NEW FAMILIES OF GRACEFUL DISCONNECTED GRAPHS

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Abstract.

In this paper we extend the definition of pseudograceful graphs given by Frucht [3] to all graphs G with vertex set V(G) and edge set E(G) such that $|V(G)| \leq |E(G)| + 1$ and we prove that if G is a pseudograceful graph, then $G \cup K_{m,n}$ is pseudograceful for $m, n \geq 2$ and $(m, n) \neq (2, 2)$, and is graceful for $m, n \geq 2$. This enables us to obtain several new families of graceful disconnected graphs.

Introduction.

All graphs in this paper are finite, simple and undirected. We follow the basic notations and terminology of graph theory as in [1].

A graph G with vertex set V(G) and edge set E(G) is said to be graceful if there exists an injective function called a graceful labelling $f : V(G) \to \{0, 1, \ldots, |E(G)|\}$ such that the induced function $f^* : E(G) \to \{1, 2, \ldots, |E(G)|\}$ defined by

$$f^*(xy) = |f(x) - f(y)| \qquad \text{for all} \qquad xy \in E(G)$$

is an injection. The image of f (Im(f)) is called the corresponding set of vertex labels. A graph which is not graceful is called a disgraceful graph. A survey about the present status of graceful graphs can be found in [5].

This paper is divided into two sections. In Section 1 we introduce the concept of pseudograceful graphs, extending the definition given by Frucht [3] to all graphs G with vertex set V(G) and edge set E(G) such that $|V(G)| \leq |E(G)| + 1$, and we investigate this new concept. In Section 2 we obtain our main theorem, Theorem 2.1, which states that if G is a pseudo-

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graceful graph then $G \cup K_{m,n}$ is pseudograceful if $m, n \geq 2$ and $(m,n) \neq (2,2)$, and is graceful if $m, n \geq 2$. Consequently, we obtain several new families of graceful disconnected graphs.

1. Pseudograceful Graphs.

The following definition of pseudograceful graphs extends the one given by Frucht [3] to all graphs G with vertex set V(G) and edge set E(G) such that $|V(G)| \leq |E(G)| + 1$.

Definition 1.1.

A graph G with vertex set V(G) and edge set E(G) such that $|V(G)| \leq |E(G)| + 1$ is said to be pseudograceful if there exists an injective function called pseudograceful labelling $f : V(G) \to \{0, 1, \ldots, |E(G)| - 1, |E(G)| + 1\}$ such that the induced function

$$f^* : E(G) \to \{1, 2, \ldots, |E(G)|\}$$

defined by

$$f^*(xy) = |f(x) - f(y)| \qquad \text{for all} \qquad xy \in E(G)$$

is an injection. The image of f (Im(f)) is called the corresponding set of vertex labels.

Frucht [3] showed that P_n $(n \geq 3)$, combs (i.e. graphs obtained by joining a single pendant edge to each vertex of a path), sparklers (i.e. graphs obtained by joining an end vertex of a path to the center of a star), $C_3 \cup P_n$ $(n \neq 3)$ and $C_4 \cup P_n$ $(n \neq 1)$ are pseudograceful while $K_{1,n}$ $(n \geq 3)$ is not.

Note that if f is a pseudograceful labelling of a graph G, then we must have $f(x) = 1$ and $f(y) = |E(G)| + 1$ for some $xy \in E(G)$.

This new concept of pseudograceful graphs is independent from that of graceful graphs since $K_{1,n}$ is graceful for all $n \geq 1$ [6, 7] but $K_{1,n}$ is not pseudograceful for $n \geq 3$ [3, Theorem 4.1] and $C_3 \cup P_2$ is disgraceful [4] but it is in fact pseudograceful [3, Theorem 6.2]. Nevertheless, we have the following result.

Proposition 1.2.

Let G be a graceful graph that has a graceful labelling f such that $1 \notin \mathrm{Im}(f)$ or $|E(G)|-1 \notin \mathrm{Im}(f)$, then G is pseudograceful graph.

Proof.

Let G and f be as stated in the proposition. If $1 \notin \mathrm{Im}(f)$, then $|E(G)|+1-f$ is a pseudograceful labelling of G and if $|E(G)|-1 \notin \mathrm{Im}(f)$, then $1+f$ is a pseudograceful labelling of G. □

We also have the following criterion for pseudograceful Eulerian graphs similar to that for graceful Eulerian graphs given by Rosa [7]. The proof of this proposition parallels that for graceful Eulerian graphs in [7] and we omit it.

Proposition 1.3.

If G is a pseudograceful Eulerian graph, then $|E(G)| \equiv 0$ or 3 (mod 4). □

Now we extend the class of known pseudograceful graphs given by Frucht [3] in the following proposition.

Proposition 1.4.

(a) C_3, C_4, C_7 and C_8 are pseudograceful graphs.

(b) K_m is pseudograceful if and only if $m \in \{1, 3, 4\}$.

(c) $K_{m,n}$, $m \le n$ is pseudograceful if and only if $m \ge 2$ or $(m, n) = (1, 2)$.

(d) $P_m + \overline{K}_n$ is pseudograceful if and only if

(1) $m \ge 2$ and $n \ge 1$ or (2) $(m, n) = (1, 2)$.

Proof.

(a) For all $C_n, n \ge 3$, let $V(C_n) = \{u_1, u_2, \ldots, u_n\}$ where $u_i u_j \in E(C_n)$ if and only if $i - j \equiv \pm 1 \pmod n$, then the following functions are pseudograceful labellings of C_3, C_4, C_7 and C_8 respectively:

$f : V(C_3) \to \{0, 1, 2, 4\}$

$f(u_1) = 1$, $f(u_2) = 2$, $f(u_3) = 4$,

$$f : V(C_4) \to \{0, 1, 2, 3, 5\}$$

$$f(u_1) = 1 \;,\; f(u_2) = 3 \;,\; f(u_3) = 2 \;,\; f(u_4) = 5,$$

$$f : V(C_7) \to \{0, 1, 2, \ldots, 6, 8\}$$

$$f(u_1) = 1 \;,\; f(u_2) = 4 \;,\; f(u_3) = 0 \;,\; f(u_4) = 5,$$

$$f(u_5) = 3 \;,\; f(u_6) = 2 \;,\; f(u_7) = 8,$$

$$f : V(C_8) \to \{0, 1, 2, \ldots, 7, 9\}$$

$$f(u_1) = 1 \;,\; f(u_2) = 7 \;,\; f(u_3) = 6 \;,\; f(u_4) = 4,$$

$$f(u_5) = 0 \;,\; f(u_6) = 5 \;,\; f(u_7) = 2 \;,\; f(u_8) = 9.$$

(b) Suppose that K_m is pseudograceful for some $m \geq 1$ and let f be a pseudograceful labelling for this K_m, then $\binom{m}{2} + 1 \in \mathrm{Im}(f)$ forces $0 \notin \mathrm{Im}(f)$ and $f - 1$ is a graceful labelling of K_m, hence $m \leq 4$ [6]. It is trivial to see that K_1 is pseudograceful while K_2 is not. $K_3 = C_3$ is pseudograceful by part (a) and K_4 is pseudograceful via any function $f : V(K_4) \to \{0, 1, 2, \ldots, 5, 7\}$ such that $\mathrm{Im}(f) = \{1, 2, 5, 7\}$.

(c) Since $K_{1,2} \cong P_3$, then $K_{1,2}$ is pseudograceful [3, Theorem 3.2] and $K_{1,n}$ is not pseudograceful for $n \geq 3$ [3, Theorem 4.1] and $K_{1,1} \cong K_2$ is not pseudograceful by part (b). There remains to prove that $K_{m,n}$ is pseudograceful for $m, n \geq 2$. Let $V(K_{m,n}) = V_1 \cup V_2$ where V_i, $i = 1, 2$ are independent sets of vertices where $V_1 = \{u_1, \ldots, u_m\}$ and $V_2 = \{v_1, \ldots, v_n\}$ and $m, n \geq 2$. Define the function

$$f : V(K_{m,n}) \to \{0, 1, \ldots, mn - 1, mn + 1\}$$

by

$$f(u_i) = ni + 1 \quad , \quad 1 \leq i \leq m$$

$$f(v_i) = i \quad , \quad 1 \leq i \leq n$$

then f is injective and f^* is injective as desired.

(d) By virtue of part (c) we have only to show that $P_m + \overline{K}_n$ is pseudograceful if $m \geq 2$ and $n \geq 1$. Let $V(P_m) = \{u_1, \ldots, u_m\}$ where $u_i u_j \in E(P_m)$ if and only if $|i - j| = 1$ and let $V(\overline{K}_n) = \{v_1, \ldots, v_n\}$, then $\left|E(P_m + \overline{K}_n)\right| = m\,n + m - 1$. Define the function

$$f : V(P_m + \overline{K}_n) \to \{0, 1, \ldots, m(n+1) - 2, m(n+1)\}$$

by

$$f(u_{2i+1}) = (m - i)(n + 1) \quad , \quad 0 \leq i \leq \left\lfloor \frac{m-1}{2} \right\rfloor$$

$$f(u_{2i}) = i(n + 1) \quad , \quad 1 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor$$

$$f(v_j) = j \quad , \quad 1 \leq j \leq n$$

then f is injective and f^* is injective as desired. □

2. New Families of Graceful Disconnected Graphs.

We first establish our main theorem.

Theorem 2.1.

Let G be a pseudograceful graph, then:

(a) $G \cup K_{m,n}$ is pseudograceful if $m, n \geq 2$ and $(m, n) \neq (2, 2)$.

(b) $G \cup K_{m,n}$ is graceful if $m, n \geq 2$.

Proof.

Let f be a pseudograceful labelling of a graph G and put $q = |E(G)|$ and let $V(K_{m,n}) = V_1 \cup V_2$ where V_i, $i = 1, 2$ are independent sets of vertices where $V_1 = \{u_1, \ldots, u_m\}$ and $V_2 = \{v_1, \ldots, v_n\}$ and assume that $m, n \geq 2$.

(a) Define the function

$$\bar{f} : V(G \cup K_{m,n}) \to \{0, 1, \ldots, mn + q - 1, mn + q + 1\}$$

by

$$\bar{f}(u_i) = ni + q + 1 \quad , \quad 1 \le i \le m$$

$$\bar{f}(v_i) = i \quad , \quad 1 \le i \le n$$

$$\bar{f}\Big|_{V(G)} = n + 1 + f$$

Note that we must have $m \ge 2$ and $(m, n) \ne (2, 2)$ since $n + 1 + q + 1 \in \bar{f}(V(G))$.

Observe that $\bar{f}$ is injective since $q \notin \mathrm{Im}(f)$ and $n \ge 2$. Then $\bar{f}^*$ is injective as equired.

(b) Reasoning similar to that in part (a) shows that the function

$$\bar{f} : V(G \cup K_{m,n}) \to \{0, 1, \ldots, mn + q\}$$

defined by

$$\bar{f}(u_i) = ni + q \quad , \quad 1 \le i \le m$$

$$\bar{f}(v_i) = i - 1 \quad , \quad 1 \le i \le n$$

$$\bar{f}\Big|_{V(G)} = n + f$$

is a graceful labelling of $G \cup K_{m,n}$. □

Note that the restrictions on m, n in the statement of Theorem 2.1 are necessary since, for example, C_3 is pseudograceful by Proposition 1.4 part (a) but $C_3 \cup K_{1,2}$ is neither pseudograceful nor graceful as can be easily checked. However, we do not know the answer to the following question (we conjecture that the answer in no).

Question: If G is a pseudograceful graph, is $G \cup K_{2,2}$ pseudograceful?

Theorem 2.1 can be used to produce numerous families of graceful disconnected graphs. We shall be mainly concerned with the families $\bigcup_{i=1}^{r} K_{m_i,n_i}$, $C_r \cup K_{m,n}$ and $K_r \cup K_{m,n}$.

Corollary 2.2.

$\bigcup_{i=1}^{r} K_{m_i,n_i}$, $r \ge 1$, $m_i, n_i \ge 2$ for all $1 \le i \le r$ and $(m_i, n_i) \ne (2,2)$ for $3 \le i \le r$ is graceful.

Proof.

If $r = 1$, $K_{m,n}$ is graceful [6, 7] and if $r = 2$, our assertion follows from Proposition 1.4 part (c) and Theorem 2.1.

If $r \ge 3$, by virtue of Theorem 2.1, the result follows by induction. □

We next consider the family $C_r \cup K_{m,n}$. We shall need the following lemma.

Lemma 2.3.

(a) If $G \cup K_3, G \ne K_1$, is graceful with graceful labelling f, then $0 \in f(V(G))$.

(b) If $G \cup K_{1,r}$ is graceful with graceful labelling f, then $0 \in f(V(G))$.

(c) $mK_3 \cup nK_{1,r}$ is disgraceful for all $m, n, r \ge 1$.

Proof.

(a) Suppose that f is a graceful labelling of $G \cup K_3$, $G \ne K_1$, such that $0 \in f(V(K_3))$ and let $q = |E(G \cup K_3)|$. Then $q-1 \in \mathrm{Im}(f^*)$ gives $f(V(K_3)) = \{0, q, 1\}$ or $\{0, q-1, q\}$. In both cases $f^*(E(K_3)) = \{1, q-1, q\}$ and $\mathrm{Im}(f^*|_{E(G)})$ is bounded by $q-3$. Since $G \ne K_1$ and $5 \le |V(G \cup K_3)| \le q+1$, we have $q-2 \ne 1$ and $q-2 \notin \mathrm{Im}(f^*)$, which is absurd.

(b) Suppose that f is a graceful labelling of $G \cup K_{1,r}$ such that $0 \in f(V(K_{1,r}))$ and let $q = |E(G \cup K_{1,r})|$ and let v be the center vertex of $K_{1,r}$ so that $f(v) = 0$ or q. Note that $G \ne K_1$ since $|V(K_1 \cup K_{1,r})| = r+2$ while $|E(K_1 \cup K_{1,r})| = r$.

Case 1 $f(v) = 0$

In this case we shall prove that $q - k \in f(V(K_{1,r}))$ for $0 \le k \le r-1$. Assume by induction that our assertion holds for all $0 \le k < s \le r-1$, then $\mathrm{Im}(f^*|_{E(G)})$ is bounded by $q-s-1$ and since $q-s \in \mathrm{Im}(f^*)$, we must have $q-s \in f(V(K_{1,r}))$ as desired. It follows that $\mathrm{Im}(f^*|_{E(G)})$ is bounded by $q-r-1$ and $q-r \notin \mathrm{Im}(f^*)$, which is absurd.

Case 2 $f(v) = q$

An argument similar to that in Case 1 shows that it is impossible to have such a graceful labelling

(c) This follows from part (a) and (b).

Corollary 2.4.

Let $1 \le m \le n$, then

(a) $C_3 \cup K_{m,n}$ is graceful if and only if $m, n \ge 2$.

(b) $C_4 \cup K_{m,n}$ is graceful if and only if $m, n \ge 2$ or $\{m, n\} = \{1, 2\}$.

(c) $C_7 \cup K_{m,n}$ and $C_8 \cup K_{m,n}$ are graceful for all m, n.

Proof.

Note that if $m, n \ge 2$, then $C_r \cup K_{m,n}$ for $r \in \{3, 4, 7, 8\}$ is graceful by Proposition 1.4 part (a) and Theorem 2.1.

(a) Observe that $C_3 \cup K_{1,n}$ is disgraceful for all $n \ge 1$ by Lemma 2.3 part (c).

(b) Since $K_{1,2} \cong P_3$ is pseudograceful [3, Theorem 3.2], then $C_4 \cup K_{1,2}$ is graceful by Theorem 2.1 since $C_4 \cong K_{2,2}$.

Let $V(C_4) = \{u_1, u_2, u_3, u_4\}$ where $u_i u_j \in E(C_4)$ if and only if $i - j \equiv \pm 1 \pmod 4$ and let v be the center vertex of $K_{1,n}$. Suppose that, for some $n \ne 2$, $C_4 \cup K_{1,n}$ is graceful and let f be a graceful labelling of this graph. We have $q = |E(C_4 \cup K_{1,n})| = 4 + n$ and $|V(C_4 \cup K_{1,n})| = 4 + n + 1$, hence both f and f^* are bijections. By Lemma 2.3 part (b), we have $0 \in f(V(C_4))$ and we may assume that $f(u_2) = 0$ and $f(u_3) = q$. Since $\mathrm{Im}(f^*|_{E(K_{1,n})})$ is bounded by $q-2$, we get $q-1 \in f^*(E(C_4))$ and we have two cases:

Case 1 $f(u_1) = q-1$

In this case $\mathrm{Im}(f^*|_{E(K_{1,n})})$ is bounded by $q-3$ and $q-2 \in f^*(E(C_4))$ gives $f(u_4) = 2$. Hence $\mathrm{Im}(f^*|_{E(K_{1,n})})$ is bounded by $q-4$ and $f(v) = 1$ or $q-3$.

If $f(v) = 1$, then $q-3 \in f^*(E(C_4)) \cap f^*(E(K_{1,n}))$, which is absurd.

If $f(v) = q-3 \neq 2$, then $n \neq 1$ and $q-2$, $q-4$ ($\neq 2$ since $n \neq 2$) are both vertex labels of $K_{1,n}$ and f^* is not injective which is absurd.

Case 2 $f(u_4) = 1$

In this case $\mathrm{Im}(f^*|_{E(K_{1,n})})$ is bounded by $q-3$ and $q-2 \in f^*(E(C_4))$ gives $f(u_1) = q-2$. Hence $\mathrm{Im}(f^*|_{E(K_{1,n})})$ is bounded by $q-4$ and $f(v) = 3$ or $q-1$.

If $f(v)=3 \neq q-2$, then $n \neq 1$ and 2, 4 ($\neq q-2$ since $n \neq 2$) are both vertex labels of $K_{1,n}$ and f^* is not injective, which is absurd.

If $f(v) = q-1$, then $q-3 \in f^*(E(C_4)) \cap f^*(E(K_{1,n}))$, which is absurd.

(c) $C_7 \cup K_{1,n}$ and $C_8 \cup K_{1,n}$ are graceful for all $n \geq 1$ [2].

We now consider the family $K_r \cup K_{m,n}$.

Corollary 2.5

Let $1 \leq m \leq n$, then

(a) $K_1 \cup K_{m,n}$ is graceful if and only if $m, n \geq 2$.

(b) $K_2 \cup K_{m,n}$ is graceful if and only if $m, n \geq 2$ and $(m, n) \neq (2, 2)$.

(c) $K_3 \cup K_{m,n}$ is graceful if and only if $m, n \geq 2$.

(d) $K_4 \cup K_{m,n}$ is graceful for all m, n.

Proof.

If $m, n \geq 2$ then $K_i \cup K_{m,n}$ for $i \in \{1, 3, 4\}$ is graceful by Proposition 1.4 part (b) and Theorem 2.1.

(a) Note that $\left|V(K_1 \cup K_{1,n})\right| = n+2$ and $\left|E(K_1 \cup K_{1,n})\right| = n$, hence $K_1 \cup K_{1,n}$ is trivially disgraceful.

(b) $K_2 \cup K_{2,2}$ is disgraceful by Corollary 2.4 part (b).

Suppose $2 \leq m \leq n$ and $(m, n) \neq (2, 2)$ and let $V(K_{m,n}) = V_1 \cup V_2$, where V_i, $i = 1, 2$ are independent sets of vertices where $\left|V_1\right| = m$, $\left|V_2\right| = n$.

Define the function

$$f : V(K_2 \cup K_{m,n}) \rightarrow \{0, 1, \ldots, mn+1\}$$

such that

$$f(V_1) = \{ni+1 : 1 \leq i \leq m\},$$

$$f(V_2) = \{j : 0 \leq j \leq n-1\},$$

and

$$f(V(K_2)) = \{mn, mn-1\},$$

then f is clearly a graceful labelling of $K_2 \cup K_{m,n}$.

(c) This follows from Corollary 2.4 part (a).

(d) By Theorem 2.1, we need only to consider the case where $m = 1$. Let v be the center vertex of $K_{1,n}$, $n \geq 1$ and define $f : V(K_4 \cup K_{1,n}) \rightarrow \{0, 1, \ldots, n+6\}$ such that $f(V(K_4)) = \{0, 1, n+4, n+6\}$, $f(v) = n+5$ and $f(V(K_{1,n})) = \{3, 4, \ldots, n+2, n+5\}$, then f is easily seen to be a graceful labelling of $K_4 \cup K_{1,n}$. □

Finally we supplement the result about the gracefulness of the family $K_r \cup K_{m,n}$ included in Corollary 2.5 by the following proposition.

Proposition 2.6

(a) $K_5 \cup K_{1,n}$ is graceful for all n.

(b) $K_6 \cup K_{1,n}$ is graceful if and only if $n \notin \{1, 3\}$.

Proof.

Let v be the center vertex of $K_{1,n}$.

(a) For $n \geq 1$ define

$$f : V(K_5 \cup K_{1,n}) \to \{0, 1, \ldots, n+10\}$$

such that

$$f(V(K_5)) = \{0, 1, 4, n+8, n+10\},$$

$$f(v) = 2,$$

and

$$f(V(K_{1,n})) = \{2, 7, 8, \ldots, n+5, n+7\},$$

then f is easily seen to be a graceful labelling of $K_5 \cup K_{1,n}$.

(b) Define

$$f : V(K_6 \cup K_{1,n}) \to \{0, 1, \ldots, n+15\}$$

such that

$$f(V(K_6)) = \{0, 1, 4, n+8, n+13, n+15\},$$

$$f(v) = n+12,$$

and

$$f(V(K_{1,n})) = \begin{cases} \{2, 6, 7, 9, 10, \ldots, n+4, n+6, n+12\} & , \quad n \geq 5 \\ \{2, 6, 7, n+6, n+12\} & , \quad n = 4 \\ \{2, 6, n+12\} & , \quad n = 2 \end{cases}$$

so that $f^*(E(K_6))=\{1, 2, \ldots, 5, 7, n+4, n+7, n+8, n+9, n+11, \ldots n+15\}$

then, $f^*\Big|_{E(K_6)}$ is injective for $n \geq 4$ and $n = 2$ and

$$f^*(E(K_{1,n})) = \begin{cases} \{6, 8, \ldots, n+3, n+5, n+6, n+10\} & , \quad n \geq 5 \\ \{6, n+5, n+6, n+10\} & , \quad n = 4 \\ \{n+6, n+10\} & , \quad n = 2 \end{cases}$$

then, $f^*\Big|_{E(K_6)}$ is injective for $n \geq 4$ and $n = 2$ also, and since $f^*(E(K_6)) \cap f^*(E(K_{1,n})) = \phi$, then f^* is injective as desired.

$K_6 \cup K_{1,1}$ is disgraceful by [8]. To verify that $K_6 \cup K_{1,3}$ is disgraceful suppose that f is a graceful labelling of $K_6 \cup K_{1,3}$, then Lemma 2.3 (b) gives $0 \in f(V(K_6))$ and hence $18 \in f(V(K_6))$. Also $17 \in \text{Im}(f^*)$ gives 1 or $17 \in f(V(K_6))$ and $16 \in \text{Im}(f^*)$ further shows that either $\{0,18,17,2\} \subseteq f(V(K_6))$ or $\{0, 18, 1, 16\} \subseteq f(V(K_6))$. We will rule out the first case. The second case is ruled out similarly. Assume $\{0, 18, 17, 2\} \subseteq f(V(K_6))$, then $14 \in \text{Im}(f^*)$ gives $14 \in f(V(K_6))$ or $\{1, 15\} \subseteq f(V(K_{1,3}))$ and $f(v)=1$ or 15.

If $14 \in f(V(K_6))$, then $13 \in \text{Im}(f^*)$ gives $\{3, 16\} \subseteq f(V(K_{1,3}))$ and $f(v) = 3$ or 16 and $11 \in \text{Im}(f^*)$ further shows that $\{5, 3, 16\} \subseteq f(V(K_{1,3}))$ and $f(v)=16$, then $10 \in \text{Im}(f^*)$ finally gives $f(V(K_{1,3})) = \{6, 5, 3, 16\}$ and $5 \notin \text{Im}(f^*)$, which is absurd.

If $\{1, 15\} \subseteq f(V(K_{1,3}))$ and $f(v) = 15$, then $13 \in \text{Im}(f^*)$ forces $13 \in f(V(K_6))$ (since $5 \in f(V(K_6))$ gives $10 \notin \text{Im}(f^*)$, which is absurd) and $12 \in \text{Im}(f^*)$ forces $3 \in f(V(K_{1,3}))$ and $10 \in \text{Im}(f^*)$ forces $5 \in f(V(K_{1,3}))$, hence $f(V(K_{1,3}))=\{1,3,5,15\}$ and $9 \notin \text{Im}(f^*)$, which is absurd.

If $\{1, 15\} \subseteq f(V(K_{1,3}))$ and $f(v) = 1$, then $13 \in \mathrm{Im}(f^*)$ gives 5 or $13 \in f(V(K_6))$ or $14 \in f(V(K_{1,3}))$. Also a $5 \in f(V(K_6))$ gives $4 \notin \mathrm{Im}(f^*)$, which is absurd, and $13 \in f(V(K_6))$ gives $12 \notin \mathrm{Im}(f^*)$, which is absurd as well. Then $14 \in f(V(K_{1,3}))$ and $12 \in \mathrm{Im}(f^*)$ gives 6 or $12 \in f(V(K_6))$ or $13 \in f(V(K_{1,3}))$. A $6 \in f(V(K_6))$ gives $5 \notin \mathrm{Im}(f^*)$, which is absurd, and $12 \in f(V(K_6))$ gives $11 \notin \mathrm{Im}(f^*)$, which is absurd as well. Then $f(V(K_{1,3})) = \{1, 13, 14, 15\}$ and $4 \notin \mathrm{Im}(f^*)$, which is absurd.

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