On 1-Z_m-well-covered and strongly Z_m-well-covered graphs

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ABSTRACT. A graph G is Z_m-well-covered if $|I| \equiv |J| \pmod{m}$, for all I, J maximal independent sets in $V(G)$. A graph G is a 1-Z_m-well-covered graph if G is Z_m-well-covered and $G\backslash\{v\}$ is Z_m-well-covered, $\forall v \in V(G)$. A graph G is strongly Z_m-well-covered if G is a Z_m-well-covered graph and $G\backslash\{e\}$ is Z_m-well-covered, $\forall e \in E(G)$. Here we prove some results about 1-Z_m-well-covered and strongly Z_m-well-covered graphs.

1 Introduction

We start with some basic definitions and notation. We denote the vertex set of a graph G by $V(G)$. $N(v)$ is the set of vertices adjacent to v in G. A set $I \subset V(G)$ is *independent* if no two vertices of I are adjacent. A graph G is *well-covered* if every maximal independent set of vertices of G has the same cardinality. These graphs were introduced by Plummer [10] in 1970. Although the recognition problem of well-covered graphs in general is Co-NP-complete [5,13], it is polynomial for some classes of graphs, for instance, claw-free [15], graphs with girth ≥ 5 [7], and chordal [12]. The reader is referred to Plummer [11], and more recently, Hartnell [8] for survey articles and further references to work on well-covered graphs. A graph H is a *parity* graph if every maximal independent set of vertices of H has the same parity. Finbow and Hartnell [6] gave a characterization of parity graphs with girth > 5. A graph G is a Z_m*-well-covered* graph if $|I_1| \equiv |I_2| \pmod{m}$ for all maximal independent sets I_1 and I_2 in $V(G)$. Caro [4] proved that the

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recognition problem of well-covered graphs is Co-NP-complete even for Z_m-well-covered graphs which are $K_{1,3m+1}$-free. In [1] it is shown that claw-free Z_m-well-covered graphs must be well-covered. Caro and Hartnell [3] give a characterization of Z_m-well-covered graphs of girth > 5.

Staples [14] defined a 1-well-covered graph G as a graph that is well-covered and $G\backslash\{g\}$ is also well-covered for any $g \in V(G)$. Pinter [9] defined a strongly well-covered graph G as a graph that is well-covered and $G\backslash\{e\}$ is well-covered for any $e \in E(G)$. In a similar way we define a 1-Z_m-well-covered graph G as a graph that is Z_m-well-covered and $G\backslash\{g\}$ is Z_m-well-covered, $\forall g \in V(G)$. A graph G is strongly Z_m-well-covered if it is Z_m-well-covered and $G\backslash\{e\}$ is Z_m-well-covered, $\forall e \in E(G)$. So, the last two definitions are natural extensions of definitions of strongly well-covered and 1-well-covered graphs. Figure 4 gives an example of a strongly Z_2-well-covered graph that is not strongly well-covered.

A clique of a graph G is a maximal complete subgraph of G. A vertex v of a graph is a *simplicial* vertex if it appears in exactly one clique of the graph. A clique of a graph G containing at least one simplicial vertex of G is called a *simplex* of G. A graph G is a *simplicial* graph if every vertex of G is a simplicial vertex of G or is adjacent to a simplicial vertex of G. A graph G is *chordal* if every cycle of G of length four or more has a chord.

In [2] a sufficient condition is given for a graph to be a Z_m-well-covered one:

Theorem 1. *If for each $g \in V(G)$, $\exists l \in N$ such that g belongs to exactly $(ml+1)$ simplices, then G is a Z_m-well-covered graph.*

The following results are also proved in [2]:

Theorem 2. *If G is a Z_m-well-covered simplicial graph, then for each vertex $g \in V(G)$, $\exists l \in N$ such that g belongs to exactly $(ml+1)$ simplices.*

Theorem 3. *G a chordal Z_m-well-covered graph $\Longrightarrow$ G is simplicial.*

2 Results

Proposition 1. *If G is Z_m-well-covered, then $\alpha(G) \equiv \alpha(G\backslash N[v]) + 1 \pmod{m}$, $\forall v \in V(G)$.*

Proof: Let I be a maximal independent set in $G\backslash N[v]$, with $|I| = \alpha(G\backslash N[v])$. $I \cup \{v\}$ is a maximal independent set in G, and since G is a Z_m-well-covered graph $\alpha(G) \equiv |I \cup \{v\}| = \alpha(G\backslash N[v]) + 1 \pmod{m}$. □

Theorem 4. *If G is 1-Z_m-well-covered, then $G\backslash N[v]$ is 1-Z_m-well-covered, $\forall v \in V(G)$.*

Proof: Since G is 1-Z_m-well-covered, $G\backslash v$ and $G\backslash N[v]$ are also Z_m-well-covered. Now, suppose there is a $w \in V(G)$ such that $(G\backslash N[v])\backslash w$ is not Z_m-well-covered. Then there are maximal independent sets I_1, I_2 in $(G\backslash N[v])\backslash w$ such that $|I_1| \not\equiv |I_2| \pmod m$. In this case, $I_1 \cup \{v\}$ and $I_2 \cup \{v\}$ would be maximal independent sets in $G\backslash w$ (note that $G\backslash w$ is Z_m-well-covered for all $w \in V(G)$), with $|I_1 \cup \{v\}| \not\equiv |I_2 \cup \{v\}| \pmod m$, a contradiction. □

Theorem 5. *If G is strongly Z_m-well-covered, then $G\backslash N[v]$ is strongly Z_m-well-covered, $\forall v \in V(G)$.*

Proof: G strongly Z_m-well-covered $\Rightarrow$ G is Z_m-well-covered $\Rightarrow$ $G\backslash N[v]$ is Z_m-well-covered, $\forall v \in V(G)$. So we have only to show that $(G\backslash N[v])\backslash e$ is Z_m-well-covered, $\forall v \in V(G)$, $e \in E(G)$.

Suppose that $\exists v \in V(G)$, $e \in E(G)$ such that $(G\backslash N[v])\backslash e$ is not Z_m-well-covered. Then $\exists I$, J maximal independent sets in $(G\backslash N[v])\backslash e$ such that $|I| \not\equiv |J| \pmod m$, but then $I \cup \{v\}$ and $J \cup \{v\}$ would be maximal independent sets in $G\backslash e$, with $|I| \not\equiv |J| \pmod m$, a contradiction since G is strongly Z_m-well-covered. □

Now, we give a sufficient condition for a graph to be a 1-Z_m-well-covered one.

Theorem 6. *Given a graph G, if every vertex of G belongs to exactly $ml + 1$ simplices and each simplex has more than one simplicial vertex, then G is a 1-Z_m-well-covered graph.*

Proof: Observe that if every simplex has more than one simplicial vertex, then when we delete a vertex g of any simplex S, $S\backslash\{g\}$ is also a simplex. Therefore every vertex of $G\backslash\{g\}$ will belong to exactly $ml+1$ simplices and by Theorem 1 $G\backslash\{g\}$ is Z_m-well-covered. □

For chordal graphs this condition is also necessary.

Theorem 7. *If G is a chordal graph which is 1-Z_m-well-covered, then every vertex of G must belong to exactly $(ml + 1)$ simplices and each simplex has more than one simplicial vertex.*

Proof: Since G is Z_m-well-covered and chordal it must be simplicial (Theorem 3), and hence every $g \in V(G)$ belongs to exactly $ml + 1$ simplices. Suppose there exists a vertex $g \in V(G)$ and a simplex S such that $g \in S$ and S has only one simplicial vertex s. Then when we remove the vertex s from G, the graph $G\backslash\{s\}$ remains chordal, but g will belong to ml simplices in $G\backslash\{s\}$, so $G\backslash\{s\}$ is not Z_m-well-covered, a contradiction. □

This result is not true for simplicial graphs as shown by the example of Figure 1.

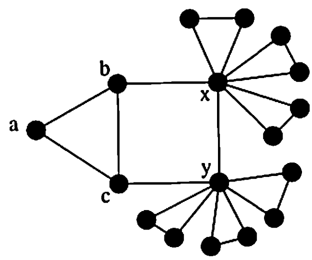

Figure 1. A simplicial Z_m-well-covered graph

Corollary 1. *K_1 and K_2 are the only connected 1-Z_m-well-covered trees.*

Proof: K_1 and K_2 are 1-Z_m-well-covered graphs. Every tree is a chordal graph, so by Theorem 3 every vertex must belong to exactly $(ml + 1)$ simplices, each one with more than one simplicial vertex, so the graph must have at least triangles and cannot be a tree. □

We prove later a more general result. It is proved that there is no vertex of degree 1 in any 1-Z_m-well-covered graph that has at least 3 vertices.

Lemma 1. *Let G be a chordal graph and s_1 a simplicial vertex of G. Then $G\backslash(s_1, g)$ is also chordal, $\forall g$ adjacent to s_1.*

Proof: Suppose there exist $s_1, g \in V(G)$, s_1 simplicial, such that $G\backslash(s_1, g)$ is not chordal. Then there is a $C_4 = s_1agb$, but in this case, since a is not adjacent to b, s_1 would not be a simplicial vertex in G, a contradiction. □

The result above is not true for simplicial graphs as shown by Figure 2.

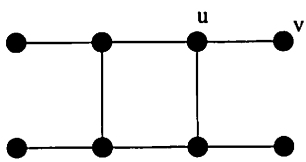

Figure 2. A simplicial graph

Proposition 2. *K_1 and K_2 are the only connected chordal strongly Z_m-well-covered graphs.*

Proof: K_1 and K_2 are strongly Z_m-well-covered graphs. Suppose $G \neq K_1$, K_2 is a chordal strongly Z_m-well-covered graph. Let S_1 be a simplex of G, and $s_1 \in S_1$, a simplicial vertex. Let g be an adjacent vertex of s_1 in G and let $e = (g, s_1)$. By Lemma 1 $G\backslash e$ is chordal, and s_1 is also simplicial in $G\backslash e$. We have to consider the following cases:

1) g is not simplicial in $G\backslash e$. In this case, when we remove e, g will not belong to S_1 and g will belong to no simplex in $G\backslash e$ or to exactly ml simplices.

2) g is simplicial in $G\backslash e$. In this case, there exists $h \in N(g) \cap N(s_1)$ and h belongs to exactly $(ml + 2)$ simplices in $G\backslash e$. □

Proposition 3. *K_1 and K_2 are the only simplicial connected strongly Z_m-well-covered graphs.*

Proof: Let G be a simplicial strongly Z_m-well-covered graph, $G \neq K_1, K_2$. Let S be a simplex in G and s one of its simplicial vertices. Let $g \in N(s)$ and $e = (s, g)$. Then we can have the following cases:

1) g belongs only to simplex S. Let $x \in N(s) \cap N(g)$. Let I_1 be the maximal independent set in $V\backslash(s, g)$ built with x and the simplicial vertices not adjacent to x. Let I_2 be the maximal independent set in $V\backslash(s, g)$ built with s, g and the simplicial vertices not adjacent to s nor g. Then $|I_1| = |I_2| - 1$, and $G\backslash e$ is not Z_m-well-covered.

2) g belongs to simplex S and to another ml simplices. In this case, when we remove the edge (s, g) from G, $G\backslash(s, g)$ remains simplicial and g belongs to exactly ml simplices in $G\backslash e$, so $G\backslash e$ cannot be Z_m-well-covered. □

Given a graph G, a critical line e in G is an edge such that $\alpha(G\backslash e) = \alpha(G) + 1$.

Lemma 2. *If G, $G \neq K_2$, is a Z_m-well-covered graph and e is a critical line in G, then $G\backslash e$ is not Z_m-well-covered.*

Proof: Let $e = (u, v)$. Since $G \neq K_2$, at least one of u and v, say u, is not a leaf, so there exists $x \in V(G)$, with $x \sim u$ and $x \neq v$. Since G is Z_m-well-covered, there exists a maximal independent set I_1 such that $|I_1| \equiv \alpha(G) \pmod m$, and $x \in I_1$. Since e is a critical line in G, $\alpha(G\backslash e) = \alpha(G) + 1$. But I_1 is also a maximal independent set in $G\backslash e$, so $|I_1| \equiv \alpha(G\backslash e)$. Then $\alpha(G) \equiv |I_1| \equiv \alpha(G\backslash e) \pmod m$, a contradiction. □

Staples [14] proved the following result:

Theorem 8. *If G is well-covered and $u \in V(G)$ such that $G\backslash u$ is also well-covered, then there is a vertex v adjacent to u such that (u, v) is a critical line.*

This result is not true, in general, for Z_m-well-covered graphs. The graph of Figure 3 is a parity graph in which $G\backslash u$ is also parity. Observe that

$\alpha(G) = \alpha(G \backslash (u, v))$, $\forall v \in N(u)$, and there is no critical line e in G such that e is incident to u.

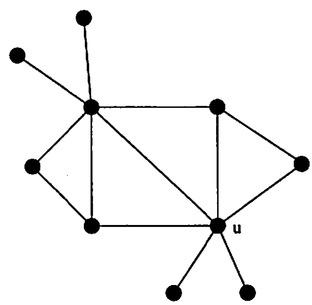

Figure 3. A graph G with an m-critical line

Given a graph G, we define an *m-critical line* e in G as an edge such that there is a maximal independent set I in $G \backslash e$ such that $|I| \not\equiv \alpha(G) \pmod m$, so every critical line is an m-critical line. Now we can prove the following result:

Theorem 9. *If G is a Z_m-well-covered graph and $u \in V(G)$ such that $G \backslash u$ is also Z_m-well-covered, then there is a vertex $v \in N(u)$ such that $e = (u, v)$ is an m-critical line.*

Proof: Extend $\{u\}$ to a maximal independent set I_1 in G, so $|I_1| \equiv \alpha(G) \pmod m$. We also have $\alpha(G \backslash u) = \alpha(G)$ (If not, $\exists I$ a maximal independent set in $G \backslash u$ with $|I| > \alpha(G)$, and I would also be a maximal independent set in G).

Then $(I_1 \backslash u)$ is not a maximal independent set in $G \backslash u$. By hypothesis, $G \backslash u$ is a Z_m-well-covered graph, so there exists a maximal independent set I_2 in $(G \backslash u)$ such that $(I_1 \backslash u) \subset I_2$.

$|I_1 \backslash u| \equiv \alpha(G) - 1 = \alpha(G \backslash u) - 1$, so

$$I_2 = (I_1 \backslash u) \cup \{v_1, v_2, \ldots, v_{km+1}\}, v_i \in G \backslash u.$$

If $v_i \not\sim u$ $\forall i = 1, \ldots, km + 1$, then $I_2 \cup \{u\}$ is independent in G and $I_1 \subset I_2 \cup \{u\}$, a contradiction because I_1 is maximal in G.

Therefore we must have $v_i \sim u$. Then, $I^* = I_1 \cup \{v_i\}$ is a maximal independent set in $G \backslash e$ with $|I^*| \equiv \alpha(G) + 1 \pmod m$, and therefore $e = (u, v)$ is an m-critical line. □

Lemma 3. *If G, $G \neq K_2$, is a Z_m-well-covered graph and e is an m-critical line, then $G \backslash e$ is not Z_m-well-covered.*

Proof: Let $e = (u, v)$ be an m-critical line in G, so there is a maximal independent set I in $G \backslash e$ such that $|I| \not\equiv \alpha(G) \pmod m$. Let $x \sim u$ and

$x \neq v$. Extend x to a maximal independent set I_1 in G. Since G is Z_m-well-covered $|I_1| \equiv \alpha(G) \pmod{m}$. I_1 is also a maximal independent set in $G\backslash e$, so $|I_1| \not\equiv |I| \pmod{m}$ and therefore $G\backslash e$ is not Z_m-well-covered. □

Theorem 10. *K_1 and K_2 are the only connected graphs that are strongly Z_m-well-covered and 1-Z_m-well-covered.*

Proof: By Theorem 9 if G is 1-Z_m-well-covered it must have an m-critical line e, but by Lemma 3 if e is an m-critical line, then $G\backslash e$ is not Z_m-well-covered, so there is no other graph, besides K_1 and K_2, with both properties. □

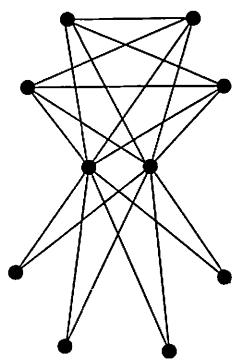

Figure 4. A strongly Z_2-well-covered graph

Theorem 11. *If G is a 1-Z_m-well-covered graph with at least 3 vertices, then $\delta \geq 2$.*

Proof: Suppose G has a leaf x. We will prove that $G\backslash\{x\}$ is not Z_m-well-covered. Let v be the unique vertex in G adjacent to x. Since G is Z_m-well-covered, there exist I_1, I_2 maximal independent sets in $V(G)$ such that $x \in I_1$, $v \in I_2$ and $|I_1| \equiv |I_2| \pmod{m}$. Now, $J_1 = I_1\backslash\{x\}$ and I_2 are also maximal independent sets in $G\backslash\{x\}$, but $|J_1| \not\equiv |I_2| \pmod{m}$, so G is not 1-Z_m-well-covered, a contradiction. □

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