Multiplicity of triangles in 2-edge coloring of a family of graphs

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Abstract

Let $\{G(n,k)\}$ be a family of graphs where $G(n,k)$ is the graph obtained from K_n, the complete graph on n vertices, by removing any set of k parallel edges. In this paper the lower bound for the multiplicity of triangles in any 2-edge coloring of the family of graphs $\{G(n,k)\}$ is calculated and it is proved that this lower bound is sharp when $n \geq 2k+4$ by explicit coloring schemes in a recursive manner. For the cases $n = 2k+1, 2k+2$ and $2k+3$ this lower bound is not sharp and exact bound in these cases also are independently calculated by explicit constructions.

1 Introduction and background results

If F and G are graphs, define $M(G,F)$ to be the minimum number of monochromatic G that occur in any 2-coloring of the edges of F. $M(G,F)$ is called the multiplicity of G in F. A. W. Goodman [2] proved that

$$\begin{aligned} M(K_3,K_n) &= \frac{1}{3}\,u\,(u-1)\,(u-2) && if\ n=2u,\\ &= \frac{2}{3}\,u\,(u-1)\,(4u+1) && if\ n=4u+1,\\ &= \frac{2}{3}\,u\,(u+1)\,(4u-1) && if\ n=4u+3, \end{aligned}$$

where u is a nonnegative integer. A set of k edges in a graph is said to be parallel if no two of them have a vertex in common. Let $\{G(n,k)\}$ be a

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family of graphs where $G(n,k)$ is the graph obtained from K_n, the complete graph on n vertices, by removing any set of k parallel edges. In this paper we determine

1. A lower bound for multiplicity of triangles in the family of graphs $\{G(n,k)\}$.
2. We also prove that the lower bound calculated is sharp for all $n \geq 2k+4$ by explicit constructions.
3. For the cases n equals $2k+1, 2k+2$ and $2k+3$ we determine sharp lower bounds separately and we give the corresponding constructions.

The case $n = 2k$ has been dealt with in earlier paper [5] where it was proved that

$$M(K_3, G(2k,k)) = 8M(K_3, K_k).$$

The idea of proof in this paper is similar to the one used in the papers of Goodman [2] and Sauve [4] but somewhat more complicated due to the fact that we are not dealing with complete graphs. We follow the method of weights given in [4]. For the basic definitions and notations used in this paper, we follow [1].

2 Method of Weights

Let $\mathbf{G} = \mathbf{G(n,k)}$. Our aim is to determine the minimum number of monochromatic triangles that exist when the edges of G are colored with two colors. For this we give weight to each pair of edges at every vertex p of G. Let $\mathcal{A}(p)$ be the set of all pairs of edges at a vertex p in G. Suppose $a \in \mathcal{A}(p)$. We define $W(a) = 2$, if both the edges are of the same color and $W(a) = -1$ otherwise. For every vertex p of G we define $W(p)$, the weight at the vertex p, to be $\sum_{a \in \mathcal{A}(p)} W(a)$. Let $W(G) = \sum_{p \in V(G)} W(p)$, where $V(G)$ is the vertex set of G. We define the weight of the graph G as $W(G)$.

Let $\mathcal{B}$ be the set of all subgraphs of G induced by any three of the vertices of the graph G. As any pair of edges at a vertex p of the graph G lies in exactly one subgraph of G induced by three vertices we get $W(G) = \sum_{B \in \mathcal{B}} W(B)$. These subgraphs in $\mathcal{B}$ fall under any one of the following 4 sets.

1. S_1, the set of all subgraphs induced by 3 vertices such that the subgraphs have 3 edges of the same color.
2. S_2, the set of all subgraphs induced by 3 vertices such that the subgraphs have 3 edges, not of the same color.

3. S_3, the set of all subgraphs induced by 3 vertices such that the subgraphs have 2 edges of the same color and a nonedge.

4. S_4, the set of all subgraphs induced by 3 vertices such that the subgraphs have 2 edges of different colors and a nonedge.

Clearly

$$\begin{aligned} W(B) &= 6, \quad \text{if } B \in S_1 \\ W(B) &= 0, \quad \text{if } B \in S_2 \\ W(B) &= 2, \quad \text{if } B \in S_3 \\ W(B) &= -1, \quad \text{if } B \in S_4 \end{aligned}$$

Hence

$$W(G) = 6|S_1| + 2|S_3| - |S_4|,$$

where for any set X, $|X|$ denotes the cardinality of the set X. Thus

$$|S_1| = \frac{1}{6}\left(W(G) - 2|S_3| + |S_4|\right).$$

Let $S_3(p)$ be the number of elements of the set S_3 where the two edges of the same color are incident at p and $S_4(p)$ be the number of subgraphs of the set S_4 where the two edges of opposite colors are incident at p. It is easy to see that $|S_3| = \sum_{p \in V(G)} S_3(p)$ and $|S_4| = \sum_{p \in V(G)} S_4(p)$. Therefore we get,

$$|S_1| = \frac{1}{6}\left(\sum_{p \in V(G)} W(p) - 2\sum_{p \in V(G)} S_3(p) + \sum_{p \in V(G)} S_4(p)\right). \tag{1}$$

We call the above equation (1) as **Weight Equation**. We will frequently refer to this Weight Equation in the following sections. Also whatever the coloring of the graph G may be, $S_3(p) + S_4(p)$ is a constant, as this is precisely the number of pairs of edges $\{pv, pw\}$ such that $vw \notin E(G)$, where $E(G)$ is the edge set of G. Therefore at any vertex p, maximizing $S_3(p)$ is equivalent to minimizing $S_4(p)$. From equation (1) the graph G will have the minimum number of monochromatic triangles if it satisfies the following two conditions.

$(*)_1$ At every vertex p of G, almost equal number of edges of each color are incident with p.

$(*)_2$ At every vertex p of G, whenever v_iv_j is a nonedge, the edges pv_i and pv_j are of the same color.

If a graph G with 2-coloring of the edges has the minimum number of monochromatic triangles then that coloring of G is said to be a **minimal coloring**. From the foregoing discussion we get the following Proposition.

Proposition 2.1 *Let G be a graph on finite number of vertices. A two coloring of the edges of G will be a minimal coloring if the coloring satisfies the conditions $(*)_1$ and $(*)_2$.*

3 Multiplicity of Triangles in G(n, k)

Let the vertices of K_n be $v_1, v_1', v_2, v_2', \ldots, v_k, v_k', x_1, x_2 \ldots x_{n-2k}$. Suppose we remove the edges $v_1v_1', v_2v_2', \ldots, v_kv_k'$ to get $G(n,k)$. We shall color the edges of $G(n,k)$ with the two colors red and blue. Let A be the subgraph of $G(n,k)$ induced by

$$v_1, v_1', v_2, v_2', \ldots, v_k, v_k'$$

and let B be the subgraph of $G(n,k)$ induced by

$$x_1, x_2, \ldots, x_{n-2k}.$$

In the graph $G(n,k)$, the degree of each vertex of A is $n-2$ and the degree of each vertex of B is $n-1$. Clearly $S_3(p)$ can take maximum value $k-1$ for any vertex p in the subgraph A and k for any vertex p in the subgraph B. When $S_3(p)$ is maximum, $S_4(p)$ is minimum and is equal to 0 for all vertices p.

We begin by determining lower bound of the number $M(K_3, G(n,k))$. For $n \geq 2k+4$, this bound will be shown to be sharp in Section 4 by explicit recursive constructions.

Theorem 3.1

$$\begin{aligned} M(K_3, G(n,k)) &\geq \frac{1}{3}(u-1)(u^2-2u-3k) && if \;\; n = 2u, \\ &\geq \frac{1}{3}(8u^3-6u^2-2u(1+3k)+3k) && if \; n = 4u+1, \\ &\geq \frac{2}{3}(u\,(4u^2+3u-(3k+1)) && if \; n = 4u+3, \end{aligned}$$

where u is a non-negative integer.

Proof : We prove this theorem by considering three cases.

Case 1 : $n = 2u$, where u is a non-negative integer.

The degree of each vertex in A is $2u-2$ and the degree of each vertex in B is $2u-1$. By Condition $(*)_1$, the weight of each vertex in A is minimum when its degree pair is $(u-1, u-1)$ and the weight of each vertex in B is minimum when its degree pair is $(u, u-1)$ or $(u-1, u)$. So, from the Weight Equation,

$$\begin{aligned}
6|S_1| &\geq (2u-2k)\left\{2\binom{u}{2} + 2\binom{u-1}{2} - u(u-1)\right\} \\
&\quad +2k\left\{2\binom{u-1}{2} + 2\binom{u-1}{2} - (u-1)^2\right\} \\
&\quad -2\left\{2k\,(k-1) + (2u-2k)\,k\right\} \\
&= (u-1)\,(2u^2-4u-6k) \\
&= 2\,(u-1)\,(u^2-2u-3k).
\end{aligned}$$

$$\text{Thus, } |S_1| \geq \frac{1}{3}(u-1)\,(u^2-2u-3k).$$

Case 2 : $n = 4u+1$, where u is a non-negative integer.

The degree of each vertex in A is $4u-1$ and the degree of each vertex in B is $4u$. By Condition $(*)_1$, the weight of each vertex in A is minimum when its degree pair is $(2u-1, 2u)$ or $(2u, 2u-1)$ and the weight of each vertex in B is minimum when its degree pair is $(2u, 2u)$. So, from the Weight equation,

$$\begin{aligned}
6|S_1| &\geq (4u+1-2k)\left\{2\binom{2u}{2} + 2\binom{2u}{2} - 2u(2u)\right\} \\
&\quad +2k\left\{2\binom{2u}{2} + 2\binom{2u-1}{2} - 2u(2u-1)\right\} \\
&\quad -2\left\{2k\,(k-1) + (4u+1-2k)\,k\right\} \\
&= 16u^3 - 12u^2 - 4u(1+3k) + 6k.
\end{aligned}$$

$$\text{Thus, } |S_1| \geq \frac{1}{3}(8u^3 - 6u^2 - 2u(1+3k) + 3k).$$

Case 3 : $n = 4u+3$, where u is a non-negative integer.

The degree of each vertex in A is $4u+1$ and the degree of each vertex in B is $4u+2$. By Condition $(*)_1$, the weight of each vertex in A is minimum when its degree pair is $(2u+1, 2u)$ or $(2u, 2u+1)$ and the weight of each vertex in B is minimum when its degree pair is $(2u+1, 2u+1)$. Since the number of vertices of B is odd, all vertex pairs of B cannot have degree $2u+1$, which is odd. So, to attain the next possible minimum, one vertex of B should have degree pair $(2u, 2u+2)$ or $(2u+2, 2u)$. So, from the Weight Equation,

$$
\begin{aligned}
6|S_1| \;\geq\;& (4u+2-2k)\left\{2\binom{2u+1}{2} + 2\binom{2u+1}{2} - (2u+1)^2\right\} \\
&+1\left\{2\binom{2u}{2} + 2\binom{2u+2}{2} - 2u(2u+2)\right\} \\
&+2k\left\{2\binom{2u}{2} + 2\binom{2u+1}{2} - 2u(2u+1)\right\} \\
&-2\left\{2k\,(k-1) + (4u+3-2k)\,k\right\} \\
=\;& 4u(4u^2 + 3u - (3k+1).
\end{aligned}
$$

Thus, $|S_1| \geq \frac{2}{3}u(4u^2+3u-(3k+1))$.

Hence the theorem. □

4 Exact Determination of $\mathrm{M(K_3, G(n,k))}$

In this section we show that for all $n \geq 2k+4$, the lower bound found in Theorem 3.1 is sharp by explicit coloring schemes. We deal with the four cases

$$k \equiv 0 \ (mod\ 4), 1\ (mod\ 4), 2\ (mod\ 4) and 3\ (mod\ 4).$$

In all the discussions below, l is a non-negative integer.

We would like to inform the reader that the proofs are tedious though they only involve recursive construction. For example, for the case $k \equiv 0\ (mod\ 4)$, Lemma 4.3 is the 'basis' of induction and Lemma 4.4 is the 'induction step'. Similar considerations apply to the other three cases.

We construct only the subgraph with red edges in all these cases, i.e. whenever we say that xy is an edge, it means that xy is an edge in the graph G(n, k) and is colored red.

Observation 4.1 *By Proposition 2.1, a coloring of $G(n,k)$ will be a minimal coloring if it satisfies Conditions $(*)_1$ and $(*)_2$. To satisfy Condition $(*)_2$, the construction is made as follows.*

Whenever v_iv_j is an edge of color red(respectively blue), where v_i and v_j are vertices of the subgraph A, the edges $v_i'v_j', v_iv_j', v_i'v_j$ are also colored red (respectively blue). This means that in the subgraph induced by A all the vertices will have even degree and the number of red edges and the number of blue edges are multiples of four. Also for each vertex x_i in B and for each vertex v_j in A, the edges x_iv_j and x_iv_j' are given the same color.

4.1 The case k $\equiv$ 0 (mod 4)

We follow the convention outlined in the beginning of Section 3 with A and B respectively denoting the set of vertices

$$\{v_1, v_1', v_2, v_2', \ldots, v_{4l}, v_{4l}'\} \text{ and } \{x_1, x_2, \ldots, x_{n-8l}\}.$$

Observation 4.2 *It is possible to construct a graph on $4l$ vertices such that $2l$ vertices have degree $2l$ and the other $2l$ vertices have degree $2l-1$.*

It is easy to check this as follows: Let $x_1, x_2, \ldots, x_{2l}$ and $y_1, y_2, \ldots, y_{2l}$ be the $4l$ vertices. Join the edges $x_i y_j$ for $1 \leq i, j \leq 2l$ and $i \neq j$. Also join the edges $x_1x_2, x_3x_4, \ldots x_{2l-1}x_{2l}$. Now the vertices $x_1, x_2, \ldots x_{2l}$ have degree $2l$ and $y_1, y_2, \ldots, y_{2l}$ have degree $2l-1$.

Condition C1 : Let $k = 4l$ and $n = 2k + 4m$, for some $m \geq 1$. Then any 2-coloring of $G(n,k)$ is said to satisfy **Condition C1** if the degrees of the vertices are as given below:

(a) the vertices of A have degree $4l + 2m - 1$;

(b) $2m$ vertices of B have degree $4l + 2m - 1$;

(c) the remaining $2m$ vertices of B have degree $4l + 2m$.

Lemma 4.3 *When $k = 4l$ and $n = 2k + 4$ there exists a minimal coloring of $G(n,k)$ satisfying Condition C1.*

Proof : Consider $G(n,k)$. Let A_1 be the subgraph induced by the vertices $v_1, v_2, \ldots, v_{4l}$ and A_2 be the subgraph induced by the vertices $v_1', v_2', \ldots, v_{4l}'$. By Observation 4.2, we construct A_1 and A_2 such that the vertices

$$v_1, v_2, \ldots, v_{2l} \text{ and } v_1', v_2', \ldots, v_{2l}'$$

have degree $2l$ and the vertices

$$v_{2l+1}, v_{2l+2}, \ldots, v_{4l} \text{ and } v_{2l+1}', v_{2l+2}', \ldots, v_{4l}'$$

have degree $2l-1$. Also, to get a minimal coloring, we construct the subgraph induced by A as dictated by Observation 4.1. Thus vertices of A have degrees $4l$ or $4l-2$ in the subgraph induced by A.

Now x_1, x_2, x_3 and x_4 are the vertices of the subgraph B. Join x_1 with

$$v_1, v_1', v_2, v_2', \ldots, v_{2l}, v_{2l}'.$$

Join x_2, x_3 and x_4 with

$$v_{2l+1}, v_{2l+1}', v_{2l+2}, v_{2l+2}', \ldots, v_{4l}, v_{4l}'.$$

Join x_3 with x_1 and x_4 with x_2. Finally join x_3 and x_4.

So, each vertex in A has degree $4l+1$, the vertices x_1 and x_2 have degree $4l+1$ and the vertices x_3 and x_4 have degree $4l+2$. This construction for $G(n,k)$ satisfies $(*)_1$ and $(*)_2$ and hence this is a minimal coloring. Also this construction satisfies Condition C1. □

Lemma 4.4 *If there exists a minimal coloring of $G(n,k)$, where $k = 4l$ and $n = 2k+4m$ for some $m \geq 1$ satisfying Condition C1 then,*

(a) there exists a minimal coloring of $G(n,k)$ for $n = 2k+4m+1$;

(b) there exists a minimal coloring of $G(n,k)$ for $n = 2k+4m+2$;

(c) there exists a minimal coloring of $G(n,k)$ for $n = 2k+4m+3$;

(d) there exists a minimal coloring of $G(n,k)$ for $n = 2k+4m+4$ satisfying Condition C1.

Proof : We prove this lemma in the following four steps where the output of each step is the input of the next step. In each step i, where $1 \leq i \leq 4$, a new vertex is added to B.

Step 0 : Suppose that there exists a minimal coloring of $G(n,k)$ for $n = 2k+4m$ satisfying Condition C1. This automatically means that all vertices of A have degree $4l+2m-1$ and $2m$ vertices of B say, $x_1, x_2, \ldots, x_{2m}$ have degree $4l+2m-1$ and the other $2m$ vertices of B say, $x_{2m+1}, x_{2m+2}, \ldots, x_{4m}$ have degree $4l+2m$. Also condition $(*)_2$ is satisfied.

Step 1 : Let $n = 2k+4m+1$. In any minimal coloring of $G(n,k)$ for $n = 2k+4m+1$, the vertices of A should have degree $4l+2m-1$ or $4l+2m$ and the vertices of B should have degree $4l+2m$ by Condition $(*)_1$.
Add the vertex x_{4m+1} to the coloring given above. Join this with the vertices

$$v_1, v_1', v_2, v_2', \ldots, v_{2l}, v_{2l}' \text{ and } x_1, x_2, \ldots, x_{2m}.$$

It then follows that the vertices v_i, v_i' where, $1 \leq i \leq 2l$ of A have degree $4l+2m$ and the remaining vertices of A still have degree $4l+2m-1$. Also all the vertices of B have degree $4l+2m$. Also Condition $(*)_2$ is satisfied. So, by Proposition 2.1 this is a minimal coloring for $n = 8l+4m+1$.
Step 2 : Let $n = 2k+4m+2$. In any minimal coloring of $G(n,k)$ for $n = 2k+4m+2$, the vertices of A should have degree $4l+2m$ and the vertices of B should have degree $4l+2m$ or $4l+2m+1$ by Condition $(*)_1$.
Add the vertex x_{4m+2} to the coloring obtained in Step 1. Join this with the vertices

$$v_{2l+1}, v_{2l+1}', v_{2l+2}, v_{2l+2}', \ldots, v_{4l}, v_{4l}' \text{ and } x_{2m+1}, x_{2m+2}, \ldots, x_{4m+1}.$$

It then follows that all the vertices of A have degree $4l + 2m$. The vertices x_is of B, $1 \leq i \leq 2m$, have degree $4l + 2m$ and the remaining vertices of B have degree $4l+2m+1$. Also Condition $(*)_2$ is satisfied. So by Proposition 2.1 this is a minimal coloring for $n = 8l + 4m + 2$.

Step 3 : Let $n = 2k + 4m + 3$. In any minimal coloring of $G(n, k)$ for $n = 2k+4m+3$, the vertices of A should have degree $4l+2m$ or $4l+2m+1$ and the vertices of B should have degree $4l + 2m + 1$ by Condition $(*)_1$.

The number of vertices of B is odd and so each vertex of B cannot have degree $4l + 2m + 1$. Hence to achieve the next possible minimal coloring, one vertex of B will have degree $4l + 2m$ or $4l + 2m + 2$. Add the vertex x_{4m+3} to the coloring obtained in Step 2. Join this with the vertices

$$v_1, v_1', v_2, v_2', \ldots, v_{2l}, v_{2l}' \text{ and } x_1, x_2, \ldots, x_{2m}.$$

It then follows that the vertices v_i, v_i' where, $1 \leq i \leq 2l$ of A have degree $4l + 2m + 1$ and the remaining vertices of A still have degree $4l + 2m$. Also all the vertices in B have degree $4l+2m+1$ except the vertex x_{4m+3} which has degree $4l+2m$. Also Condition $(*)_2$ is satisfied. So, by Proposition 2.1 this is a minimal coloring for $n = 8l + 4m + 3$.

Step 4 : Let $n = 2k + 4m + 4$.In any minimal coloring of $G(n, k)$ for $n = 2k+4m+4$, the vertices of A should have degree $4l+2m+1$ and the vertices of B should have degree $4l + 2m + 1$ or $4l + 2m + 2$ by Condition $(*)_1$. Add the vertex x_{4m+4} to the coloring obtained in Step 1. Join this with the vertices

$$v_{2l+1}, v_{2l+1}', v_{2l+2}, v_{2l+2}', \ldots, v_{4l}, v_{4l}', x_1, x_2, \ldots, x_{2m+1} \text{ and } x_{4m+3}.$$

It then follows that all the vertices of A have degree $4l + 2m + 1$. The vertices $x_i, 1 \leq i \leq 2m + 1$ and x_{4m+4} of B have degree $4l + 2m + 2$ and the remaining vertices of B have degree $4l+2m+1$. Also Condition $(*)_2$ is satisfied. By Proposition 2.1, this is a minimal coloring for $n = 8l+4m+4$. Also this construction satisfies Condition C1.

Hence the lemma. □

For the sake of clarity, each step of the above lemma is explained in each column of Table 1 at the end of Section 4. We explain the similar lemma of the remaining cases $k \equiv 1$ (*mod* 4), 2 (*mod* 4) and 3 (*mod* 4) in Tables 2, 3 and 4.

Using Lemmas 4.3 and 4.4 and by induction hypothesis, we get the following result.

Theorem 4.5 *When $k = 4l$ and $n \geq 2k + 4$, equality is attained in Theorem 3.1.*

4.2 The case $k \equiv 1 \pmod 4$

Let A and B respectively denote the set of vertices

$$\{v_1, v'_1, v_2, v'_2, \dots, v_{4l+1}, v'_{4l+1}\} \text{ and } \{x_1, x_2, \dots, x_{n-8l-2}\}.$$

Observation 4.6 *It is possible to construct a graph on $4l+1$ vertices such that all the vertices have degree $2l$.*

This follows from Observation 4.2. Just take a new vertex and join it to all the vertices of degree $2l-1$.

Condition C2 : Let $k = 4l+1$ and $n = 2k+4m$, for some $m \geq 1$. Then any 2-coloring of $G(n,k)$ is said to satisfy **Condition C2** if the degrees of the vertices are as given below:

(a) the vertices of A have degree $4l+2m$;

(b) $2m$ vertices of B have degree $4l+2m$;

(c) the remaining $2m$ vertices of B have degree $4l+2m+1$.

Lemma 4.7 *When $k = 4l+1$ and $n = 2k+4$ there exists a minimal coloring of $G(n,k)$ satisfying Condition C2.*

Proof : Let A_1 be the subgraph induced by the vertices $v_1, v_2, \dots, v_{4l+1}$ and A_2 be the subgraph induced by the vertices $v'_1, v'_2, \dots, v'_{4l+1}$. By Observation 4.6, we construct A_1 and A_2 such that the vertices

$$v_1, v_2, \dots, v_{4l+1}$$

have degree $2l$ and the vertices

$$v'_1, v'_2, \dots, v'_{4l+1}$$

have degree $2l$. Also, to get a minimal coloring, we construct the subgraph induced by A as dictated by Observation 4.1. Thus vertices of A have degrees $4l$ in the subgraph induced by A.

Now x_1, x_2, x_3 and x_4 are the vertices of the subgraph B. Join x_1 and x_2 with

$$v_1, v'_1, v_2, v'_2, \dots, v_{2l}, v'_{2l}.$$

Join x_3 and x_4 with

$$v_{2l+1}, v'_{2l+1}, v_{2l+2}, v'_{2l+2}, \dots, v_{4l+1}, v'_{4l+1}.$$

Now join x_3 with x_1 and x_4 with x_2. Finally join x_1 and x_2.

So, each vertex in A has degree $4l+2$, the vertices x_1 and x_2 have degree $4l+2$ and the vertices x_3 and x_4 have degree $4l+3$. This construction for $G(n,k)$ satisfies $(*)_1$ and $(*)_2$ and hence this is a minimal coloring. Also this construction satisfies Condition C2. □

Lemma 4.8 *If there exists a minimal coloring of $G(n,k)$, where $k = 4l+1$ and $n = 2k+4m$ for some $m \geq 1$ satisfying Condition C2 then,*

(a) there exists a minimal coloring of $G(n,k)$ for $n = 2k+4m+1$;

(b) there exists a minimal coloring of $G(n,k)$ for $n = 2k+4m+2$;

(c) there exists a minimal coloring of $G(n,k)$ for $n = 2k+4m+3$;

(d) there exists a minimal coloring of $G(n,k)$ for $n = 2k+4m+4$ satisfying Condition C2.

Proof : The proof is similar to the proof of Lemma 4.4. This is explained in Table 2. □

Using Lemma 4.7 and Lemma 4.8 and by induction hypothesis, we get the following result.

Theorem 4.9 *When $k = 4l+1$ and $n \geq 2k+4$, equality is attained in Theorem 3.1.*

4.3 The case k ≡ 2 (mod 4)

Let A denote the set of vertices $v_1, v_1', v_2, v_2', \ldots, v_{4l+2}, v_{4l+2}'$ and B denote the vertices $x_1, x_2, \ldots x_{n-8l-4}$.

Observation 4.10 *It is possible to construct a graph on $4l+2$ vertices such that $2l$ vertices have degree $2l$ and the remaining $2l+2$ vertices have degree $2l+1$.*

This follows from Observation 4.6. Just add a new vertex and join it with $2l+1$ vertices.

Condition C3 : Let $k = 4l+2$ and $n = 2k+4m$, for some $m \geq 1$. Then any 2-coloring of $G(n,k)$ is said to satisfy **Condition C3** if the degrees of the vertices are as given below:

(a) the vertices of A have degree $4l+2m+1$;

(b) $2m$ vertices of B have degree $4l+2m+1$;

(c) the remaining $2m$ vertices of B have degree $4l+2m+2$.

Lemma 4.11 *When $k = 4l + 2$ and $n = 2k + 4$ there exists a minimal coloring of $G(n, k)$ satisfying Condition C3.*

Proof : Let A_1 be the subgraph induced by the vertices $v_1, v_2, \ldots, v_{4l+2}$ and A_2 be the subgraph induced by the vertices $v'_1, v'_2, \ldots, v'_{4l+2}$. By Observation 4.10, we construct A_1 and A_2 such that the vertices

$$v_1, v_2 \ldots v_{2l} \text{ and } v'_1, v'_2, \ldots, v'_{2l}$$

have degree $2l$ and the vertices

$$v_{2l+1}, v_{2l+2}, \ldots, v_{4l+2} \text{ and } v'_{2l+1}, v'_{2l+2}, \ldots, v'_{4l+2}$$

have degree $2l + 1$. Also, to get a minimal coloring, we construct the subgraph induced by A as dictated by Observation 4.1. Thus vertices of A have degrees $4l$ or $4l + 2$ in the subgraph induced by A.

Now x_1, x_2, x_3 and x_4 are the vertices of the subgraph B.
Join x_1, x_2 and x_3 with

$$v_1, v'_1, v_2, v'_2, \ldots, v_{2l}, v'_{2l}.$$

Join x_4 with

$$v_{2l+1}, v'_{2l+1}, v_{2l+2}, v'_{2l+2}, \ldots, v_{4l+1}, v'_{4l+1}.$$

Join x_3 with v_{4l+2} and v'_{4l+2}. Now join x_1 with x_2, x_3 and x_4. Also join x_2 with x_3 and x_4.

So, each vertex in A has degree $4l+3$, the vertices x_1 and x_2 have degree $4l + 3$ and the vertices x_3 and x_4 have degree $4l + 4$. This construction for $G(n, k)$ satisfies $(*)_1$ and $(*)_2$ and hence this is a minimal coloring. Also this construction satisfies Condition C3. □

Lemma 4.12 *If there exists a minimal coloring of $G(n, k)$, where $k = 4l+2$ and $n = 2k + 4m$ for some $m \geq 1$ satisfying Condition C3 then,*

(a) there exists a minimal coloring of $G(n, k)$ for $n = 2k + 4m + 1$;

(b) there exists a minimal coloring of $G(n, k)$ for $n = 2k + 4m + 2$;

(c) there exists a minimal coloring of $G(n, k)$ for $n = 2k + 4m + 3$;

(d) there exists a minimal coloring of $G(n, k)$ for $n = 2k+4m+4$ satisfying Condition C3.

Proof : The proof is explained in Table 3. □

Using Lemma 4.11 and Lemma 4.12 and by induction hypothesis, we get the following result.

Theorem 4.13 *When $k = 4l + 2$ and $n \geq 2k + 4$, equality is attained in Theorem 3.1.*

4.4 The case k $\equiv$ 3 (mod 4)

Let A denote the set of vertices

$$v_1, v'_1, v_2, v'_2, \ldots, v_{4l+3}, v'_{4l+3}$$

and B denote

$$x_1, x_2, \ldots x_{n-8l-6}.$$

Observation 4.14 *It is possible to construct a graph on $4l+3$ vertices such that $4l+2$ vertices have degree $2l+1$ and the remaining one vertex has degree $2l$ or $2l+2$.*

This follows from Observation 4.10. Add a new vertex and join it with the vertices of degree $2l$.

Condition C4 : Let $k = 4l+3$ and $n = 2k+4m$, for some $m \geq 1$. Then any 2-coloring of $G(n,k)$ is said to satisfy **Condition C4** if the degrees of the vertices are as given below:

(a) the vertices of A have degree $4l+2m+2$;

(b) $2m$ vertices of B have degree $4l+2m+2$;

(c) the remaining $2m$ vertices of B have degree $4l+2m+3$.

Lemma 4.15 *When $k = 4l+3$ and $n = 2k+4$ there exists a minimal coloring of $G(n,k)$ satisfying Condition C4.*

Proof : Let A_1 be the subgraph induced by the vertices $v_1, v_2, \ldots, v_{4l+3}$ and let A_2 be the subgraph induced by the vertices $v'_1, v'_2, \ldots, v'_{4l+3}$. By Observation 4.14, we construct A_1 and A_2 such that the vertices

$$v_1, v_2, \ldots, v_{4l+2} \text{ and } v'_1, v'_2 \ldots v'_{4l+2}$$

have degree $2l+1$ and the vertices v_{4l+3}, v'_{4l+3} have degree $2l$. Also, to get a minimal coloring, we construct the subgraph induced by A as dictated by Observation 4.1. Thus vertices of A have degrees $4l+2$ or $4l$ in the subgraph induced by A.

Now x_1, x_2, x_3 and x_4 are the vertices of the subgraph B. Join x_1 and x_2 with

$$v_1, v'_1, v_2, v'_2, \ldots, v_{2l+1}, v'_{2l+1}.$$

Join x_3 and x_4 with

$$v_{2l+1}, v'_{2l+1}, v_{2l+2}, v'_{2l+2}, \ldots, v_{4l+2}, v'_{4l+2}.$$

Now join x_1, x_2, x_3 and x_4 with v_{4l+3} and v'_{4l+3}. Also join x_1 with x_3, x_2 with x_4 and x_3 with x_4.

So, each vertex in A has degree $4l+4$, the vertices x_1 and x_2 have degree $4l+4$ and the vertices x_3 and x_4 have degree $4l+5$. This construction for $G(n,k)$ satisfies $(*)_1$ and $(*)_2$ and hence is a minimal coloring. Also this construction satisfies Condition C4. □

Lemma 4.16 *If there exists a minimal coloring of $G(n,k)$, where $k = 4l+3$ and $n = 2k+4m$ for some $m \geq 1$ satisfying Condition C4 then,*

(a) there exists a minimal coloring of $G(n,k)$ for $n = 2k+4m+1$;

(b) there exists a minimal coloring of $G(n,k)$ for $n = 2k+4m+2$;

(c) there exists a minimal coloring of $G(n,k)$ for $n = 2k+4m+3$;

(d) there exists a minimal coloring of $G(n,k)$ for $n = 2k+4m+4$ satisfying Condition C4.

Proof : The proof is explained in Table 4. □

Using Lemma 4.15 and Lemma 4.16 and by induction hypothesis, we get the following result.

Theorem 4.17 *When $k = 4l+3$ and $n \geq 2k+4$, equality is attained in Theorem 3.1.*

Theorem 4.18 *(cf. Goodman's result) When $n \geq 2k+4$,*

$$\begin{aligned} M(K_3, G(n,k)) &= \frac{1}{3}(u-1)(u^2-2u-3k) && \text{if } n = 2u, \\ &= \frac{1}{3}(8u^3-6u^2-2u(1+3k)+3k) && \text{if } n = 4u+1, \\ &= \frac{2}{3}(u(4u^2+3u-(3k+1)) && \text{if } n = 4u+3, \end{aligned}$$

where u is a non-negative integer.

Proof : Follows immediately by combining Theorems 4.5, 4.9, 4.13 and 4.17. □

Vertices	Degrees of Vertices when n is equal to				
	$2k+4m$	$2k+4m+1$	$2k+4m+2$	$2k+4m+3$	$2k+4m+4$
v_1	$4l+2m-1$	$4l+2m$	$4l+2m$	$4l+2m+1$	$4l+2m+1$
v'_1	$4l+2m-1$	$4l+2m$	$4l+2m$	$4l+2m+1$	$4l+2m+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_{2l}	$4l+2m-1$	$4l+2m$	$4l+2m$	$4l+2m+1$	$4l+2m+1$
v'_{2l}	$4l+2m-1$	$4l+2m$	$4l+2m$	$4l+2m+1$	$4l+2m+1$
v_{2l+1}	$4l+2m-1$	$4l+2m-1$	$4l+2m$	$4l+2m$	$4l+2m+1$
v'_{2l+1}	$4l+2m-1$	$4l+2m-1$	$4l+2m$	$4l+2m$	$4l+2m+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_{4l}	$4l+2m-1$	$4l+2m-1$	$4l+2m$	$4l+2m$	$4l+2m+1$
v'_{4l}	$4l+2m-1$	$4l+2m-1$	$4l+2m$	$4l+2m$	$4l+2m+1$
x_1	$4l+2m-1$	$4l+2m$	$4l+2m$	$4l+2m+1$	$4l+2m+2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{2m}	$4l+2m-1$	$4l+2m$	$4l+2m$	$4l+2m+1$	$4l+2m+2$
x_{2m+1}	$4l+2m$	$4l+2m$	$4l+2m+1$	$4l+2m+1$	$4l+2m+2$
x_{2m+2}	$4l+2m$	$4l+2m$	$4l+2m+1$	$4l+2m+1$	$4l+2m+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{4m}	$4l+2m$	$4l+2m$	$4l+2m+1$	$4l+2m+1$	$4l+2m+1$
x_{4m+1}	-	$4l+2m$	$4l+2m+1$	$4l+2m+1$	$4l+2m+1$
x_{4m+2}	-	-	$4l+2m+1$	$4l+2m+1$	$4l+2m+1$
x_{4m+3}	-	-	-	$4l+2m$	$4l+2m+1$
x_{4m+4}	-	-	-	-	$4l+2m+2$

Table 1 : The case k = 4l

	Degrees of Vertices when n is equal to				
Vertices	$2k+4m$	$2k+4m+1$	$2k+4m+2$	$2k+4m+3$	$2k+4m+4$
v_1	$4l+2m$	$4l+2m+1$	$4l+2m+1$	$4l+2m+2$	$4l+2m+2$
v'_1	$4l+2m$	$4l+2m+1$	$4l+2m+1$	$4l+2m+2$	$4l+2m+2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_{2l}	$4l+2m$	$4l+2m+1$	$4l+2m+1$	$4l+2m+2$	$4l+2m+2$
v'_{2l}	$4l+2m$	$4l+2m+1$	$4l+2m+1$	$4l+2m+2$	$4l+2m+2$
v_{2l+1}	$4l+2m$	$4l+2m$	$4l+2m+1$	$4l+2m+1$	$4l+2m+2$
v'_{2l+1}	$4l+2m$	$4l+2m$	$4l+2m+1$	$4l+2m+1$	$4l+2m+2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_{4l+1}	$4l+2m$	$4l+2m$	$4l+2m+1$	$4l+2m+1$	$4l+2m+2$
v'_{4l+1}	$4l+2m$	$4l+2m$	$4l+2m+1$	$4l+2m+1$	$4l+2m+2$
x_1	$4l+2m$	$4l+2m+1$	$4l+2m+1$	$4l+2m+2$	$4l+2m+3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{2m}	$4l+2m$	$4l+2m+1$	$4l+2m+1$	$4l+2m+2$	$4l+2m+3$
x_{2m+1}	$4l+2m+1$	$4l+2m+1$	$4l+2m+1$	$4l+2m+2$	$4l+2m+3$
x_{2m+2}	$4l+2m+1$	$4l+2m+1$	$4l+2m+2$	$4l+2m+2$	$4l+2m+2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{4m}	$4l+2m+1$	$4l+2m+1$	$4l+2m+2$	$4l+2m+2$	$4l+2m+2$
x_{4m+1}	-	$4l+2m$	$4l+2m+1$	$4l+2m+2$	$4l+2m+2$
x_{4m+2}	-	-	$4l+2m+2$	$4l+2m+2$	$4l+2m+2$
x_{4m+3}	-	-	-	$4l+2m+2$	$4l+2m+2$
x_{4m+4}	-	-	-	-	$4l+2m+3$

Table 2 : The case $k = 4l + 1$

Vertices	Degrees of Vertices when n is equal to				
	$2k+4m$	$2k+4m+1$	$2k+4m+2$	$2k+4m+3$	$2k+4m+4$
v_1	$4l+2m+1$	$4l+2m+2$	$4l+2m+2$	$4l+2m+3$	$4l+2m+3$
v'_1	$4l+2m+1$	$4l+2m+2$	$4l+2m+2$	$4l+2m+3$	$4l+2m+3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_{2l+1}	$4l+2m+1$	$4l+2m+2$	$4l+2m+2$	$4l+2m+3$	$4l+2m+3$
v'_{2l+1}	$4l+2m+1$	$4l+2m+2$	$4l+2m+2$	$4l+2m+3$	$4l+2m+3$
v_{2l+2}	$4l+2m+1$	$4l+2m+1$	$4l+2m+2$	$4l+2m+2$	$4l+2m+3$
v'_{2l+2}	$4l+2m+1$	$4l+2m+1$	$4l+2m+2$	$4l+2m+2$	$4l+2m+3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_{4l+2}	$4l+2m+1$	$4l+2m+1$	$4l+2m+2$	$4l+2m+2$	$4l+2m+3$
v'_{4l+2}	$4l+2m+1$	$4l+2m+1$	$4l+2m+2$	$4l+2m+2$	$4l+2m+3$
x_1	$4l+2m+1$	$4l+2m+2$	$4l+2m+2$	$4l+2m+3$	$4l+2m+3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{2m}	$4l+2m+1$	$4l+2m+2$	$4l+2m+2$	$4l+2m+3$	$4l+2m+3$
x_{2m+1}	$4l+2m+2$	$4l+2m+2$	$4l+2m+3$	$4l+2m+3$	$4l+2m+3$
x_{2m+2}	$4l+2m+2$	$4l+2m+2$	$4l+2m+3$	$4l+2m+3$	$4l+2m+4$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{4m}	$4l+2m+2$	$4l+2m+2$	$4l+2m+3$	$4l+2m+3$	$4l+2m+4$
x_{4m+1}	-	$4l+2m+2$	$4l+2m+3$	$4l+2m+3$	$4l+2m+4$
x_{4m+2}	-	-	$4l+2m+3$	$4l+2m+3$	$4l+2m+4$
x_{4m+3}	-	-	-	$4l+2m+2$	$4l+2m+3$
x_{4m+4}	-	-	-	-	$4l+2m+4$

Table 3 : The case $k = 4l + 2$

Vertices	Degrees of Vertices when n is equal to				
	$2k+4m$	$2k+4m+1$	$2k+4m+2$	$2k+4m+3$	$2k+4m+4$
v_1	$4l+2m+2$	$4l+2m+3$	$4l+2m+3$	$4l+2m+4$	$4l+2m+4$
v'_1	$4l+2m+2$	$4l+2m+3$	$4l+2m+3$	$4l+2m+4$	$4l+2m+4$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_{2l+1}	$4l+2m+2$	$4l+2m+3$	$4l+2m+3$	$4l+2m+4$	$4l+2m+4$
v'_{2l+1}	$4l+2m+2$	$4l+2m+3$	$4l+2m+3$	$4l+2m+4$	$4l+2m+4$
v_{2l+2}	$4l+2m+2$	$4l+2m+2$	$4l+2m+3$	$4l+2m+3$	$4l+2m+4$
v'_{2l+2}	$4l+2m+2$	$4l+2m+2$	$4l+2m+3$	$4l+2m+3$	$4l+2m+4$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_{4l+2}	$4l+2m+2$	$4l+2m+2$	$4l+2m+3$	$4l+2m+3$	$4l+2m+4$
v'_{4l+2}	$4l+2m+2$	$4l+2m+2$	$4l+2m+3$	$4l+2m+3$	$4l+2m+4$
v_{4l+3}	$4l+2m+2$	$4l+2m+3$	$4l+2m+3$	$4l+2m+4$	$4l+2m+4$
v'_{4l+3}	$4l+2m+2$	$4l+2m+3$	$4l+2m+3$	$4l+2m+4$	$4l+2m+4$
x_1	$4l+2m+2$	$4l+2m+3$	$4l+2m+3$	$4l+2m+4$	$4l+2m+4$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{2m-1}	$4l+2m+2$	$4l+2m+3$	$4l+2m+3$	$4l+2m+4$	$4l+2m+4$
x_{2m}	$4l+2m+2$	$4l+2m+3$	$4l+2m+4$	$4l+2m+4$	$4l+2m+4$
x_{2m+1}	$4l+2m+3$	$4l+2m+3$	$4l+2m+4$	$4l+2m+4$	$4l+2m+4$
x_{2m+2}	$4l+2m+3$	$4l+2m+3$	$4l+2m+4$	$4l+2m+4$	$4l+2m+5$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{4m}	$4l+2m+3$	$4l+2m+3$	$4l+2m+4$	$4l+2m+4$	$4l+2m+5$
x_{4m+1}	-	$4l+2m+4$	$4l+2m+4$	$4l+2m+4$	$4l+2m+5$
x_{4m+2}	-	-	$4l+2m+3$	$4l+2m+4$	$4l+2m+5$
x_{4m+3}	-	-	-	$4l+2m+4$	$4l+2m+5$
x_{4m+4}	-	-	-	-	$4l+2m+4$

Table 4 : The case $k = 4l + 3$

5 The cases $\mathbf{n = 2k+1, 2k+2}$ and $\mathbf{2k+3}$

These cases could not be contained in Section 4 as $2k$ is very close to n. **We construct only the red graph in all these cases.**

Case 1 : $n = 2k + 1$

Sub-case (a) : $k = 4l$'

In this case $n = 8l + 1$. In any minimal coloring, the vertices of A should have degree $4l - 1$ or $4l$ and the vertex of B should have degree $4l$. Also $S_3(p)$ should be $4l - 1$ for all p in A and $4l$ for all p in B. This can be achieved and hence the minimum number of monochromatic triangles is given by Theorem 3.1. A minimal construction is as below.

Construct the subgraph A in such a way that the vertices

$$v_1, v_1', v_2, v_2', \ldots, v_{2l}, v_{2l}'$$

have degree $4l$ and the vertices

$$v_{2l+1}, v_{2l+1}', v_{2l+2}, v_{2l+2}', \ldots, v_{4l}, v_{4l}'$$

have degree $4l - 2$. Now join x_1 with

$$v_{2l+1}, v_{2l+1}', v_{2l+2}, v_{2l+2}', \ldots, v_{4l}, v_{4l}'.$$

This is a minimal coloring.

Sub-case (b) : $k = 4l + 1$.

In this case $n = 8l+3$. In any minimal coloring, the vertices of A should have degree $4l$ or $4l+1$ and the vertex of B should have degree $4l$ or $4l+2$. Also $S_3(p)$ should be $4l$ for all p in A and $4l + 1$ for all p in B. This can be achieved and hence the minimum number of monochromatic triangles is given by Theorem 3.1. A minimal coloring is as below. Construct the subgraph A in such a way that all the vertices

$$v_1, v_1', v_2, v_2', \ldots, v_{4l+1}, v_{4l+1}'$$

have degree $4l$. Now join x_1 with

$$v_1, v_1', v_2, v_2', \ldots, v_{2l}, v_{2l}'.$$

This is a minimal coloring.

Sub-case (c): $k = 4l + 2$.

In this case $n = 8l+5$. In any minimal coloring, the vertices of A should have degree $4l+2$ or $4l+1$ and the vertex of B should have degree $4l+2$. Also $S_3(p)$ should be $4l+1$, for all p in A, and $4l+2$, for all p in B. This can not be achieved and the reason is as follows. Suppose the vertex x_1 of B is to be joined to $4l+2$ vertices, say

$$v_1, v_1', v_2, v_2', \ldots, v_{2l+1}, v_{2l+1}'.$$

In the induced subgraph of A all vertices should have even degree by Observation 4.1. Hence the vertices

$$v_1, v_1', v_2, v_2', \ldots, v_{2l+1}, v_{2l+1}'$$

have degree $4l$ and the remaining $4l+2$ vertices of A have degree $4l+2$ in the induced subgraph of A. But, this is not possible since the number of edges in A should be a multiple of 4.

Hence the next possible minimum number of monochromatic triangles is obtained as follows. The vertex of B will have degree $4l$ or $4l+4$.

$$\begin{aligned} \text{Now } 6|S_1| &\geq (8l+4)\left\{2\binom{4l+1}{2} + 2\binom{4l+2}{2} - (4l+1)(4l+2)\right\} \\ &\quad +1\left\{2\binom{4l}{2} + 2\binom{4l+4}{2} - (4l)(4l+4)\right\} \\ &\quad -2((8l+4)(4l+1)+1(4l+2)). \\ \text{Thus, } |S_1| &\geq \frac{1}{3}(64l^3+24l^2-16l). \end{aligned}$$

A minimal coloring is given as follows. Color the edges of A in such a way that $4l$ vertices, say

$$v_1, v_1', v_2, v_2', \ldots, v_{2l}, v_{2l}'$$

have degree $4l$ and

$$v_{2l+1}, v_{2l+1}', v_{2l+2}, v_{2l+2}', \ldots, v_{4l+2}, v_{4l+2}'$$

have degree $4l+2$. Now join x_1 with $v_1, v_1', v_2, v_2', \ldots, v_{2l}, v_{2l}'$. This is a minimal coloring.

Sub-case (d) : $k = 4l + 3$.

In this case $n = 8l+7$. In any minimal coloring, the vertices of A should have degree $4l+2$ or $4l+3$ and the vertex of B should have degree $4l+2$

or $4l+4$. Also $S_3(p)$ should be $4l+2$ for all p in A and $4l+3$ for all p in B. This can not be achieved because the vertex x_1 of B has to be joined with $4l+2$ or $4l+4$ vertices of A and so all the vertices of A should have degree $4l+2$ in the induced subgraph of A. This implies that the number of edges in A is not a multiple of 4. Hence the next possible minimum number of monochromatic triangles is obtained as follows. The vertex of B will have degree $4l+2$ or $4l+4$ and two vertices of A will have degree $4l+1$ or $4l+4$.

$$\begin{aligned}
\text{Now } 6|S_1| \geq\ & (8l+4)\left\{2\binom{4l+2}{2} + 2\binom{4l+3}{2} - (4l+2)(4l+3)\right\} \\
& +2\left\{2\binom{4l+1}{2} + 2\binom{4l+4}{2} - (4l+1)(4l+4)\right\} \\
& +1\left\{2\binom{4l+2}{2} + 2\binom{4l+4}{2} - (4l+2)(4l+4)\right\} \\
& -2((8l+6)(4l+2) + 1(4l+3)).
\end{aligned}$$

$$\text{Thus, } |S_1| \geq \frac{1}{3}(64l^3 + 72l^2 + 8l).$$

A minimal coloring is given as below. Color the edges of A in such a way that $8l+4$ vertices say

$$v_1, v_1', v_2, v_2', \ldots, v_{4l+2}, v_{4l+2}'$$

have degree $4l+2$ and v_{4l+3}, v_{4l+3}' have degree $4l$.
Now join x_1 with

$$v_1, v_1', v_2, v_2', \ldots, v_{2l}, v_{2l}', v_{4l+3}, v_{4l+3}'.$$

This is a minimal coloring.

Case 2 : $n = 2k+2$.

Elaborate explanations in most cases are omitted as these are very similar to the situation $n = 2k+1$ handled earlier.

Sub-case (a) : $k = 4l$.

In this case $n = 8l+2$. In any minimal coloring the vertices of A should have degree $4l$ and the vertices of B should have degree $4l$ or $4l+1$. Also $S_3(p)$ should be $4l-1$ for all p in A and $4l$ for all p in B. This can be achieved and hence the minimum number of monochromatic triangles is given by Theorem 3.1. A minimal coloring is as below.

Construct the subgraph A in such a way that the vertices

$$v_1, v'_1, v_2, v'_2, \ldots, v_{2l}, v'_{2l}$$

have degree $4l$ and the vertices

$$v_{2l+1}, v'_{2l+1}, v_{2l+2}, v'_{2l+2}, \ldots, v_{4l}, v'_{4l}$$

have degree $4l-2$. Join the vertices x_1 and x_2 with

$$v_{2l+1}, v'_{2l+1}, v_{2l+2}, v'_{2l+2}, \ldots, v_{4l}, v'_{4l}.$$

This is a minimal coloring.

Sub-case (b) : $k = 4l+1$.

In this case $n = 8l+4$. In any minimal coloring the vertices of A should have degree $4l+1$ and the vertices of B should have degree $4l+1$ or $4l+2$. Also $S_3(p)$ should be $4l$ for all p in A and $4l+1$ for all p in B. This can not be achieved and the next possible minimum number of monochromatic triangles is obtained in the following manner. One vertex of B will have degree $4l$ and the other will have $4l+2$.

$$\begin{aligned}
\text{Now } 6|S_1| \geq\ & (8l+2)\left\{2\binom{4l+1}{2} + 2\binom{4l+1}{2} - (4l+1)^2)\right\} \\
& +1\left\{2\binom{4l+2}{2} + 2\binom{4l+1}{2} - (4l+2)(4l+1)\right\} \\
& +1\left\{2\binom{4l}{2} + 2\binom{4l+3}{2} - (4l)(4l+3)\right\} \\
& -2((8l+2)(4l) + 2(4l+1)).
\end{aligned}$$

$$\text{Thus, } |S_1| \geq \frac{16}{3} l(4l^2-1).$$

Construct the subgraph A in such a way that the vertices

$$v_1, v'_1, v_2, v'_2, \ldots, v_{4l+1}, v'_{4l+1}$$

have degree $4l$. Join the vertex x_1 with the vertices

$$v_1, v'_1, v_2, v'_2, \ldots, v_{2l}, v'_{2l}$$

and the vertex x_2 with

$$v_{2l+1}, v'_{2l+1}, v_{2l+2}, v'_{2l+2}, \ldots, v_{4l+1}, v_{4l+1}.$$

This is a minimal coloring.

Sub-case (c) : $k = 4l + 2$

In this case $n = 8l + 6$. In any minimal coloring the vertices of A should have degree $4l + 2$ and the vertices of B should have degree $4l + 2$ or $4l + 3$. Also $S_3(p)$ should be $4l + 1$ for all p in A and $4l + 2$ for all p in B. This can not be achieved and the next possible minimum number of monochromatic triangles is obtained as follows. Both the vertices of B will have degree $4l + 1$ or $4l + 4$.

$$\begin{aligned}\text{Now } 6|S_1| \geq\ & (8l+4)\left\{2\binom{4l+2}{2} + 2\binom{4l+2}{2} - (4l+2)^2\right\} \\ & +2\left\{2\binom{4l+1}{2} + 2\binom{4l+4}{2} - (4l+1)(4l+4)\right\} \\ & -2((8l+4)(4l+1) + 2(4l+2)). \\ \text{Thus, } |S_1| \geq\ & \frac{1}{3}(64l^3 + 48l^2 - 4l).\end{aligned}$$

A minimal coloring is given as below. Color the edges of A in such a way that $4l$ vertices say

$$v_1, v_1', v_2, v_2', \ldots, v_{2l}, v_{2l}'$$

have degree $4l + 2$ and

$$v_{2l+1}, v_{2l+1}', v_{2l+2}, v_{2l+2}', \ldots, v_{4l+2}, v_{4l+2}'$$

have degree $4l$. Now join x_1 and x_2 with

$$v_{2l+1}, v_{2l+1}', v_{2l+2}, v_{2l+2}', \ldots, v_{4l+2}, v_{4l+2}'.$$

Sub-case (d) : $k = 4l + 3$.

In this case $n = 8l + 8$. In any minimal coloring the vertices of A should have degree $4l + 3$ and the vertices of B should have degree $4l + 3$ or $4l + 4$. Also $S_3(p)$ should be $4l + 2$ for all p in A and $4l + 3$ for all p in B. This can not be achieved and the next possible minimum number of monochromatic triangles is obtained as follows. Two vertices of A will have degree $4l + 2$ or $4l + 4$.

$$
\begin{aligned}
\text{Now } 6|S_1| \geq\ & (8l+4)\left\{2\binom{4l+3}{2} + 2\binom{4l+3}{2} - (4l+3)^2\right\} \\
& +2\left\{2\binom{4l+2}{2} + 2\binom{4l+4}{2} - (4l+2)(4l+4)\right\} \\
& +2\left\{2\binom{4l+3}{2} + 2\binom{4l+4}{2} - (4l+3)(4l+4)\right\} \\
& -2((8l+6)(4l+2) + 2(4l+3)).
\end{aligned}
$$

$$
\text{Thus, } |S_1| \geq \frac{1}{3}(64l^3 + 96l^2 + 32l).
$$

A minimal coloring is given as below. Color the edges of A in such a way that $8l+4$ vertices say

$$v_1, v_1', v_2, v_2', \ldots, v_{4l+2}, v_{4l+2}'$$

have degree $4l+2$ and v_{4l+3}, v_{4l+3}' have degree $4l$. Now join x_1 with

$$v_1, v_1', v_2, v_2', \ldots, v_{2l+1}, v_{2l+1}'$$

and with v_{4l+3} and v_{4l+3}' and join x_2 with

$$v_{2l+2}, v_{2l+2}', v_{2l+3}, v_{2l+3}', \ldots, v_{4l+3}, v_{4l+3}'.$$

This is a minimal coloring.

Case 3 : $n = 2k+3$.

Sub-case (a) : $k = 4l$.

In this case $n = 8l+3$. In any minimal coloring the vertices of A should have degree $4l$ or $4l+1$, two vertices of B should have degree $4l+1$ and one vertex should have degree $4l$ or $4l+2$. Also $S_3(p)$ should be $4l-1$ for all p in A and $4l$ for all p in B. This can be achieved and hence the minimum number of monochromatic triangles is given by Theorem 3.1. A minimal coloring is as below.

Construct the subgraph A in such a way that the vertices

$$v_1, v_1', v_2, v_2', \ldots, v_{2l}, v_{2l}'$$

have degree $4l-2$ and the vertices

$$v_{2l+1}, v_{2l+1}', v_{2l+2}, v_{2l+2}', \ldots, v_{4l}, v_{4l}'$$

have degree $4l$. Now join x_1 and x_2 with

$$v_1, v_1', v_2, v_2', \ldots, v_{2l}, v_{2l}'$$

and x_3 with

$$v_{2l+1}, v_{2l+1}', v_{2l+2}, v_{2l+2}', \ldots, v_{4l}, v_{4l}'.$$

Join x_1 and x_2. This is a minimal coloring.

Sub-case (b) : $k = 4l + 1$.

In this case $n = 8l + 5$. In any minimal coloring the vertices of A should have degree $4l + 1$ or $4l + 2$ and the vertices of B should have degree $4l + 2$. Also $S_3(p)$ should be $4l$ for all p in A and $4l + 1$ for all p in B. This can not be achieved and the next possible minimum number of monochromatic triangles is got as follows. Two vertices of B will have degree $4l + 1$ and the other will have $4l + 2$.

$$\begin{aligned}
\text{Now } 6|S_1| \geq\ & (8l+2)\left\{2\binom{4l+1}{2} + 2\binom{4l+2}{2} - (4l+1)(4l+2))\right\} \\
& +2\left\{2\binom{4l+1}{2} + 2\binom{4l+3}{2} - (4l+1)(4l+3)\right\} \\
& +1\left\{2\binom{4l+2}{2} + 2\binom{4l+2}{2} - (4l+2)^2\right\} \\
& -2((8l+2)(4l) + 3(4l+1)).
\end{aligned}$$

$$\text{Thus, } |S_1| \geq \frac{2}{3}l(32l^2 + 12l - 2).$$

Construct the subgraph A in such a way that all the vertices have degree $4l$. Join the vertices x_1 and x_2 with the vertices

$$v_1, v_1', v_2, v_2', \ldots, v_{2l}, v_{2l}'$$

and x_3 with

$$v_{2l+1}, v_{2l+1}', v_{2l+2}, v_{2l+2}', \ldots, v_{4l+1}, v_{4l+1}'.$$

Finally join x_1 with x_2. This is a minimal coloring.

Sub-case (c) : $k = 4l + 2$

In this case $n = 8l + 7$. In any minimal coloring the vertices of A should have degree $4l + 2$ or $4l + 3$ and the vertices of B should have degree $4l + 3$. Also $S_3(p)$ should be $4l + 1$ for all p in A and $4l + 2$ for all p in B. This can be achieved and hence the minimum number of monochromatic triangles is given by Theorem 3.1. A minimal coloring is as below.

Color the edges of A in such a way that $4l$ vertices, say

$$v_1, v_1', v_2, v_2', \ldots, v_{2l}, v_{2l}'$$

have degree $4l+2$ and

$$v_{2l+1}, v_{2l+1}', v_{2l+2}, v_{2l+2}', \ldots, v_{4l+2}, v_{4l+2}'$$

have degree $4l$. Now join x_1 and x_2 with

$$v_{2l+1}, v_{2l+1}', v_{2l+2}, v_{2l+2}', \ldots, v_{4l}, v_{4l}'.$$

Also join x_1 with v_{4l+1}, v_{4l+1}' and x_2 with v_{4l+2}, v_{4l+2}'. Also join x_3 with

$$v_{2l+1}, v_{2l+1}', v_{2l+2}, v_{2l+2}', \ldots, v_{4l+2}, v_{4l+2}'.$$

Finally join x_1 and x_2 also. This is a minimal coloring.

Sub-case (d) : $k = 4l+3$

In this case $n = 8l+9$. In any minimal coloring the vertices of A should have degree $4l+3$ or $4l+4$ and the vertices of B should have degree $4l+4$ Also $S_3(p)$ should be $4l+2$ for all p in A and $4l+3$ for all p in B. This can be achieved and hence the minimum number of monochromatic triangles is given by Theorem 3.1. The minimal coloring is as below.

Color the edges of A in such a way that $4l+4$ vertices say

$$v_1, v_1', v_2, v_2', \ldots, v_{4l+2}, v_{4l+2}'$$

have degree $4l+2$ and v_{4l+3}, v_{4l+3}' have degree $4l$. Now join x_1 with

$$v_1, v_1', v_2, v_2', \ldots, v_{2l+1}, v_{2l+1}'$$

and with x_{4l+3} and x_{4l+3}' and join x_2 and x_3 with

$$v_{2l+2}, v_{2l+2}', v_{2l+3}, v_{2l+3}', \ldots, v_{4l+3}, v_{4l+3}'.$$

This is a minimal coloring.

Next we summarise the results obtained in this section.

Theorem 5.1 *Following is the summary of the results obtained in all the above cases.*

(i) *When* $n = 2k + 1$,

$$
\begin{aligned}
M(K_3, G(n,k)) &= \tfrac{1}{3}(64l^3 - 72l^2 + 8l) && \text{if } k = 4l,\\
&= \tfrac{1}{3}(64l^3 - 24l^2 - 16l) && \text{if } k = 4l + 1,\\
&= \tfrac{1}{3}(64l^3 + 24l^2 - 16l) && \text{if } k = 4l + 2,\\
&= \tfrac{1}{3}(64l^3 + 72l^2 + 8l) && \text{if } k = 4l + 3.
\end{aligned}
$$

(ii) *When* $n = 2k + 2$,

$$
\begin{aligned}
M(K_3, G(n,k)) &= \tfrac{1}{3}(64l^3 - 48l^2 - 4l) && \text{if } k = 4l,\\
&= \tfrac{1}{3}(64l^3 - 16l) && \text{if } k = 4l + 1,\\
&= \tfrac{1}{3}(64l^3 + 48l^2 - 4l) && \text{if } k = 4l + 2,\\
&= \tfrac{1}{3}(64l^3 + 96l^2 + 32l) && \text{if } k = 4l + 3.
\end{aligned}
$$

(iii) *When* $n = 2k + 3$,

$$
\begin{aligned}
M(K_3, G(n,k)) &= \tfrac{1}{3}(64l^3 - 24l^2 - 4l) && \text{if } k = 4l,\\
&= \tfrac{1}{3}(64l^3 + 24l^2 - 4l) && \text{if } k = 4l + 1,\\
&= \tfrac{1}{3}(64l^3 - 8l^2 - 20l) && \text{if } k = 4l + 2,\\
&= \tfrac{1}{3}(64l^3 + 120l^2 + 68l + 9) && \text{if } k = 4l + 3.
\end{aligned}
$$

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