Domination Sequences

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Abstract

Let $I(G)$ be a graphical invariant defined for any graph G. For several choices of I representing domination parameters, we characterize sequences of positive integers $a_1, a_2, \ldots, a_n$ which have an associated sequence of graphs $G_1, G_2, \ldots, G_n$ such that G_i has i vertices, G_i is an induced subgraph of G_{i+1}, and $I(G_i) = a_i$.

1 Introduction

Consider a graphical invariant $I(G)$ defined for any graph G. We are interested in determining sequences of positive integers $a_1, a_2, \ldots, a_n$ for which there is an associated sequence of graphs $G_1, G_2, \ldots, G_n$ such that G_i is an induced subgraph of G_{i+1} for $i = 1, 2, \ldots, n-1$; G_i has i vertices $v_1, v_2, \ldots, v_i$; and $I(G_i) = a_i$ for $i = 1, 2, \ldots, n$. We will refer to such sequences as *achievable* (under I) and all other sequences as *unachievable* (under I). Also, we will use the subscript i to specify objects associated with G_i. Thus, $I(G_i)$ often will be written I_i.

This work was motivated by that of Harary and Kabell [4] who answered the question posed for monotonic sequences involving vertex connectivity, edge connectivity, minimum degree, maximum degree, chromatic index, diameter, and number of edges. They point out that the restriction to monotonic sequences is unnecessary.

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A different type of graph sequence has been studied by Rasmussen [8] and Odom and Rasmussen [5]. They deal with sequences $G_0, G_1, \ldots, G_k$ where $k = \binom{n}{2} - q$, G_i has order n for all i, G_0 has size q, and G_{i+1} is obtained from G_i by adding an edge. The sequence of edges added is called a *P-completion sequence*, where P is some graphical property, if each graph G_i has P. Their work examines the existence of P-completion sequences for various categories of graphs G_0.

A great deal of research has involved the domination number and its many associated versions. This is partially due to the importance these parameters play in applications. In work which is somewhat related to that presented here, Vasumathi and Vangipuram [9] have demonstrated graphs which, given a positive integer r, have r as the domination number, edge domination number, total domination number, or total edge domination number. A deeper concern, however, is the changes domination related parameters undergo when an application being modeled by a graph loses the equivalent of a vertex, so studies of possible sequences involving these parameters become interesting. This paper initiates one such study by taking I to be a domination related parameter. In general, the graphs G_i have no restrictions imposed on them, other than those which are a part of the definition of the problem. For some of our results, however, we will insist that the G_i be connected.

2 Three Domination Numbers

In this section we investigate three choices of I: the *domination number* $\gamma(G)$, the *independent domination number* $\gamma_I(G)$, and the *connected domination number* $\gamma_c(G)$ of graph G. A set of vertices in G dominates the graph G if every vertex in G is either in the set or is adjacent to a vertex in the set. The domination number is the minimum size of a set of vertices which dominates. The independent domination number is the minimum size of an independent set of vertices which dominates. The connected domination number is the minimum size of a dominating set in G which induces a connected graph. When considering connected domination numbers, the graphs G_i are necessarily connected. It is straightforward to show that $1 \leq \gamma_i \leq i$ and $\gamma_{i+1} \leq \gamma_i + 1$, with corresponding inequalities applying to γ_{Ii}. The second inequality also holds for connected domination, but the first changes to $1 \leq \gamma_{ci} \leq i - 2$, if $i \geq 3$. Our first result shows that any sequence, limited only by these elementary restrictions, is achievable under γ.

Theorem 1 *Any sequence which satisfies $1 \leq a_i \leq i$ and $a_{i+1} \leq a_i + 1$ is achievable under γ.*

Proof. Given a sequence satisfying the two criteria, we inductively construct the associated graphs G_i in such a way that each G_i will have a_i components, each dominated by a single vertex. Clearly $a_1 = 1$ and G_1 is an isolated vertex. Now suppose we have a graph G_i which consists of a_i components, each dominated by one vertex. Let $a_{i+1} = a_i - k$ where $-1 \leq k \leq a_i - 1$. To construct G_{i+1}, arbitrarily choose $k+1$ components and add a new vertex v_{i+1} with edges to all vertices of each of the chosen components. Thus, if $k = -1$, that is, $a_{i+1} = a_i + 1$, then the new vertex v_{i+1} is isolated, forming a new component, and $\gamma_{i+1} = \gamma_i + 1$. If $k \geq 0$, then v_{i+1} dominates in G_{i+1} a single component consisting of v_{i+1} and the vertices of the $k+1$ chosen components. Each of the remaining $\gamma_i - (k+1)$ components is left unchanged and so is still dominated by one vertex. Hence, $\gamma_{i+1} = 1 + \gamma_i - (k+1) = \gamma_i - k$ as required. □

Notice that this construction actually produces a minimum independent dominating set. Thus, we have the following corollary.

Corollary 2 *Any sequence which satisfies* $1 \leq a_i \leq i$ *and* $a_{i+1} \leq a_i + 1$ *is achievable under* γ_I.

The case of connected domination is only slightly more complicated.

Theorem 3 *Any sequence which satisfies* $a_1 = a_2 = 1$, $1 \leq a_i \leq i - 2$ *for* $i \geq 3$, *and* $a_{i+1} \leq a_i + 1$ *is achievable under* γ_c.

Proof. The graph G_i will include a path, P_i, with end vertices L_i and R_i, where possibly L_i dominates a subgraph distinct from the path. For the special case $i = 1$, we have $a_1 = 1$ and G_1 is the isolated vertex, v_1, called L_1 for the left end of the path. To construct G_2, we connect the next vertex, v_2, with an edge to L_1. Set L_2 equal to L_1 and R_2 to v_2; so, $\gamma_{c,2} = 1$. Our restrictions also imply $a_3 = 1$ and we obtain G_3 by connecting a new vertex, v_3, to L_2, setting L_3 and R_3 equal to L_2 and R_2, respectively, and renaming v_3 as s.

For $3 \leq i \leq n-1$, we assume that G_i is as in Figure 1 where there are $a_i + 1$ vertices along a path, P_i, from L_i to R_i inclusive. The subgraph H_i consists of one or more vertices all with edges to L_i and includes the vertex s which is not adjacent to any other vertex in P_i. The above construction establishes this form when $i = 3$ with v_3 being the only vertex in H_3. Observe that the vertices of $V(P_i) - \{R_i\}$ form a minimum connected dominating set of G_i when $i = 3$, and this property will be retained for larger indices.

If $a_{i+1} = a_i$, we set L_{i+1} equal to L_i, R_{i+1} equal to R_i and connect the new vertex, v_{i+1}, to L_{i+1}. If $a_{i+1} = a_i + 1$, v_{i+1} is connected to R_i and becomes R_{i+1} while L_{i+1} equals L_i. The domination number increases by one because R_i must be added to the connected dominating set to dominate R_{i+1}. Finally, if $a_{i+1} = a_i - k$ with $k > 0$, then v_{i+1} is added to the graph with edges to all vertices in H_i, to L_i, and to $k+1$ consecutive vertices along P_i beginning with the vertex adjacent to L_i. In addition, v_{i+1} becomes L_{i+1} and H_{i+1} now includes H_i, L_i, and the next k former path vertices.

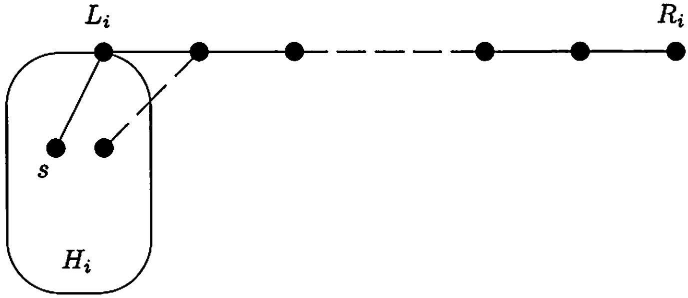

Figure 1: G_i. Note that dotted edge is not present before any decreases in the sequence.

Note that the dotted edge shown in the figure from a vertex of H_i to the vertex of P_i adjacent to L_i is not present until the first decrease in the sequence of a_i's. After any such decrease, H_{i+1} still contains s which is not adjacent to any vertex in $V(P_{i+1}) - \{L_{i+1}\}$. Furthermore, any connected dominating set must include every vertex in P_{i+1}, except R_{i+1} and L_{i+1}. At least one more vertex also must be included to dominate s, and L_{i+1} does this. Since L_{i+1} also dominates H_{i+1}, it and the path vertices other than R_{i+1} form a connected dominating set for G_{i+1}. Thus, $\gamma_{c,i+1} = a_i - (k+1) + 1 = a_{i+1}$. It is straightforward to see that G_{i+1} still satisfies the properties indicated by Figure 1 and the result follows by induction. □

3 Domination Number and Connected Graphs

In this section we restrict our attention to the domination number $\gamma(G)$ of graph G with the additional constraint that each of the graphs G_i must be connected. This restriction might correspond to a step by step construction of a communication network that remains connected at every stage.

In this situation, $a_i = \gamma_i$, and we need the following results from Ore [6] and Payan and Xuong [7].

Theorem 4 *For any graph G with $n \geq 2$ vertices,*

1. [6] $\gamma(G) \leq \lfloor \frac{n}{2} \rfloor$ *if G has no isolates, and*

2. [7] *if* $\gamma(G) = \frac{n}{2}$, *then each component of* G *is the cycle* C_4 *or the corona* $H \circ K_1$ *for some graph* H.

Theorem 4, Part 1, indicates that all achievable sequences must have, in addition to the standard restriction on the domination number of $a_{i+1} \leq a_i + 1$, that $1 \leq a_i \leq \lfloor \frac{i}{2} \rfloor$ if $i \geq 2$. We designate such a sequence as *legitimate* and restrict our attention to sequences of this type. Notice that any legitimate sequence with three or more entries must begin 1, 1, 1. In the following lemma, we employ the notation $\hat{\beta}(G)$ for the size of a maximum independent set of edges in a graph G such that each edge in the set is incident to a degree one vertex (we call a vertex of degree d a *degree d vertex*).

Lemma 5

1. *For any graph* G, $\hat{\beta}(G) \leq \gamma(G)$.
2. *If* $\gamma_{i+1} < \gamma_i$, *then* v_{i+1} *must be in any minimum dominating set of* G_{i+1}.
3. *If* $\gamma_{i+1} < \gamma_i$, *then* $\hat{\beta}_{i+1} \leq \gamma_{i+1} - 1$.
4. *If* $\gamma_{i+1} < \gamma_i$ *and* $\gamma_{i+1+k} = \gamma_{i+1} + k$ *for all* $k \geq 0$, *then* $\hat{\beta}_{i+1+k} \leq \gamma_{i+1+k} - 1$ *for all* $k \geq 0$.

Proof. A unique vertex is required to dominate the degree one vertex of each edge in the set counted by $\hat{\beta}(G)$, which proves Part 1. For Part 2, suppose a minimum dominating set of G_{i+1} exists which does not contain v_{i+1}. Then some γ_{i+1} vertices of G_i dominate G_{i+1} and hence also G_i, a contradiction to the fact that $\gamma_i > \gamma_{i+1}$. Using Part 2 we know v_{i+1} is in every minimum dominating set of G_{i+1} which implies v_{i+1} can not have degree one. Also, v_{i+1} can not have a degree one neighbor since then that neighbor would have been isolated in G_i. Thus at most $\gamma_{i+1} - 1$ vertices in the dominating set can dominate degree one vertices and Part 3 is shown. Finally, the result of Part 4 is true when $k = 0$ by Part 3. For each increase of k by one, $\hat{\beta}$ can increase by at most one while the domination number increases by exactly one. □

Using the previous lemma, we can show that certain legitimate sequences are not achievable.

Lemma 6 *If the sequence contains an index* i *such that* $\gamma_{i+1} < \gamma_i$ *and* $\gamma_{2(i+1-\gamma_{i+1})} = i + 1 - \gamma_{i+1}$, *then the sequence is unachievable.*

Proof. Observe that the hypothesis $\gamma_{i+1} < \gamma_i$ can occur only if $i \geq 4$. The index $2(i+1-\gamma_{i+1})$ is $i+1-2\gamma_{i+1}$ greater than the index $i+1$. Furthermore, $\gamma_{i+1} + (i+1-2\gamma_{i+1}) = i+1-\gamma_{i+1}$. Thus, if $\gamma_{2(i+1-\gamma_{i+1})} = i+1-\gamma_{i+1}$, it means $\gamma_{i+1+k} = \gamma_{i+1} + k$ for $0 \leq k \leq i+1-2\gamma_{i+1}$. By Lemma 5 Part 4, $\hat{\beta}_{2(i+1-\gamma_{i+1})} \leq \gamma_{2(i+1-\gamma_{i+1})} - 1$, contradicting Theorem 4 Part 2 which requires $\hat{\beta}_{2(i+1-\gamma_{i+1})} = \gamma_{2(i+1-\gamma_{i+1})}$. □

As an example, notice that Lemma 6 can be used to show that the sequence $1,1,1,2,2,3,2,3,4,5$ is unachievable by letting $i = 6$. We will show that the only legitimate sequences which are unachievable are ones of the type covered by the lemma. Toward this end we present the following algorithm which we will show correctly generates the graphs G_i for any achievable sequence. We partition the vertices of G_i into sets D_i containing a current minimum dominating set, P_i containing vertices which are potential dominating vertices of G_{i+1}, and U_i containing vertices which may not become dominating vertices of G_{i+1}. We also define $D_{i,0}$ to be the set of vertices in D_i which have no degree one neighbor.

Algorithm for generating the graphs G_i

1. Set G_1 to be an isolated vertex v_1. Set $D_1 = \{v_1\}$ and $D_{1,0} = P_1 = U_1 = \emptyset$.

2. Obtain G_2 by adding a pendant vertex v_2 to v_1. Set $D_2 = D_1$, $U_2 = \{v_2\}$, $D_{2,0} = P_2 = \emptyset$.

3. For $i \geq 2$, obtain G_{i+1} from G_i by adding a vertex v_{i+1} according to the following steps:

   (a) If $\gamma_{i+1} = \gamma_i$, add pendant vertex v_{i+1} adjacent to any vertex $w \in D_i$ with the added proviso that w must be selected from $D_{i,0}$ if $D_{i,0} \neq \emptyset$. Let M be the neighbors of w which are in U_i. Set $D_{i+1} = D_i$, $D_{i+1,0} = \emptyset$, $P_{i+1} = P_i \cup M$, and $U_{i+1} = (U_i - M) \cup \{v_{i+1}\}$.

   (b) If $\gamma_{i+1} = \gamma_i + 1$, add pendant vertex v_{i+1} adjacent to any vertex $w \in P_i$. Set $D_{i+1} = D_i \cup \{w\}$, $D_{i+1,0} = D_{i,0}$, $P_{i+1} = P_i - \{w\}$, and $U_{i+1} = U_i \cup \{v_{i+1}\}$.

   (c) If $\gamma_{i+1} = \gamma_i - k$ for $k \geq 1$, define $\hat{D}$ to be a set of any $k+1$ vertices of D_i with the added proviso it must include any vertex in $D_{i,0}$, $\hat{N}$ to be $\bigcup_{x \in \hat{D}} N[x] - (D_i - \hat{D})$, y to be a degree one vertex in $\hat{N}$, and z to be y's neighbor. Add vertex v_{i+1} adjacent to all vertices in $\hat{N}$. Set $D_{i+1} = \{v_{i+1}\} \cup (D_i - \hat{D})$, $D_{i+1,0} = \{v_{i+1}\}$, $P_{i+1} = P_i \cup \hat{N} - \{y,z\}$, and $U_{i+1} = (U_i - \hat{N}) \cup \{y,z\}$.

We will show that the algorithm correctly generates all legitimate sequences of graphs except for the type described in Lemma 6. We require the following lemma.

Lemma 7 *After each step of the execution of the algorithm,* $|D_{i,0}| \leq 1$. *For* $i \geq 2$, *every vertex in* $D_i - D_{i,0}$ *is adjacent to a pendant vertex in* U *and* $\hat{\beta}_i \geq \gamma_i - 1$. *Finally, if* $\gamma_{i+1} = \gamma_i$, *then* $\hat{\beta}_{i+1} = \gamma_{i+1}$.

Proof. By construction $|D_{1,0}| = |D_{2,0}| = 0$. For $i \geq 2$, either $D_{i+1,0} = \emptyset$ or $D_{i+1,0} = \{v_{i+1}\}$ or $D_{i+1,0} = D_{i,0}$. Thus $|D_{i+1,0}| \leq 1$ by induction. For $i = 2$, $D_2 - D_{2,0} = \{v_1\}$ and v_2 is a pendant vertex in U. Vertices are added to $D_i - D_{i,0}$ only in Step 3b which simultaneously adds an adjacent pendant vertex in U. Step 3a assures every vertex already in $D_i - D_{i,0}$ has a pendant vertex in U. Hence, for $i > 2$ every vertex in $D_i - D_{i,0}$ is adjacent to a pendant vertex in U by induction. Given that every element of $D_i - D_{i,0}$ is adjacent to a pendant vertex, $|D_{i,0}| \leq 1$ implies $\hat{\beta}_i \geq \gamma_i - 1$. If $\gamma_{i+1} = \gamma_i$, G_{i+1} is obtained using Step 3a; hence, $|D_{i+1,0}| = 0$ which in turn implies $\hat{\beta}_{i+1} \geq \gamma_{i+1}$. Equality follows from Lemma 5. □

The next lemma is needed to justify our construction of D_i.

Lemma 8 *After each step of the execution of the algorithm,* D_i *is a minimum dominating set.*

Proof. The result is immediate from Steps 1 and 2 for $i \leq 2$. Suppose that D_i is a minimum dominating set. If G_{i+1} is obtained via Step 3a, the result follows by induction since adding a pendant vertex to a member of a minimum dominating set leaves the same minimum dominating set. Suppose G_{i+1} is obtained via Step 3b. By Lemma 7 every vertex in $D_i - D_{i,0}$ is adjacent to a pendant vertex in U_i; hence, $w \notin U_i$ implies all of these vertices are also in a minimum dominating set of G_{i+1}. By construction, w also is adjacent to a pendant vertex and thus can be placed in any minimum dominating set. Further, if the vertex y exists due to an earlier execution of Step 3c, meaning $D_{i,0} \neq \emptyset$, it is not adjacent to w or any of the vertices in $D_i - D_{i,0}$ so one additional vertex is required to dominate it. The result now follows. Finally, suppose that G_{i+1} is obtained via Step 3c. As before, every vertex in $D_i - \hat{D}$ is adjacent to a pendant vertex in U_i which is also a pendant vertex in U_{i+1}. None of these vertices are adjacent to y; hence, they and v_{i+1} form a minimum dominating set and once again the result follows. □

A straightforward count reveals that the minimum dominating sets generated by the algorithm are all of the correct size, that is, $\gamma_i = a_i$. All that remains is to determine conditions which terminate the algorithm, which we do in the following theorem.

Theorem 9 *When each G_i is connected, every legitimate sequence is achievable under γ except any in which, for some j, $\gamma_{j+1} < \gamma_j$ and $\gamma_{2(j+1-\gamma_{j+1})} = j + 1 - \gamma_{j+1}$.*

Proof. Suppose that we have generated a sequence of graphs up to G_i. Clearly, we will be able to generate G_{i+1} if $\gamma_{i+1} = \gamma_i$. Suppose $\gamma_{i+1} = \gamma_i - k$ with $k \geq 1$. In this case, $|D_{i,0}| \leq 1$ and $k + 1 \geq 2$ imply, by Lemma 7, that at least one vertex in $\hat{D}$ is adjacent to a degree one vertex. Thus, the vertex y always exists and we will be able to generate G_{i+1}.

The only remaining possibility, $\gamma_{i+1} = \gamma_i + 1$, can be executed as long as P_i is not empty. Notice that the set U_i always contains exactly one degree one vertex adjacent to each vertex in $D_i - D_{i,0}$ and, when $D_{i,0}$ is not empty, contains two additional vertices adjacent to the vertex in $D_{i,0}$. If $\hat{\beta}_i = \gamma_i$ then every vertex is D_i has a degree one neighbor. Therefore, $D_{i,0}$ is empty, $|U_i| = \gamma_i$ and $|P_i| = i - |U_i| - \gamma_i = i - 2\gamma_i$. If P_i is empty we have $\gamma_i = i/2$. Hence, i is even, and $\gamma_{i+1} = i/2 + 1 = \lfloor \frac{i+1}{2} \rfloor + 1$ which violates the condition on legitimate sequences given in Theorem 4, Part 1. On the other hand, if $\hat{\beta}_i = \gamma_i - 1$, then $D_{i,0}$ is not empty and $|U_i| = \hat{\beta}_i + 2 = \gamma_i + 1$. In this case, $|P_i| = i - |U_i| - \gamma_i = i - 2\gamma_i - 1 = i + 1 - 2\gamma_{i+1}$ and there exists an index j such that $\gamma_{j+1} < \gamma_j$ followed by a strictly monotonic increase (possibly of zero terms) in the sequence values. Letting k be the number of increments in the monotonically increasing sequence, we have $i + 1 = j + 1 + k$ and $\gamma_{i+1} = \gamma_{j+1} + k$. If $|P_i| = 0$, then $j + 1 + k = 2\gamma_{i+1} = 2(\gamma_{j+1} + k) = 2\gamma_{j+1+k}$. From the equality involving the first and third terms of the previous string, we see that $k = j + 1 - 2\gamma_{j+1}$. Substituting this value for k in the preceding equalities we obtain $i + 1 = 2(j + 1 - \gamma_{j+1}) = 2\gamma_{i+1} = 2\gamma_{2(j+1-\gamma_{j+1})}$. Hence, $j + 1 - \gamma_{j+1} = \gamma_{2(j+1-\gamma_{j+1})}$ which is the case excluded by the hypothesis. □

4 Total Domination Number

The results for total domination, the invariant discussed in this section, are similar to those considered in the previous section for domination of connected graphs. A *total dominating set* of graph $G = (V, E)$ is a subset D of V such that every vertex of V, including those of D, has a neighbor in D. The cardinality of a smallest total dominating set is the *total domination number* of G, denoted $\gamma_t(G)$, and we are interested in those sequences for which $a_i = \gamma_{t,i}$.

Note that any achievable sequences must begin with $i = 2$ since any total dominating set must contain at least two vertices. This fact also shows that each of the graphs G_i must be connected. If this weren't the case, there would be a first G_i which is disconnected. This G_i must have a component composed of v_i alone, and this vertex cannot be totally dominated. As a vertex is added, the total domination number can either decrease by any amount as long as the result is at least two, stay the same, or increase by one. To verify this last statement, suppose that $\gamma_{t,i+1} > \gamma_{t,i}$ and let D be a total dominating set of G_i. Clearly no vertex of D is adjacent to v_{i+1}. If w is any neighbor of v_{i+1} in G_{i+1}, then $D \cup \{w\}$ is a total dominating set of G_{i+1}.

An upper bound for the total domination number of a connected graph has been found by Cockayne, Dawes, and Hedetniemi [2] to be $\lfloor 2n/3 \rfloor$. It follows that we must restrict attention to sequences such that $2 \le a_i \le \lfloor 2i/3 \rfloor$ and $a_{i+1} \le a_i + 1$. As in Section 3, we call such sequences *legitimate* and we will show that not all legitimate sequences are achievable.

The motivation for the present proof is found in the characterization of connected graphs for which $\gamma_t = \lfloor 2n/3 \rfloor$ and $n = 2 \pmod 3$ given by Brigham, Carrington, and Vitray [1].

Theorem 10 *A connected graph G with $n = 3k+2$ vertices has $\gamma_t = \lfloor 2n/3 \rfloor$ if and only if G is C_5 or is obtained from a connected graph L on $k+1$ vertices by identifying an end vertex of a distinct P_3 with each of k vertices of L and identifying one vertex of a P_2 with the remaining vertex of L.*

When constructing the sequence of G_i's, we may eliminate the C_5 possibility by using P_5 as our graph with total domination number three when $n = 5$. This choice allows us to assume that all the graphs in our sequence fit the general characterization given in Theorem 10. Note that the graphs characterized by the theorem have a minimum size total domination set composed of all vertices of degree at least two. In particular, this set is constructed from two vertices from each of the P_3's and one from the single P_2.

Motivated by the structure of the graphs given in the theorem, we define a P_2 formed by a pendant vertex to be a *pendant* P_2 with the vertex adjacent to the degree one vertex being called the *base vertex*. Similarly, a P_3 having a degree one vertex with an adjacent degree two vertex is a *pendant* P_3, the neighbor of the degree one vertex is the *central vertex* of the P_3, and the remaining vertex of the P_3 is the *base vertex*. In either a pendant P_2 or P_3, the degree one vertex is called an *end vertex*.

Suppose a graph G has a maximum size subgraph of the form $sP_2 \cup tP_3$ where the paths are restricted to be pendant ones with distinct base vertices. We define a parameter r of G, denoted $r(G)$, by $r(G) = s + 2t$. In any total dominating set of G, we may include without loss of generality all of the $r(G)$ base and central vertices of the pendant P_2's and P_3's. For an extremal graph having $3k+2$ vertices of the type described by Theorem

10, it is easy to see that $r(G) = \gamma_t(G) = 2k+1$. In the following lemma we derive a number of important facts regarding r, γ_t, and their relation to each other. As before, we use subscripts on graphical parameters to indicate properties of G_i so, for example, $r_i = r(G_i)$. The following basic results will be fundamental to our argument.

Lemma 11

1. *For any graph G, $r(G) \leq \gamma_t(G)$.*
2. *For any legitimate sequence, $r_{i+1} \leq r_i + 1$.*
3. *If $\gamma_{t,i+1} < \gamma_{t,i}$, then v_{i+1} must be in any minimum dominating set of G_{i+1} and no neighbor of v_{i+1} has degree one in G_{i+1}.*
4. *If $\gamma_{t,i+1} < \gamma_{t,i}$, then $r_{i+1} \leq \gamma_{t,i+1} - 1$.*
5. *If $\gamma_{t,i+1} < \gamma_{t,i}$ and $\gamma_{t,i+1+k} = \gamma_{t,i+1} + k$ for all $k \geq 0$, then $r_{i+1+k} \leq \gamma_{t,i+1+k} - 1$ for all $k \geq 0$.*

Proof. Part 1 follows from comments before the lemma. Part 2 is immediate from the fact that r can increase only by appending v_{i+1} either to the end vertex of a pendant P_2 or to a vertex not in a pendant P_2 or P_3, and in both cases r increases by 1. For Part 3, if D is a minimum total dominating set of G_{i+1} which does not include v_{i+1}, then D totally dominates G_i, contradicting the fact that $\gamma_{t,i+1} < \gamma_{t,i}$. Furthermore, no neighbor u of v_{i+1} can have degree one, for then u would have been isolated in G_i. To show Part 4, notice that v_{i+1} can not be the base vertex of a pendant P_2 or P_3. The former is banned by Part 3. The latter also is impossible because, for that situation to occur, v_{i+1} must be adjacent to an end vertex of a P_2 which must form a component in G_i, an impossibility since i must be greater than three if the total domination number is to decrease. The result of Part 5 is true for $k = 0$ by Part 4. The domination number, by hypothesis, increases by one for each increase of k by one, and, by Part 2, r increases by at most one for each increase of k by one. Thus Part 5 follows by induction. □

Lemma 11 can be employed to show certain legitimate sequences are unachievable.

Lemma 12 *Any sequence which contains an index i such that $\gamma_{t,i+1} < \gamma_{t,i}$ and*
$\gamma_{t,3(i-\gamma_{t,i+1})+2} = 2(i - \gamma_{t,i+1}) + 1$ *is unachievable.*

Proof. The index $3(i - \gamma_{t,i+1}) + 2$ is $2i + 1 - 3\gamma_{t,i+1}$ greater than the index $i + 1$. Furthermore, $\gamma_{t,i+1} + 2i + 1 - 3\gamma_{t,i+1} = 2(i - \gamma_{t,i+1}) + 1$. Since $\gamma_{t,3(i-\gamma_{t,i+1})+2} = 2(i-\gamma_{t,i+1})+1$, it follows that $\gamma_{t,i+1+k} = \gamma_{t,i+1}+k$ for $0 \leq k \leq 2i + 1 - 3\gamma_{t,i+1}$. By Lemma 11 Part 5, $r_{3(i-\gamma_{t,i+1})+2} \leq \gamma_{t,3(i-\gamma_{t,i+1})+2} - 1 = 2(i-\gamma_{t,i+1})+1-1 = 2(i-\gamma_{t,i+1})$. However, a graph with $3(i-\gamma_{t,i+1})+2$ vertices and total domination number $2(i-\gamma_{t,i+1})+1$ satisfies the conditions of Theorem 10. Since the total domination number cannot decrease unless the graph has at least five vertices, G_{i+1} cannot be C_5 and the comments following Theorem 10 indicate that $r_{3(i-\gamma_{t,i+1})+2} = 2(i - \gamma_{t,i+1}) + 1$ which provides our desired contradiction. □

We present an algorithm which generates the graphs G_i for all legitimate sequences except those described by Lemma 12. For each i, we construct a minimum dominating set D_i which is partitioned into four subsets $D_{i,0}$, $D_{i,1}$, $D_{i,2}$, and $D_{i,C}$. We will show that $D_{i,1}$ is composed of the base vertices of pendant P_2's in G_i, and $D_{i,2}$ similarly contains the base vertices of pendant P_3's whose central vertices are the members of $D_{i,C}$. The set $D_{i,0}$ is the set of remaining vertices of D_i. We will also demonstrate that, for $i \geq 4$, $|D_{i,0}| \leq 1$, and, if $D_{i,0}$ contains a vertex, it will be adjacent either to a vertex of degree two, or to a vertex of degree three which is in $D_{i,1}$. We call this vertex which is adjacent to the vertex in $D_{i,0}$ the *constrained vertex.* If the constrained vertex has degree three, then its degree one neighbor is called the *constrained neighbor.* The neighbor of the constrained vertex which is not the constrained neighbor and not the vertex in $D_{i,0}$ is the *reserved neighbor.* Finally, we define a fifth set Q_i to be the set $V_i - D_i$ minus the end vertices of pendant P_3's, the reserved neighbor and the constrained neighbor. The algorithm generates G_{i+1} from G_i by adding a single vertex, v_{i+1}. Note that we begin with $i = 2$ since there is no total dominating set for a graph with one vertex.

Algorithm for generating the graphs G_i

1. Set $G_2 = P_2$ with end vertices v_1 and v_2, $D_{2,0} = \{v_1, v_2\}$, and all other sets to empty.
2. For $i \geq 3$ add vertex v_{i+1} according to the following rules, where all vertex sets in G_{i+1} not explicitly mentioned are set equal to the corresponding sets in G_i, except for Q_i which always is assumed to be as defined above:
   (a) If $\gamma_{t,i+1} = \gamma_{t,i}$, add a pendant vertex v_{i+1} to a vertex w of $D_i - D_{i,C}$ where the order of selection is a vertex in $D_{i,0}$, a vertex in $D_{i,2}$, and finally a vertex in $D_{i,1}$. In addition,

i. if $w \in D_{i,0}$, set $D_{i+1,0} = D_{i,0} - \{w\}$ and $D_{i+1,1} = D_{i,1} \cup \{w\}$; then remove any existing designation of a constrained vertex, a constrained neighbor, and a reserved neighbor,

ii. if $w \in D_{i,2}$, let u be the central vertex of the P_3 in G_i which has w as its base and set $D_{i+1,2} = D_{i,2} - \{w\}$, $D_{i+1,1} = D_{i,1} \cup \{w, u\}$, and $D_{i+1,C} = D_{i,C} - \{u\}$.

(b) If $\gamma_{i+1} = \gamma_i + 1$, add pendant vertex v_{i+1} to any vertex $w \in Q_i$ with the proviso that a vertex with degree at least two is selected if it exists. In addition,

i. if w has degree at least two in G_i, set $D_{i+1,1} = D_{i,1} \cup \{w\}$,

ii. if w has a single neighbor u in G_i, then

A. if w is the only degree one neighbor u, set $D_{i+1,2} = D_{i,2} \cup \{u\}$, $D_{i+1,1} = D_{i,1} - \{u\}$, and $D_{i+1,C} = D_{i,C} \cup \{w\}$,

B. if u has at least two degree one neighbors, set $D_{i+1,1} = D_{i,1} \cup \{w\}$.

(c) If $\gamma_{i+1} = \gamma_i - k$ for $1 \leq k \leq \gamma_i - 2$, define $\hat{D}$ to be a set of $k+1$ vertices selected from D_i according to the following ordering: the vertex in $D_{i,0}$ if it exists, both the base and central vertices of the same pendant P_3 repeated as long as unused pendant P_3's remain and at least two more vertices must be placed in $\hat{D}$, and then vertices in $D_{i,1}$. If a single vertex remains for inclusion into $\hat{D}$ and only pendant P_3's remain, that is, $D_{i,1}$ is empty, select the base vertex of a P_3. Define $\hat{N}$ to be $\cup_{x \in D_i - \hat{D}} N(x)$, that is, $\hat{N}$ is the set of vertices dominated by $D_i - \hat{D}$. Add vertex v_{i+1} adjacent to all vertices of $V(G_i) - \hat{N}$. Thus, v_{i+1} dominates all vertices not dominated by $D_i - \hat{D}$. If v_{i+1} is not dominated by $D_i - \hat{D}$, add an edge between v_{i+1} and a vertex of $D_i - \hat{D}$, chosen from $D_{i,2}$ if possible, in order to totally dominate v_{i+1}. In addition,

i. set $D_{i+1,0} = \{v_{i+1}\}$ and $D_{i+1,2} = D_{i,2} - \hat{D}$,

ii. if $\hat{D}$ contains a base vertex of a P_3 but not the associated central vertex u, set $D_{i+1,1} = (D_{i,1} - \hat{D}) \cup \{u\}$ and $D_{i+1,C} = D_{i,C} - (\hat{D} \cup \{u\})$; otherwise, set $D_{i+1,1} = D_{i,1} - \hat{D}$ and $D_{i+1,C} = D_{i,C} - \hat{D}$,

iii. remove any existing designation of a constrained vertex, a constrained neighbor, and a reserved neighbor,

iv. select the constrained vertex w to be a vertex in $V(G_i) - \hat{N}$ which has degree two if possible and otherwise has degree three with a pendant vertex,

v. if the vertex w in Step 2(c)iv has degree three, select the pendant vertex adjacent to w as the constrained neighbor,

vi. select the neighbor of the constrained vertex found in Step 2(c)iv which is neither v_{i+1} nor the constrained neighbor to be the reserved neighbor.

Notice that, in Step 2a, $D_{i+1} = D_i$; in Step 2b, $D_{i+1} = D_i \cup \{w\}$; and, in Step 2c, $D_{i+1} = (D_i - \hat{D}) \cup \{v_{i+1}\}$. An easy check reveals that neither w in Step 2b nor v_{i+1} in Step 2c is a degree one vertex in G_{i+1}. Also, D_4 contains no degree one vertices which leads, by induction, to the following.

Observation 13 *For $i \geq 4$, D_i contains no degree one vertices.*

Our first task is to show that the above algorithm correctly generates a sequence of graphs having the desired total domination numbers. We begin with the following lemma which justifies our descriptions of $D_{i,0}$, $D_{i,1}$, $D_{i,2}$, $D_{i,C}$, and D_i.

Lemma 14

1. *For $i \geq 2$, the sets $D_{i,0}$, $D_{i,1}$, $D_{i,2}$, and $D_{i,C}$ partition D_i.*

2. *For $i \geq 2$, every vertex in $D_{i,2}$ is the base vertex of a pendant P_3 whose central vertex is in $D_{i,C}$, and every vertex in $D_{i,1}$ is the base vertex of a pendant P_2.*

Proof. An easy check verifies that the lemma holds for $i \in \{2, 3\}$. We proceed by induction on i. Assuming the result holds for some $i \geq 3$, we consider three cases depending on the step used to generate G_{i+1} from G_i. In all three cases it is straightforward to check that, if $D_{i,0}$, $D_{i,1}$, $D_{i,2}$, and $D_{i,C}$ partition D_i, then $D_{i+1,0}$, $D_{i+1,1}$, $D_{i+1,2}$, and $D_{i+1,C}$ also partition D_{i+1} which implies Part 1.

1. If $\gamma_{t,i+1} = \gamma_{t,i}$, we obtain G_{i+1} by adding a pendant vertex to $w \in D_i - D_{i,C}$. By construction, w is the base vertex of a pendant P_2 in G_{i+1} whose end vertex is v_{i+1} and, if Step 2(a)ii is employed, u is the base vertex of a pendant P_2. Thus, by induction, every vertex in $D_{i+1,1}$ is the base vertex of a pendant P_2. By Observation 13, w is not a degree one vertex. It follows that a path is a pendant P_3 in G_i if and only if it is a pendant P_3 in G_{i+1}, with the possible exception of a pendant P_3 in G_i having w as its base vertex. Therefore, if v is in $D_{i+1,2}$, then by construction $v \in D_{i,2}$ and hence is the base vertex of a pendant P_3 in G_i which is also a pendant P_3 in G_{i+1}. Further, w is not in $D_{i+1,2}$; hence, the central vertex of the pendant P_3 which is in $D_{i,C}$ must also be in $D_{i+1,C}$ and Part 2 follows.

2. If $\gamma_{t,i+1} = \gamma_{t,i} + 1$, we obtain G_{i+1} by adding a pendant vertex v_{i+1} to $w \in Q_i$. If w has degree at least two or w has degree one and $N(w)$ has at least two degree one neighbors, then $D_{i+1,1} = D_{i,1} \cup \{w\}$. As before w is the base vertex of a pendant P_2 whose other vertex is v_{i+1}. All of the other corresponding sets are equal and the results of the lemma follow. Alternatively, suppose w has degree one in G_i with u as its only neighbor. The path $\langle u, w, v_{i+1}\rangle$ is a pendant P_3 in G_{i+1} with u as its base and w as its central vertex. In this case, u is placed in $D_{i+1,2}$ and w is placed in $D_{i+1,C}$ and the result follows by induction.

3. Finally, suppose that $\gamma_{t,i+1} = \gamma_{t,i} - k$ for $k \geq 1$. If $v \in D_{i+1,2}$, then $v \in D_{i,2} - \hat{D}$ and so is the base vertex of a pendant P_3 in G_i with central vertex $c \in D_{i,C}$ and degree one vertex $m \in V_i - D_i$. By construction, since v is not in $\hat{D}$, neither is c nor can c be the central vertex u of Step 2(c)ii. Thus, $c \in D_{i,C} - \hat{D} - \{u\} = D_{i+1,C}$. Further, neither c nor m is adjacent to v_{i+1}; hence, v is still the base vertex of a pendant P_3 in G_{i+1}. It follows that every vertex in $D_{i+1,2}$ is the base vertex of a pendant P_3 in G_{i+1} whose central vertex is in $D_{i+1,C}$.

   If $v \in D_{i+1,1}$, then, from Step 2(c)ii, either $v \in D_{i,1} - \hat{D}$ or v is the central vertex of a pendant P_3 in G_i whose base vertex b is in $\hat{D}$. In the first case, v is the base vertex of a pendant P_2 in G_i whose end vertex m is in $\hat{N}$ and is not in D_i. Hence, m is not adjacent to v_{i+1} and v is still the base vertex of a pendant P_2 in G_{i+1}. In the second case, v_{i+1} is adjacent to both b and v but not to the end vertex of the pendant P_3 and once again v is the base vertex of a pendant P_2 in G_{i+1}. Thus, every vertex in $D_{i+1,1}$ is the base vertex of a pendant P_2.

□

It will be helpful to have a notation for those indices which may be part of a subsequence which leads to an index of the type described in Lemma 12. To this end, we partition the indices of the sequence into two sets, PN (potentially nonachievable) and PA (potentially achievable). An index j is in PN if and only if there is an index $i < j$ such that $\gamma_{t,i+1} < \gamma_{t,i}$ and $\gamma_{t,j} = \gamma_{t,i+1} + j - i - 1$, that is, the value of γ has increased by one at each step since its last decrease. This definition leads immediately to the following observation.

Observation 15 *For any sequence,*

1. *if* $\gamma_{t,i+1} = \gamma_{t,i}$, *then* $i + 1 \in PA$,

2. *if* $\gamma_{t,i+1} = \gamma_{t,i} + 1$, *then* i *and* $i+1$ *are either both in* PA *or both in* PN, *and*

3. *if* $\gamma_{t,i+1} < \gamma_{t,i}$, *then* $i + 1 \in PN$.

We can use Observation 15 to obtain the following useful lemma regarding $D_{i,0}$.

Lemma 16 *For* $i \geq 4$, *the graph* G_i *generated by the algorithm has* $|D_{i,0}| = 1$ *if* i *is in* PN *and* $|D_{i,0}| = 0$ *otherwise.*

Proof. We proceed by induction on i, the result being true when $i = 4$ for which $|D_{i,0}| = 0$. After Step 2a of the algorithm, $D_{i,0}$ is the empty set. Furthermore, by Part 1 of Observation 15, $i + 1 \in PA$. Note that, in Step 2b, $|D_{i,0}| = |D_{i+1,0}|$ and, by Part 2 of Observation 15, $i + 1 \in PN$ if and only if $i \in PN$. Finally, Step 2c sets $D_{i,0}$ to $\{v_{i+1}\}$ so $|D_{i,0}| = 1$ and, by Part 3 of Observation 15, $i + 1 \in PN$. □

We need one additional fact regarding $D_{i,0}$.

Lemma 17 *For* $i \geq 4$, *if* $|D_{i,0}| = 1$ *then the constrained vertex* $r_i \in G_i$ *is not adjacent to any vertex in* $D_i - D_{i,0}$.

Proof. We proceed by induction on i, the result being vacuously true when $i = 4$ for which $|D_{i,0}| = 0$. If G_{i+1} is generated from G_i via Step 2a of the algorithm, then Lemma 16 implies $|D_{i+1,0}| = 0$ and the result is again vacuously true. If G_{i+1} is generated from G_i via Step 2b, then $|D_{i,0}| = |D_{i+1,0}|$. If they both have size 0, we are done. Otherwise, by Lemma 16, we have that $|D_{i,0}| = |D_{i+1,0}| = 1$. Further, the neighborhood of the constrained vertex in G_i consists of the vertex in $D_{i,0} = D_{i+1,0}$, the reserved neighbor, and the constrained neighbor if one exists. Thus, none of the neighbors of the constrained vertex are in Q_i. It follows that the constrained vertex satisfies the conditions of the lemma for G_{i+1}. Finally, suppose G_{i+1} is generated from G_i via step 2c. In this case the constrained vertex satisfies the conditions of the lemma by construction. □

The above lemmas leads to the following theorem.

Theorem 18 *For* $i \geq 2$,

1. D_i *is a minimum total dominating set of* G_i *and*

2. *the algorithm correctly alters the total domination number in each of Steps 2a, 2b, and 2c.*

Proof:

1. As has been pointed out previously, there exists a minimum total dominating set which contains every base vertex of a P_2, every base vertex of a P_3, and every central vertex of a P_3. Hence, by Lemma 14, there exists a minimum total dominating set which contains $D_{i,1} \cup D_{i,2} \cup D_{i,C}$. Furthermore, if $|D_{i,0}| = 1$, then the existence of r_i implies by Lemma 17 that the minimum total dominating set must contain at least one more vertex, and the element in $D_{i,0}$ serves this purpose. All that remains is to show that D_i actually totally dominates G_i. This follows easily by induction since v_{i+1} is dominated by w in steps 2a and 2b while, in step 2c, every vertex not dominated by $D_i - \hat{D}$ is dominated by v_{i+1} and some vertex of $D_i - \hat{D}$ dominates v_{i+1}.

2. In Step 2a, $D_{i+1} = D_i$; hence, by Part 1, $\gamma_{t,i+1} = \gamma_{t,i}$. In Step 2b, $D_{i+1} = D_i \cup \{w\}$ where $w \in Q_i$. By the definition of Q_i, the vertex w is not in D_i; hence, $\gamma_{t,i+1} = \gamma_{t,i} + 1$ since Lemma 14 guarantees that the vertex dominating w in G_i is still in the total dominating set in G_{i+1}. Finally, in Step 2c, $D_{i+1} = (D_i - \hat{D}) \cup \{v_{i+1}\}$. Thus, since $|\hat{D}| = k + 1$ and v_{i+1} is not in D_i, we have $|D_{i+1}| = |D_i| - |\hat{D}| + 1 = \gamma_i - (k+1) + 1 = \gamma_i - k$.

□

We are left with the task of determining those sequences for which the algorithm can be executed. Notice that Steps 2a and Step 2c can always be accomplished. On the other hand, Step 2b can be completed only if the set Q_i is nonempty. The next lemma shows this is indeed the case for any legitimate sequence if $D_{i,0}$ is empty and for any sequence not eliminated by Lemma 12 if $D_{i,0}$ contains a vertex.

Lemma 19

1. *If $i \in PA$ and $\gamma_{t,i} < 2i/3$, then $|Q_i| > 0$.*

2. *If $i \in PN$ and $\gamma_{t,i} < 2(i+2)/3 - 2$, then $|Q_i| > 0$.*

Proof.

1. By definition, $|Q_i| = i - \gamma_{t,i} - |D_{i,2}|$. Since $|D_{i,2}| \leq \gamma_{t,i}/2$, we have $|Q_i| \geq i - \gamma_{t,i} - \gamma_{t,i}/2 = i - 3\gamma_{t,i}/2 > 0$.

2. Clearly Q_i is nonempty if it contains the constrained vertex. Otherwise the constrained vertex is in $D_{i,1}$ and the constrained neighbor exists. In this case, it follows that $|Q_i| = i - \gamma_{t,i} - |D_{i,2}| - 2$ and $|D_{i,2}| \leq \gamma_{t,i}/2 - 1$. Hence, we have $|Q_i| \geq i - \gamma_{t,i} - (\gamma_{t,i}/2 - 1) - 2 = i - 3\gamma_{t,i}/2 - 1 > i - (3/2)[2/3(i+2) - 2] - 1 = i - i - 2 + 3 - 1 = 0$. □

Combining the results in the previous lemmas leads to the following characterization theorem for sequences associated with the total domination number.

Theorem 20 *Every legitimate sequence is achievable except any in which, for some index i,* $\gamma_{t,i+1} < \gamma_{t,i}$ *and* $\gamma_{t,3(i-\gamma_{t,i+1})+2} = 2(i - \gamma_{t,i+1}) + 1$.

5 Domatic Number and Other Domination Related Parameters

First we consider I to be *domatic number*, $d(G)$, defined by Cockayne and Hedetniemi [3] to be the maximum number of disjoint dominating sets in G. Like domination number, the domatic number can increase by at most one with the addition of a vertex. To see this, let D_1, D_2,..., $D_{d(G_{i+1})}$ be a maximum collection of dominating sets of G_{i+1}. At most one of these sets contains v_{i+1} and the rest dominate G_i; so, $d(G_{i+1}) \leq d(G_i) + 1$. We show the same sequences are achievable for $d(G)$ as for γ.

Theorem 21 *Any sequence which satisfies* $1 \leq a_i \leq i$ *and* $a_{i+1} \leq a_i + 1$ *is achievable under* $d(G)$.

Proof. Again, we inductively construct a sequence of graphs associated with a given sequence beginning with $a_1 = 1$. As always, G_1 is an isolated vertex and the only dominating set is the vertex itself. Thus, $d(G_1) = 1$. Now assume we have a graph G_i containing i vertices and with $d(G_i) = a_i$. If $a_{i+1} = a_i + 1$, we add the new vertex, v_{i+1}, with edges in G_{i+1} to all vertices of G_i. All of the $d(G_i)$ dominating sets of G_i also dominate G_{i+1}, and $\{v_{i+1}\}$ forms an additional dominating set. If $a_{i+1} = a_i - k$, $0 \leq k \leq a_i - 1$, then v_{i+1} is joined to exactly one vertex in each of $a_i - k - 1$ dominating sets of G_i. Each of these $a_i - k - 1$ sets now dominates G_{i+1} and is disjoint from the union of v_{i+1} with the vertices of the other $k + 1$ sets, a union which also forms a dominating set for G_{i+1}. Thus, $d(G_{i+1}) \geq a_i - k$. Suppose $d(G_{i+1}) \geq a_i - k + 1$. By disjointness, at most one of the dominating sets in a maximum collection includes v_{i+1}, so at least $a_i - k$ of them do not. However, the degree of v_{i+1} is $a_i - k - 1$; hence, at least one of the sets not containing v_{i+1} fails to dominate v_{i+1}. Thus, $d(G_{i+1}) \leq a_i - k$ and the result follows. □

Our approach yields results for a number of additional parameters. The *independence number*, $\beta(G)$, is the maximum size of a maximal independent set while the *lower independence number*, $i(G)$, is the minimum size of such a set. The *upper domination number*, $\Gamma(G)$ is a maximum size minimal dominating set. A set S of vertices is irredundant if $v \in S$ implies $N[S] -$

$N[S - \{v\}] \neq \emptyset$. The *irredundance number*, $ir(G)$, is the minimum size of a maximal irredundant set. The *upper irredundance number*, $IR(G)$, is the maximum size of a maximal irredundant set.

As first noticed by Cockayne and Hedetniemi [3], these parameters along with domination number are related by the following string of inequalities:

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta(G) \leq \Gamma(G) \leq IR(G).$$

Theorem 22 *Any sequence which satisfies* $1 \leq a_i \leq i$ *and* $a_{i+1} \leq a_i + 1$ *is achievable under* $ir(G)$ *and under* $i(G)$.

Proof. In both cases we use the construction in the proof to Theorem 1. Notice that any maximal irredundant set and any maximal independent set must contain at least one vertex from each component. Thus, $ir(G) \leq \gamma(G)$ implies the result for $ir(G)$. Also, the dominating vertices from the construction form a maximal independent set which implies the result for $i(G)$. □

The sequences which are achievable under $\beta(G)$, $\Gamma(G)$, and $IR(G)$ are monotonically nondecreasing. To see this, note that any independent set in G_i is also independent in G_{i+1}, any set which dominates G_i dominates G_{i+1} with the possible exception of v_{i+1}, and a set which is irredundant in G_i remains so in G_{i+1}.

Theorem 23 *Any sequence which satisfies* $1 \leq a_i \leq i$ *and* $a_i \leq a_{i+1} \leq a_i + 1$ *is achievable under* $\beta(G)$, $\Gamma(G)$, *and* $IR(G)$.

Proof. We again begin with an isolated vertex which is a maximum independent set, a maximum minimal dominating set, and a maximum irredundant set. If $a_{i+1} = a_i$, we add v_{i+1} with edges to all vertices of G_i, maintaining the same values for β, Γ, and IR. If $a_{i+1} = a_i + 1$, then v_{i+1} is added as an isolated vertex and thus must be included in any maximal independent set, minimal dominating set, or maximal irredundant set. □

As seen in Section 3, characterizing sequences achievable under γ became substantially more complicated when graphs were constrained to be connected. Interestingly, however, the upper domination number is just as simple when graphs are connected and, in fact, once again, the same sequences are achievable under all three upper parameters, $\beta(G)$, $\Gamma(G)$, and $IR(G)$, for connected graphs. These sequences differ from those achievable under the same parameters for unrestricted graphs only that for connected graphs with n vertices, any independent set, minimal dominating set, and irredundant set contains no more than $n - 1$ vertices.

Theorem 24 *Any sequence which satisfies* $a_1 = 1$ *and* $1 \leq a_i \leq i - 1$ *for* $i \geq 2$ *and* $a_i \leq a_{i+1} \leq a_i + 1$ *is achievable under* $\beta(G)$, $\Gamma(G)$, *and* $IR(G)$ *where each* G_i *is connected.*

Proof. The proof is the same as that for Theorem 23 except, if $a_{i+1} = a_i + 1$, then v_{i+1} is pendant to a vertex which is not contained in at least one maximum independent set (maximum minimal dominating set or maximum irredundant set). □

Work is in progress regarding sequences for the lower parameters $ir(G)$ and $i(G)$ for the connected graph case which, so far, appear more difficult.

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