Generalized Index of Boolean Matrices

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ABSTRACT. In this paper we introduce a new parameter related to the index of convergence of Boolean matrix – the generalized index. The parameter is motivated by memoryless communication system. We obtain the values of this parameter for reducible, irreducible and symmetric matrices.

1 Introduction

The set, B_n of $n \times n$ Boolean matrices forms a finite multiplicative semigroup of order 2^{n^2}.

Note that we use Boolean arithmetic when calculating the powers of a matrix. Let $A \in B_n$. Since the sequence of powers $A^0 = I,\ A, A^2, \ldots$ forms a finite subsemigroup $\langle A \rangle$ of B_n, then there exists a least nonnegative integer $k = k(A)$ such that $A^k = A^{k+t}$ for some $t > 0$, and there exists a

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least positive integer $p = p(A)$ such that $A^k = A^{k+p}$. We call the integer $k = k(A)$ the index of convergence of A and the integer $p = p(A)$ the period of A.

A matrix $A \in B_n$ is reducible if there is an $n \times n$ permutation matrix P such that

$$PAP^{-1} = \begin{bmatrix} B & 0 \\ C & D \end{bmatrix},$$

where B and C are square (nonvacuous) matrices. The matrix A is irreducible if it is not reducible. Let IB_n and RB_n denote the set of irreducible and reducible matrices of order n, respectively.

If $A \in B_n$ $(n > 1)$ is irreducible, then $p(A) = 1$ if and only iff A is primitive and in this case $k(A)$ is just the primitive exponent of A, i.e., the least positive integer k such that A^k is the matrix of all 1's. Let P_n denote the set of primitive matrices in B_n.

There is a natural $1-1$ correspondence between the set of $n \times n$ Boolean matrices and the set of labeled digraphs of order n. We associate with the matrix $A = (a_{ij})$ the digraph $D = D(A)$ with vertex $V(D) = \{1, 2, \ldots, n\}$ and arc set $\{(i,j)\colon a_{ij} \neq 0\}$. Thus the study of the properties of Boolean matrices can be turned into that of the corresponding associated digraphs. The index of convergence, period of A are called the index of convergence, period of $D(A)$, and is denoted by $k(D(A))$ and $p(D(A))$ equivalently. It is well known that if A is irreducible, then $p(A)$ is the greatest common divisor of the distinct lenghts of the circuits of $D(A)$, and that if A is reducible, then $p(A)$ is the least multiple of $p(A_1), \ldots, p(A_m)$, where $A_1, \ldots, A_m$ are the irreducible constituents of A. It is also well known that Boolean matrix A is irreducible if and only if $D(A)$ is strongly connected.

Let $A = (a_{ij}) \in B_n$. The (i,j)-entry of A^m is denoted by $a_{ij}^{(m)}$. Then $a_{ij}^{(m)} \neq 0$ if and only if there is a walk of length m from i to j in $D(A)$.

Recently, R.A. Brualdi and Bolian Liu [1] introduced generalized exponents of primitive matrices from memoryless communication system. Note that an adjacency matrix of a network need not be primitive. In this paper we introduce a new parameter that is a generalization of the exponent for the sequence of powers of Boolean matrices in [1].

Let D be a digraph with period $p = p(D)$ where $D = D(A)$ for some $A = (a_{ij}) \in B_n$. Define $k_D(i,j) := \min\{k\colon$ for any integer $m \geq k$, there exists a walk of length m from i to j in D if and only if there exists a walk of length $m+p$ from i to j in $D\}$.

Clearly, if A is primitive, then $k_D(i,j)$ is the local exponent in [1].

$k_D(i,j)$ is called the local index of convergence (or simply local index) from i to j. Clearly, $k_D(i,j)$ is the integer k such that the sequence $\{a_{ij}^{(m)}\colon m = 0, 1, 2, \ldots\}$ is with period p from a beginning term $a_{ij}^{(k)}$, i.e.,

$k_D(i,j)$ is the least nonnegative integer k such that $a_{ij}^{(t)} = a_{ij}^{(t+p)}$ for all $t \geq k$. Clearly $k(D) = \max_{1\leq i,j\leq n} k_D(i,j)$ (see [2]).

Let $D = D(A)$ where $A = (a_{ij}) \in B_n$. The least positive period of the sequence $\{a_{ij}^{(m)}: m = 0, 1, 2, \dots\}$ is called the local period from i to j of digraph D, denoted by p_{ij}. Clearly $p_{ij}|p(A)$.

For $i = 1, 2, \dots, n$, define

$k_D(i) := \max_{j\in D} k_D(i,j)$;

$p_D(i) := \min\{p$: there exists a walk of length m for every integer $m \geq k_D(i,j)$ from i to j in D if and only if there exists a walk of length $m+p$ from i to j, $j \in \{1, 2, \dots, n\}$ in $D\}$.

$k_D(i)$ and $p_D(i)$ are called the index and period of vertex i of D respectively.

Lemma 1. ([2] *Let*

$$a_0, a_1, a_2, \dots, a_m, \dots \tag{1}$$

be a sequence with period d (> 0) from a beginning term a_{m_d}, i.e., m_d is the least nonnegative integer such that $a_{t+d} = a_t$ for all $t \geq m_d$. If $d|p$, and (1) is also with period p from a beginning term a_{m_p}, then $m_d = m_p$.

Since $p_{ij}|p(A)$, $p_A(i)|p(A)$ holds. It follows from Lemma 1 that the sequence $\{a_{ij}^{(m)}: m = 0, 1, \dots\}$ is with period $p_A(i)$ and $p(A)$ from the same beginning term. Hence we use period $p(A)$ to define $k_A(i,j)$ and $k_A(i)$.

The numbers $k_D(i)$ have an interpretation in terms of a memoryless communication system associated with D. Suppose that at time $t = 1$ each vertex of D with some information passes the information to each of its neighbours (those vertices reachable by a walk of length 1) and then forgets its information. But it may receive information from another vertex. The system continues in this way. Vertex i always passes the information to the same vertices, the set of which is denoted by $N(i)$, every period of time after $t = t_0$. This time t_0 is $k_D(i)$. The period of time is $p_D(i)$. Clearly if $N(i) = V(D)$, then $k_D(i)$ is the generalized exponent in [1].

We choose to order the vertices of D in such a way that

$$k_D(1) \leq k_D(2) \leq \cdots \leq k_D(n),$$

and call $k_D(i)$ the ith generalized index of D, denoted by $k(D), i)$. We write $k(A,i) = k(D(A),i)$ for any $A \in B_n$ with $1 \leq i \leq n$ and call $k(A,i)$ the ith generalized index of A. Clearly $k(A,i)$ is the smallest nonnegative integer k such that i rows of A^k and A^{k+p} are equal.

Thus for all $n \times n$ Boolean matrices A,

$$k(A,n) = k(A) = k(D(A)) \leq (n-1)^2 + 1. \quad \text{(see [1], [5])}$$

Define $k(n,i)$, $k^R(n,i)$, and $k^I(n,i)$, $k^S(n,i)$ to be the maximum of $k(A,i)$ where the maximum is taken over all Boolean matrices, all reducible Boolean matrices, all irreducible Boolean matrices and all symmetric Boolean matrices of order n, respectively.

The numbers $k(n,i)$, $k^R(n,i)$, $k^I(n,i)$, $k^S(n,i)$ are called generalized index of the corresponding classes of Boolean matrices of order n.

2 $k^I(n,i)$

We first establish the following.

Lemma 2. *There exists a matrix $\Gamma_n \in IB_n$ such that $k(\Gamma_n,i) = n^2-3n+2+i$ for $1 \le i \le n$.*

Proof: By Theorem 2.3 of [1], there exists $\Gamma_n \in P_n \subseteq IB_n$ such that $k(\Gamma_n,i) = k_{\Gamma_n}(i) = \exp_{\Gamma_n}(i) = n^2-3n+2+i$. Thus Lemma 2 follows □

Lemma 3. ([2]) *Suppose that A is an irreducible Boolean matrices with period p, and the length of a shortest circuit of $D(A)$ is s, then*

$$k(A) \le n + s(\frac{n}{p} - 2).$$

Theorem 1. $k^I(n,i) = n^2 - 3n + i + 2$, $1 \le i \le n$.

Proof: For any $A \in IB_n$, let $D(A)$ be the associated digraph of A. We consider the following two cases.

Case 1. $A \in P_n$. By Theorem 3.4 of [1],

$$k(A,i) = k(D(A),i) \le n^2 - 3n + i + 2.$$

Case 2. $A \in IB_n \backslash P_n$. Then $p = p(A) \ge 2$ and $D(A)$ is strongly connected. Let s be the length of a shortest circuit of $D(A)$. By Lemma 3,

$$\begin{aligned} k(A,i) &\le k(A,n) = k(A) \\ &\le n + s(\frac{n}{p} - 2) \\ &\le n + n(\frac{n}{2} - 2) \\ &\le n^2 - 3n + 3 \\ &\le n^2 - 3n + i + 2. \end{aligned}$$

Summarizing the above conclusions, we have $k(A,i) \le n^2 - 3n + i + 2$ for any $A \in IB_n$, and by Lemma 2 there exists a $\Gamma_n \in IB_n$ such that $k(\Gamma_n,i) = n - 3n + i + 2$. Hence

$$k^I(n,i) = \max_{A \in IB_n} k(A,i) = n - 3n + i + 2.$$

□

3 $k^R(n,i)$

Lemma 4. ([2]) *If* $X \in B_n$ *has the following form*

$$X = \begin{bmatrix} B & 0 \\ \alpha & a \end{bmatrix},$$

where B *is an* $(n-1) \times (n-1)$ *Boolean matrix. Then*

$$\begin{cases} k(B) \leq k(X) \leq k(B) + 1 & \text{if } a = 0, \\ k(B) \leq k(X) \leq \max\{k(B), n-1\} & \text{if } a = 1. \end{cases}$$

We now show the following.

Theorem 2. $k^R(n,i) = (n-3)(n-2) + i$, $1 \leq i \leq n$, $n \geq 2$.

Proof: By Lemma 2, there exists a strongly connected digraph $G = (V, E)$ (of order $n-1$) with vertex set $V = \{1, 2, \ldots, n-1\}$ such that $k(G,i) = (n-2)(n-3) + i$ for all $1 \leq i \leq n-1$. Let G' be (of order n) obtained from G by adding a new vertex named n and an arc $(n, n-1)$ to G. Then G' is not strongly connected, and it is easy to verify that

$$k(G', i) = \begin{cases} k(G,i) = (n-2)(n-3) + i & \text{if } 1 \leq i \leq n-1, \\ k(G, n-1) + 1 = (n-2)(n-3) + n & \text{if } i = n, \end{cases}$$

That is $k(G', i) = (n-2)(n-3) + i$ for all $1 \leq i \leq n$. Thus Theorem 2 follows immediately from the following

Theorem 2'. *If* A *is a reducible Boolean matrix of order* n*, then* $k(A,i) \leq (n-2)(n-3) + i$ *for all* $1 \leq i \leq n$.

Proof: We use induction on n to prove the theorem. Theorem 2' is true for all reducible Boolean matrices of order less that n and $p = p(A)$.

Claim 1. $k(B,i) \leq (|B|-1)(|B|-2) + i$, $1 \leq i \leq |B|$, for any Boolean matrix B of order less than n. (Here we use $|B|$ to denote the order of B for convenience). To justify this claim, if B is irreducible, then Claim 1 follows from Theorem 1; otherwise if B is reducible, then Claim 1 follows from the induction hypothesis. (Theorem 2' holds for $|B| < n$).

The proof is now divided into the following three cases.

Case 1. There exists an $n \times n$ permutation matrix P such that $PAP^{-1} = \begin{bmatrix} B & 0 \\ \alpha & a \end{bmatrix}$, where B is an $(n-1) \times (n-1)$ matrix. Then, for all $t \geq 1$,

$$PA^tP^{-1} = \begin{bmatrix} B^t & 0 \\ \alpha B^{t-1} + a\alpha \sum_{i=0}^{t-2} B^i & a \end{bmatrix}.$$

Since $k(A,i)$ is the smallest nonnegative integer k such that i rows of A^k and A^{k+p} are equal, $k(A,i) \le k(B,i) \le (n-2)(n-3)+i$ for all $1 \le i \le n-1$. (The last inequality follows from Claim 1). Also by Lemma 4 and Claim 1, $k(A,n) \le \max\{k(B,n-1)+1, n-1\} \le (n-2)(n-3)+n$.

Case 2. There exists an $n \times n$ permutation matrix P such that $PAP^{-1} = \begin{bmatrix} B & \beta \\ 0 & a \end{bmatrix}$, where B is an $(n-1) \times (n-1)$ matrix. Then, for all $t \ge 1$,

$$PA^tP^{-1} = \begin{bmatrix} B^t & (1-a)B^{t-1}\beta a(\sum_{i=0}^{t-1} B^i)\beta \\ 0 & a \end{bmatrix}.$$

Since the nth row of PA^tP^{-1} is independent of $t \ge 1$, we have $k(A,1) \le 1$. Recall that B is an $(n-1) \times (n-1)$ matrix. Thus, for all $2 \le i \le n$ and $t \ge \max\{k(B,i-1)+1, n-1\}$, $\sum_{i=0}^{t-1} B^i = \sum_{i=0}^{n-2} B^i$ and so

$$PA^tP^{-1} = \begin{bmatrix} B^t & (1-a)B^{t-1}\beta + a(\sum_{i=0}^{n-2} B^i)\beta \\ 0 & a \end{bmatrix}.$$

Since $k(A,i)$ is the smallest nonnegative integer k such that i rows of A^k and A^{k+p} are equal, $k(A,i) \le \max\{k(B,i-1)+1, n-1\} \le (n-2)(n-3)+i$. for all $2 \le i \le n$. (The last inequality follows from Claim 1).

Case 3. A does not satisfy the conditions in Cases 1 or 2. By [2, Lemma 7], $k(A,i) \le k(A) \le n^2 - 5n + 9 \le (n-2)(n-3)+i$ for all $3 \le i \le n$. Now suppose $1 \le i \le 2$. Since A is reducible, there exists an $n \times n$ permutation matrix P such that $PAP^{-1} = \begin{bmatrix} C & 0 \\ E & D \end{bmatrix}$, where C and D are square matrices woth orders at most $n-2$. Then $n \ge 4$. By Claim 1, $k(C,i) \le (|C|-1)(|C|-2)+i \le (n-3)(n-4)+2 \le (n-2)(n-3)+i$. Again since $k(A,i)$ is the smallest nonnegative integer k such that i rows of A^k and A^{k+p} are equal, $k(A,i) \le k(C,i) \le (n-2)(n-3)+i$ for $1 \le i \le 2$.

Combining the above three cases, we complete the proof of Theorem 2' □

It follows from Theorems 1 and 2 that

$$k(n,i) = \max\{k^I(n,i), k^R(n,i)\} = n^2 - 3n + 2 + i$$

for all $1 \le i \le n$.

4 $k^S(n,s)$

We turn to discuss symmetric irreducible matrices. Let ISB_n be the set of symmetric irreducible matrices and let

$$k^{IS}(n,i) := \max\{k(A,i) \colon A \in ISB_n\}.$$

Lemma 5. ([1]) *If A is an $n\times n$ symmetric primitive matrix, then $k(A,i) \leq n-2+i$ and there is an $n \times n$ symmetric primitive matrix A such that $k(A,i) = n-2+i$.*

By the way, we want to point out a minor error in Figure 2 of [1]. Its labeling of vertices $(n, n-1, \ldots, 1)$ should be $(1, 2, \ldots, n)$.

Note that if $A \in ISB_n \backslash P_n$, then $p(A) = 2$, and the length of any circuit of $D(A)$ is even.

Lemma 6. ([4]) *Let $A \in ISB_n \backslash P_n$. Then $k(A) \leq n-2$.*

By Lemmas 5 and 6, for any $A \in ISB_n$, we have $k(A,i) \leq n-2+i$ if A is primitive and $k(A,i) \leq k(A,n) = k(A) \leq n-2 < n-2+i$ otherwise. And by Lemma 5 again, there is a matrix $A_0 \in ISB_n \cap P_n$ such that $k(A_0, i) = n-2+i$. Hence we have

Theorem 3. $k^{IS}(n,i) = n-2+i$.

Theorem 4. *For any $A \in ISB_n \backslash P_n$, $k(A,i) \leq \lceil \frac{n-1}{2} \rceil + i - 2$, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.*

Proof: Let $A \in ISB_n \backslash P_n$. Then $p(A) = 2$. And let T be a spanning tree of $D(A)$. Then T has one or two centres (see [6] page 27), one of which is labeled u. Let $d = \max_{v \in V(D)} d(u,v)$, where $d(u,v)$ denotes the distance from vertex u to v in T, and $D = D(A)$.

Now we consider $k_D(u)$. Take any vertex $v \in V(D)$.

If there is a walk W of length $d-1$ from u to v in D, then there is a walk of length $d-1+2 = d+1$ from u to v by attaching a circuit of length 2 to W.

On the other hand, if there is a walk of length $d+1$ from u to v in D, since $d(u,v)$ and $d+1$ have the same parity and $d(u,v) \leq d$, we have $d(u,v) \leq d-1$. Assume $d-1 = d(u,v) + 2b$ for some nonnegative integer b. By attaching circuits of length 2 to a path of length $d(u,v)$ from u to v, we obtain a walk of length $d(u,v) + 2b = d-1$ from u to v.

Hence there is a walk of length $d-1$ from u to v in D if and only if there is a walk of length $d-1+2 = d+1$ from u to v. By the arbitrary of v, we have $k(A,1) \leq k_D(u) \leq d-1 \leq \lceil \frac{1}{2} d(A) \rceil - 1$, where $d(A)$ denotes the diameter of $D(A)$.

And we show that $k(A,i) \leq k(A,i-1)+1$ $(2 \leq i \leq n)$ as follows.

Suppose $k(A,j) = k_D(v_j)$ for $j = 1, 2, \ldots, i-1$. Since $A \in ISB_n$, D is strongly connected. There is an arc from some vertex $v \in V(D) \backslash \{v_1, v_2, \ldots, v_{i-1}\}$ to some vertex $v_j \in \{v_1, v_2, \ldots, v_{i-1}\}$. So $k_D(v) \leq k(A, i-1) + 1$, which implies that $k(A,i) \leq k(A,i-1)+1$ for $2 \leq i \leq n$.

This follows that

$$\begin{aligned} k(A,i) &\le k(A,1)+i-1 \\ &\le \lceil \frac{1}{2}d(A)\rceil -1+i-1 \\ &\le \lceil \frac{1}{2}(n-1)\rceil + i - 2. \end{aligned}$$

□

Remark: We consider the digraph D_1 obtained from an indirected path of order n with any edge uv replaced by arcs (u,v) and (v,u). The matrix A with associated digraph D_1 satisfies $d(A) = n-1$, $\lceil \frac{1}{2}d(A)\rceil = \lceil \frac{n-1}{2}\rceil$, $k(A,1) = \lceil \frac{1}{2}d(A)\rceil - 1 = \lceil \frac{n-1}{2}\rceil - 1$.

Finally, we show our main result in this section.

Theorem 5. $k^S(n,i) = n-2+i$.

Proof: Let $A \in B_n$ is symmetric. If $A \in ISB_n$, then by Theorem 3, we have $k(A,i) \le n-2+i$. Suppose A is reducibe. Denote by $D_1, D_2, \ldots, D_t$ $(t \ge 2)$ all strong components of $D(A)$, the number of vertices are $n_1, n_2, \ldots, n_t$, respectively. Then for every D_j $(1 \le j \le t)$, there is no walk from any vertex of D_j to any vertex of another component. We choose to order the vertices of D_j $(1 \le j \le t)$ such that

$$k_{D_j}(1^{(j)}) \le k_{D_j}(2^{(j)}) \le \cdots \le k_{D_j}(n_j^{(j)}).$$

By Theorem 3, we have $k_{D_j}(r^{(j)}) \le n_j - 2 + r < n-2+r$ for $1 \le j \le t$, $1 \le r \le n_j$ with $r^{(j)} \in V(D_j)$, which implies that $k(A,i) = k(D(A),i) < n-2+i$ for $1 \le i \le n$.

Hence we have $k(A,i) \le n-2+i$ no matter A is irreducible or not. By Lemma 5, ther is a symmetric matrix A_0 in B_n such that $k(A_0,i) = n-2+i$. It follows that $k^S(n,i) = n-2+i$.

We complete the proof. □

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