii result in terms of n. However, an example reveals that no such theorem is possible independent of n.

Throughout the paper, bdry S will denote the boundary of set S. The reader may refer to Valentine[11], to Lay [8], to Danzer, Grünbaum, Klee [5], and to Eckhoff [6] for discussions concerning visibility via segments and Krasnosel'skii-type theorems.

2. The results. The following theorem will yield a Krasnosel'skii-type result for certain non simply connected orthogonal polygons.

Theorem 1. *Let $S = T \sim (\cup\{A : A \text{ in } \mathcal{A}\})$, where T is a simply connected orthogonal polygon and $\mathcal{A}$ is a collection of n pairwise disjoint open rectangular regions contained in T. Point x belongs to Ker S if and only if x belongs to Ker T and neither the horizontal nor the vertical line through x meets any A in $\mathcal{A}$.*

Proof. To establish the necessity, let x belong to Ker S. For A in $\mathcal{A}$, since x sees via staircase paths in S each vertex of A, it is easy to show that x sees via staircase paths in T each point of A. Hence $x \in$ Ker T. Furthermore, if the horizontal line h through x were to meet some set A, then x could not see via staircase paths in S points of $h \cap S$ beyond A from x. Thus the second condition holds as well.

For the sufficiency, let point x in Ker T satisfy the condition above. Define set $S_x = \{y\colon x \text{ sees } y \text{ via staircase paths in } S\}$. Observe that set S is connected. Also, since $x \in S_x$, $S_x \neq \emptyset$. We will show that $S = S_x$, using induction on the number n of sets A in $\mathcal{A}$. If $n = 0$, then $S=T$ and $S_x=S$. Assume that the result holds for $0 \leq n \leq k-1$, and let $n = k$. We will prove that the nonempty set S_x is both open and closed relative to the connected set S and thus $S=S_x$.

By an argument like the one in [3, Lemma 1], set S_x is a finite union of closed rectangular regions, hence closed and certainly closed relative to S. To prove that S_x is open relative to S, choose y in S_x to show that for some neighborhood N of y, x sees via staircase paths in S all points of $N \cap S$. By [2, Lemma 1], it suffices to show that for horizontal line h through y and vertical line v through y, x sees (via staircase paths in S) each point of $h \cup v$ near y. Without loss of generality, assume x is northwest of y, and let $x = x_o, x_1, \ldots, x_m, y$ be distinct consecutive vertices of a staircase path in S from x to y, with $[x_m, y]$ vertical and $m \geq 1$. Then x sees (via staircase paths in S) all points near y which lie either on v or on h east of y.

Assume that there are points of h west of y and near y. Such points are visible from x (via staircase paths in S) when S contains a nondegenerate rectangle along $[x_m, y]$ and west of $[x_m, y]$. If S contains no such rectangle, then some subset of $[x_m, y]$ must meet bdry S and must bound on the east a rectangle in $\sim S$ (See Figure 1.).

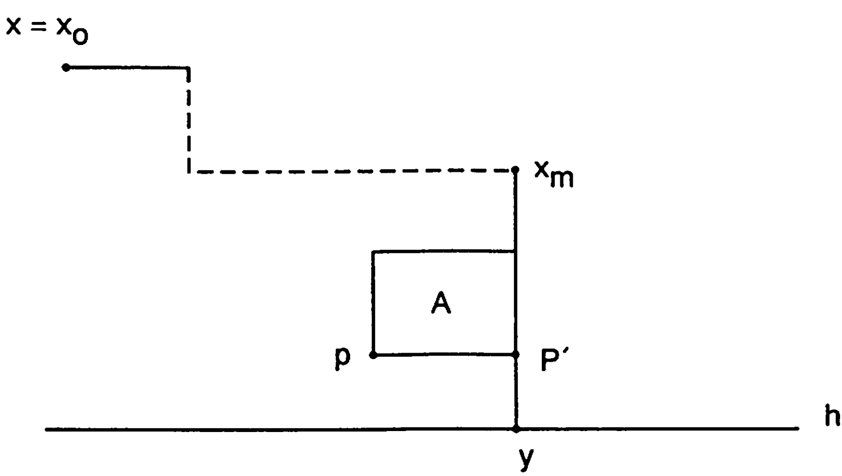

Figure 1.

Since $x \epsilon$ Ker T and T is simply connected, this can occur if and only if for some set A in $\mathcal{A}$, (x_m, y) meets bdry A on the east. Assume that such a set A has been chosen as close as possible to h. Then A lies between lines $L(x_{m-1}, x_m)$ and h, so A is south of the horizontal line through x. Furthermore, since no vertical line at x meets A, A is east of the vertical line through x.

Consider set $S \cup A$. Since its complement contains only k-1 members of $\mathcal{A}$, we may apply our induction hypothesis to conclude that point x sees via staircase paths in $S \cup A$ each point of $S \cup A$. Hence x sees the southwest corner p of A via a staircase path λ in $S \cup A$. Since set A is southeast of x, p is either directly south of x or southeast of x. Thus λ is disjoint from A, and $\lambda \subseteq S$. Let $[p, p']$ be the south edge of A. By our choice of A, either $p' = y$ or S contains a nondegenerate rectangle along $[p', y]$ and west of $[p', y]$. In either case, x sees points of h west of y and near y (via staircase paths in S). We conclude that S_x is open relative to S and therefore $S_x = S$ when $n = k$, finishing the induction and completing the proof.

Corollary 1. *Let $S = T \sim (\cup\{A : A \text{ in } \mathcal{A}\})$, where T is a simply connected orthogonal polygon and $\mathcal{A}$ is a collection of n pairwise disjoint open rectangular regions contained in T. Set S is starshaped via staircase paths if and only if every $4n$ points of S see via staircase paths in S a common point of Ker T.*

Proof. Choose one nonvertex point along each edge of A for every A in $\mathcal{A}$. If point x in Ker T sees (via staircase paths in S) each of these $4n$ points, then neither the horizontal nor the vertical line through x can meet any A set. Hence $x \epsilon$ Ker S. The converse is immediate.

However, without taking into account the cardinality of $\mathcal{A}$, no Krasnosel'skii member exists to guarantee that an arbitrary orthogonal polygon will be starshaped via staircase paths. The following definition and theorem will be useful in constructing a counterexample.

Definition. Let S be an orthogonal polygon with $A_1, \ldots, A_n$ the bounded components of $\mathbb{R}^2 \sim S$. We say that A is a *south domino* for point x in S if and only if for an appropriate labeling of certain bounded components $A_1, \ldots, A_m = A$, some point of A_1 is directly south of x and, for $2 \leq i \leq m$, some point of A_i is directly south of a point of A_{i-1}. The *south domino region* for x is the set of all y in S such that y is directly south of a point of A for some south domino A of x. Similar definitions can be made for north,east, west dominos and corresponding regions.

Theorem 2. *Let $S = T \sim (\cup\{A : A \text{ in } \mathcal{A})$, where T is an orthogonally convex polygon and $\mathcal{A}$ is a collection of n pairwise disjoint open rectangular regions contained in T. For points x, y in S, x sees y via S if and only if y is not in a domino region for x.*

Proof. The necessity is obvious. For the sufficiency, assume that y is not in a domino region for x. (Of course, this implies that x is not in a domino region for y, either.) We proceed by induction on the number n of sets A in $\mathcal{A}$. The result is clear when $n = 0$, so assume that it is true for $0 \leq n \leq k - 1$, and let $n = k$. If y is directly south, north, east, or west of x, the result is immediate. Without loss of generality, suppose that y is southeast of x.

For the moment, suppose that east dominos of x exist, and let A be an east domino of x whose south edge is as far south as possible. Let s be any nonvertex of A on this edge. Consider the set $S \cup A$. Since y does not belong to any domino region of x in S, certainly y does not belong to any domino region of x in $S \cup A$. Since $T \sim (S \cup A)$ is a union of $k - 1$ rectangular regions, by our induction hypothesis there is a staircase path λ in $S \cup A$ from x to y. If $\lambda \subseteq S$, this part of the argument is finished, so assume that $\lambda \cap A \neq \emptyset$. Observe that point s is either directly west or northwest of y since y is not in a domino region for x. Let a, a' be the endpoints of path $\lambda \cap A$. Then neither a nor a' is a vertex of A, and for an appropriate labeling, point a is on the west edge of A. If a' is on the south edge of A, then $\lambda \cap A$ may be replaced by a subset of bdry A to produce a staircase path from x to y in S. If a' is not on the south edge of A, then a' is on the west edge of A. Let t be any point of λ lying directly east of s. (See Figure 2.) Observe that by our choice of A, for A' in $\mathcal{A}$, A' is disjoint from $[s, t]$. Since T is orthogonally convex, $[s, t] \subseteq T$, and since $[s, t]$ is disjoint from $\cup\{A : A \text{ in } \mathcal{A}\}$, it follows that $[s, t] \subseteq S$. Then λ from a to t may be replaced by a (southeast) subset of bdry A from a to s, followed by $[s, t]$, to produce an $x - y$ staircase path in S.

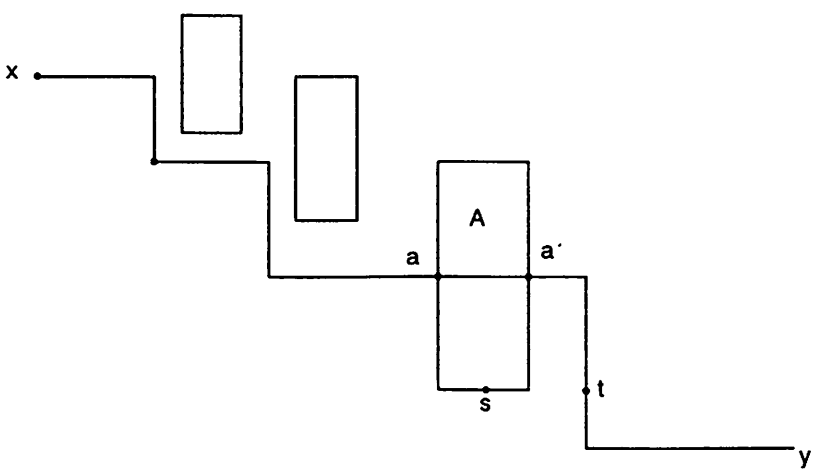

Figure 2.

By symmetry, an analogous argument may be used if there is a south domino of x, a north domino of y, or a west domino of y. If none of these occur, let R denote the rectangular region of the plane having x, y as two of its vertices. Select x' at x or east of x, y' at y or north of y, such that segments $[x, x']$ and $[y, y']$ lie in S and have maximal length. If these segments meet at point z, then $[x, z] \cup [z, y]$ is an $x - y$ staircase path in S. If the segments are disjoint, then since T is orthogonally convex and bdry R is disjoint from $\cup\{A : A \text{ in } \mathcal{A}\}$, both x' and y' belong to R. Again using the orthogonal convexity of T, an appropriate subset of bdry T from x' to y', together with $[x, x']$ and $[y, y']$, produces an $x - y$ staircase path in S. Thus the result is true when $T \sim S$ is a union of k rectangular regions, and by induction the proof is complete.

Comments. Let S be an orthogonal polygon in the plane, with $A_1, \ldots, A_n$ the bounded components of $\mathbb{R}^2 \sim S$. For x in S, certainly neither the horizontal nor the vertical line through x meets any A_i if and only if x is not in any domino region in S. Thus these two concepts are interchangeable in Theorem 1. However, the result in Theorem 2 concerns specific points x and y, neither of which needs to belong to Ker S. In fact, Ker S may be empty. Furthermore, it is interesting to notice that Theorem 2, unlike Theorem 1, fails without the requirement that set $T = S \cup A \cup \cdots \cup A_n$ be orthogonally convex. Consider the following example.

Example 1. Let T be the simply connected orthogonal polygon in Figure 3, with $S = T \sim A$. For labeled points x and y, $x \in$ Ker T and y is not in a domino region for x in S. However, x cannot see y via staircase paths in S.

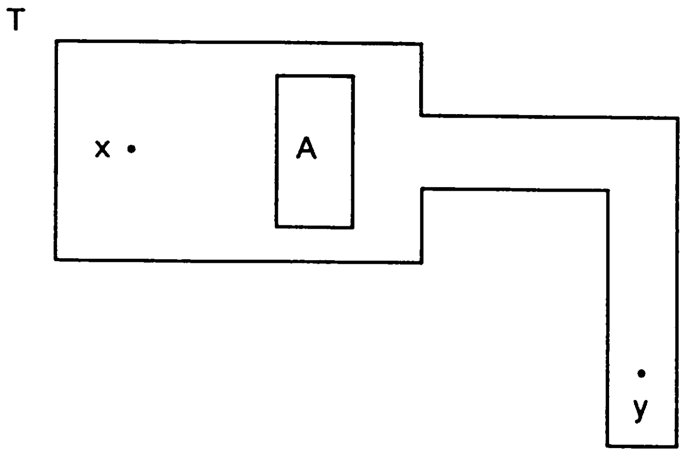

Figure 3.

Our second example shows that no general Krasnosel'skii number exists for orthogonal polygons which are not simply connected.

Example 2. Let T be the simply connected orthogonal polygon in Figure 4. For n odd, $n \geq 5$, a collection $\mathcal{A}$ of $4n$ open rectangular regions may be arranged in an "alternating domino" pattern according to the scheme in that figure. Observe that region A_2 is an east domino for points on the south edge of A_1, A_3 is a south domino for points on the east edge of A_2, and so forth, with directions changing at A_n, A_{2n}, A_{3n}, A_{4n}. (Figure 4 illustrates the construction for $n = 7$.) Let $S = T \sim (\cup\{A: A \text{ in } \mathcal{A}\})$. By inspection, each point in S has an associated family of at most 10 dominos. (Point x, for example, has 10 dominos.) Hence for $10k < 4n$ and for any k points $x_1, \ldots, x_k$ in S, there is at least one A in $\mathcal{A}$ which is not a domino for any $x_i, 1 \leq i \leq k$. For y in bdry A, clearly y cannot belong to a domino region for any $x_i, 1 \leq i \leq k$, so by Theorem 2 y sees $x_1, \ldots, x_k$ via staircase paths in S. However, for every p in S, either the horizontal or the vertical line at p meets some A in $\mathcal{A}$, and thus by Theorem 1 S is not starshaped via staircase paths. Since n may be chosen as large as we like, no Krasnosel'skii number exists for such orthogonal polygons S.

Concluding remarks. As Victor Chepoi points out in [4], a staircase path is a shortest rectilinear path (l_1 - path) between its endpoints. Hence Example 2 shows that there is no general Krasnosel'skii number to characterize l_1-starshaped sets.

Finally, it would be interesting to replace $4n$ in Corollary 1 with the best bound and to find an appropriate n-dimensional analogue of the result.

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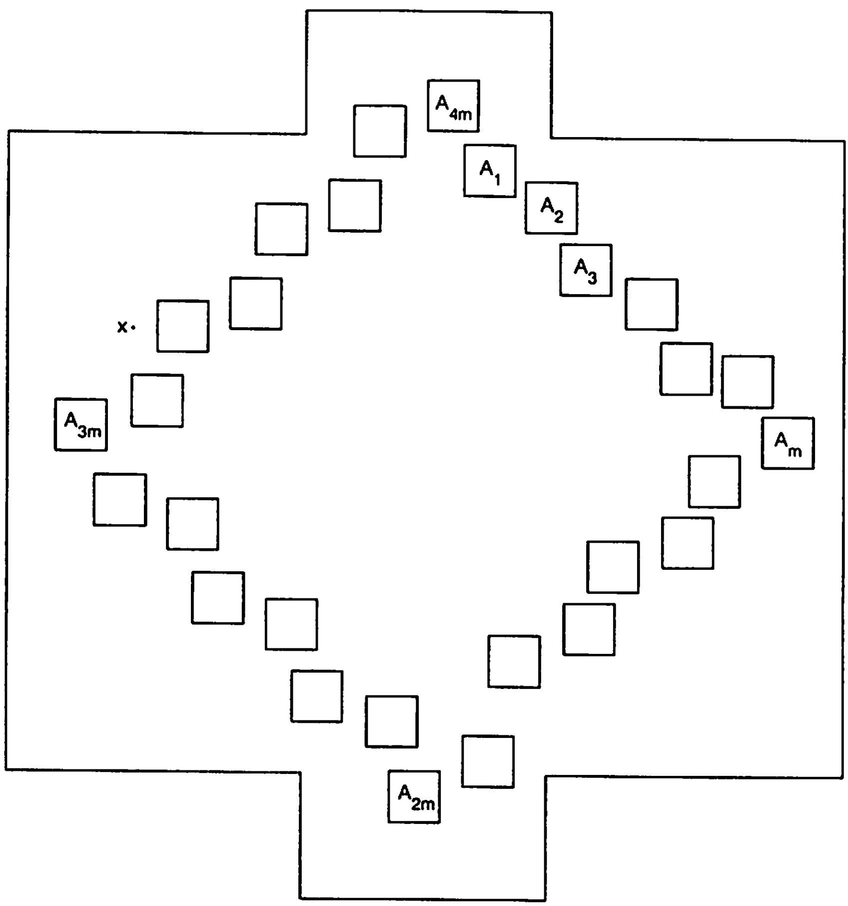

Figure 4.