# SPECIAL FACE NUMBERING OF PLANE QUARTIC GRAPHS

MARTIN BAČA

ABSTRACT. In this paper we concentrate on those graphs which are $(a,d)$-face antimagic, and we show that the graphs $D_n$ from a special class of convex polytopes consisting of $4-$ *sided* faces are $(6n+3,2)$-face antimagic and $(4n+4,4)$-face antimagic. It is worth a conjecture, we feel, that $D_n$ are $(2n+5,6)$-face antimagic.

## 1. INTRODUCTION AND NOTATIONS

We shall be dealing with simple undirected connected plane graphs (no loops or multiple edges). The vertex (respectively edge) set of a graph $G$ will be denoted $V(G)$ (respectively $E(G)$). A graph is said to be plane if it is drawn on the Euclidean plane such that edges do not cross each other except at vertices of the graph. For a plane graph $G=(V,E,F)$, it makes sense to determine its faces, including the unique face of infinite area. Let $F(G)$ be the face set and $|F(G)|$ be the number of faces of $G$.

Hartsfield and Ringel [6] introduced the concept of an antimagic graph. Bodendiek and Walther defined (see [2,4]) the concept of an $(a,d)$-antimagic graph as a special antimagic graph. They present [3] that the theory of linear Diophantine equations and other concepts of number theory can be applied to determine the set of all connected $(a,d)$-antimagic graphs. $(a,d)$-antimagic labelings of the special graphs called parachutes are described in [4,5]. In [1] are characterized all $(a,d)$-antimagic graphs of prisms $D_n$ when $n$ is even and it is shown that if $n$ is odd the prisms $D_n$ are $(\frac{5n+5}{2},2)$-antimagic.

Now, let us define the weight of a face and $(a,d)$-face antimagic labeling of the plane graph $G=(V,E,F)$. The weight $w(f)$ of a face $f\in F(G)$ under a edge labeling $g:E\to\{1,2,...,|E|\}$ is the sum of the labels of edges surrounding that face.

A connected plane graph $G=(V,E,F)$ is said to be $(a,d)$-face antimagic if there exist positive integers $a,d\in N$ and bijection $g:E(G)\to\{1,2,...,|E(G)|\}$ such that the induced mapping $\delta_g:F(G)\to W$ is also a bijection, where $W=\{w(f):f\in F(G)\}=\{a,a+d,a+2d,...,a+(|F(G)|-1)d\}$ is the set of weights of faces.

If $G=(V,E,F)$ is $(a,d)$-face antimagic and $g:E(G)\to\{1,2,...,|E(G)|\}$ is a corresponding bijective mapping of $G$ then $g$ is said to be an $(a,d)$-face antimagic labeling of $G$. The $(a,d)$-face antimagic graph is modification of $(a,d)$-antimagic graph.

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Let $I = \{1,2,3,...,n\}$ and $J = \{1,2\}$ be index set and $D_n$ be the graph of a prism. The prism $D_n, n \geq 3$, is a trivalent graph which can be defined as the cartesian product $P_2 \times C_n$ of a path on two vertices with a cycle on $n$ vertices, embedded in the plane. Let us denote the vertex set of $D_n$ by $V(D_n) = \{x_{j,i} : j \in J \text{ and } i \in I\}$ and edge set by $E(D_n) = \{x_{j,i}x_{j,i+1} : j \in J \text{ and } i \in I\} \cup \{x_{1,i}x_{2,i} : i \in I\}$. We make the convention that $x_{j,n+1} = x_{j,1}$ and $x_{j,n+2} = x_{j,2}$ for $j \in J$.

The face set $F(D_n)$ contains $n$ $4-sided$ faces and two $n-sided$ faces (internal and external). We insert exactly one vertex $y$ ($z$) into the internal (external) $n-sided$ face of $D_n$. Suppose that $n$ is even, $n \geq 4$, and consider the graph $\mathbb{D}_n$ with the vertex set $V(\mathbb{D}_n) = V(D_n) \cup \{y,z\}$ and the edge set $E(\mathbb{D}_n) = E(D_n) \cup \{x_{1,2k-1}y : k = 1,2,...,\frac{n}{2}\} \cup \{x_{2,2k}z : k = 1,2,...,\frac{n}{2}\}$. The $\mathbb{D}_n, n \geq 4$, is the plane graph on $|V(\mathbb{D}_n)| = 2n+2$ vertices, $|E(\mathbb{D}_n)| = 4n$ edges and consisting of $|F(\mathbb{D}_n)| = 2n$ faces. Let its vertices be labeled as in Figure 1.

In this paper, we prove that if $n$ is even, $n \geq 4$, then the plane graph $\mathbb{D}_n$ is $(6n+3,2)$-face antimagic and $(4n+4,4)$-face antimagic. A conjecture, that $\mathbb{D}_n$ is $(2n+5,6)$-face antimagic, is proposed at the end of the paper.

## 2. Linear Diophantine equation

Assume that $\mathbb{D}_n$, $n \geq 4$, is $(a,d)$-face antimagic and $W = \{w(f) : f \in F(\mathbb{D}_n)\} = \{a, a+d, a+2d, ..., a+(2n-1)d\}$ is the set of weights of faces. Clearly, the sum of weights in the set $W$ is $2na + dn(2n-1)$. Since the edges of $\mathbb{D}_n$ are labeled by the set of integers $\{1,2,...,4n\}$ and since each of these labels is used twice in the computation of the weights of faces, the sum of all the edge labels used to calculate the weights of faces is equal to $4n(1+4n)$.
Thus the following equation holds

$$4(1+4n) = 2a + d(2n-1).$$

The equation is a linear Diophantine equation.

By putting $a \geq 10$ ($a = 10$ is the minimal value of weight which can be assigned to a $4-sided$ face) we get the upper bound on the value $d$, i.e. $0 < d < 8$. Since $n$ is even then it follows from linear Diophantine equation that $d$ is even. This implies that the Diophantine equation has exactly the three different solutions $(a,d) = (6n+3,2)$ or $(4n+4,4)$ or $(2n+5,6)$, respectively.

## 3. Face antimagic labelings

Define the edge labelings $g_1$ and $g_2$ of a plane graph $\mathbb{D}_n$ as follows.

$$g_1(x_{1,i}x_{2,i}) = 2n + i, \qquad g_2(x_{1,i}x_{2,i}) = 2i - 1,$$

$$g_1(x_{1,i}y) = (3n+i)\sigma(i), \qquad g_2(x_{1,i}y) = (2n+2i-1)\sigma(i),$$

$$g_1(x_{2,i}z) = (3n+i)\sigma(i+1), \qquad g_2(x_{2,i}z) = (2n+2i-1)\sigma(i+1),$$

where $i \in I$ and

$$\sigma(t) = \begin{cases} 1 & \text{if } t \equiv 1 \pmod 2, \\ 0 & \text{if } t \equiv 0 \pmod 2. \end{cases}$$

If $n \equiv 2 \pmod 4$, then

$$g_1(x_{1,i}x_{1,i+1}) = \begin{cases} i & \text{if } i \in I \text{ is even}, \\ \frac{n}{2} & \text{if } i = \frac{n}{2}, \\ 2n - i & \text{if } i \in I - \{\frac{n}{2}\} \text{ is odd}. \end{cases}$$

$$g_2(x_{1,i}x_{1,i+1}) = \begin{cases} 2i & \text{if } i \in I \text{ is even}, \\ 4n - 2i & \text{if } i \in I - \{\frac{n}{2}\} \text{ is odd}, \\ n & \text{if } i = \frac{n}{2}. \end{cases}$$

$$g_1(x_{2,i}x_{2,i+1}) = \begin{cases} 2n - i & \text{if } i \in I - \{n\} \text{ is even}, \\ 2n & \text{if } i = n, \\ i & \text{if } i \in I - \{\frac{n}{2}\} \text{ is odd}, \\ \frac{3n}{2} & \text{if } i = \frac{n}{2}. \end{cases}$$

$$g_2(x_{2,i}x_{2,i+1}) = \begin{cases} 4n - 2i & \text{if } i \in I - \{n\} \text{ is even}, \\ 4n & \text{if } i = n, \\ 2i & \text{if } i \in I - \{\frac{n}{2}\} \text{ is odd}, \\ 3n & \text{if } i = \frac{n}{2}. \end{cases}$$

If $n \equiv 0 \pmod 4$, then

$$g_1(x_{1,i}x_{1,i+1}) = \begin{cases} i & \text{if } i \in I - \{\frac{n}{2}\} \text{ is even}, \\ \frac{3n}{2} & \text{if } i = \frac{n}{2}, \\ 2n - i & \text{if } i \in I \text{ is odd}. \end{cases}$$

$$g_2(x_{1,i}x_{1,i+1}) = \begin{cases} 2i & \text{if } i \in I - \{\frac{n}{2}\} \text{ is even}, \\ 3n & \text{if } i = \frac{n}{2}, \\ 4n - 2i & \text{if } i \in I \text{ is odd}. \end{cases}$$

$$g_1(x_{2,i}x_{2,i+1}) = \begin{cases} 2n-i & \text{if } i \in I - \{\frac{n}{2}, n\} \text{ is even,} \\ \frac{n}{2} & \text{if } i = \frac{n}{2}, \\ 2n & \text{if } i = n, \\ i & \text{if } i \in I \text{ is odd.} \end{cases}$$

$$g_2(x_{2,i}x_{2,i+1}) = \begin{cases} 4n-2i & \text{if } i \in I - \{\frac{n}{2}, n\} \text{ is even,} \\ n & \text{if } i = \frac{n}{2}, \\ 4n & \text{if } i = n, \\ 2i & \text{if } i \in I \text{ is odd.} \end{cases}$$

Let us denote the weights of $4-$ *sided* faces of $\mathbb{D}_n$ (under an edge labeling $g$) by

$w_i^1 = g(x_{1,i}x_{1,i+1}) + g(x_{2,i}x_{2,i+1}) + g(x_{1,i}x_{2,i}) + g(x_{1,i+1}x_{2,i+1})$ if $i \in I$,

$w_i^2 = g(x_{1,i}x_{1,i+1}) + g(x_{1,i+1}x_{1,i+2}) + g(x_{1,i}y) + g(x_{1,i+2}y)$ if $i \in I$ is odd and

$w_i^3 = g(x_{2,i}x_{2,i+1}) + g(x_{2,i+1}x_{2,i+2}) + g(x_{2,i}z) + g(x_{2,i+2}z)$ if $i \in I$ is even.

## 4. The results

**Theorem 1.** *For $n \geq 4, n \equiv 0 \pmod 2$, the plane graph $\mathbb{D}_n$ has a $(6n+3, 2)$-face antimagic labeling.*

*Proof.* Label the edges of $\mathbb{D}_n$ by the edge labeling $g_1$. It is a matter of routine checking to see that the edge labeling $g_1$ uses each integer $1, 2, ..., 4n$ exactly once and this implies that the labeling $g_1$ is the bijection from the edge set $E(\mathbb{D}_n)$ onto the set $\{1, 2, ..., 4n\}$. Under the edge labeling $g_1$ the weights of all $4-$ *sided* faces constitute set

$W = \{w_i^1 : i \in I\} \cup \{w_i^2 : i \in I \textit{ is odd}\} \cup \{w_i^3 : i \in I \textit{ is even}\} = \{6n+1+2i : i = 1, 2, ..., 2n\}$.

We can see that each $4-$ *sided* face of $\mathbb{D}_n$ receives exactly one label of weight from $W$ and each number from $W$ is used exactly once as a label of weight. This proves that $g_1$ is $(6n+3, 2)$-face antimagic labeling.

**Theorem 2.** *If $n$ is even, $n \geq 4$, then the plane graph $\mathbb{D}_n$ has a $(4n+4, 4)$-face antimagic labeling.*

*Proof.* Label the edges of $\mathbb{D}_n$ by the edge labeling $g_2$. It is not difficult to check that the values of $g_2$ are $1, 2, ..., 4n$. By direct computation we obtain that the weights of $4-$ *sided* faces constitute the sets $W_1$ and $W_2$:

$W_1 = \{w_i^1 : i \in I\} = \{4n+4i : i \in I\}$,

$$W_2 = \{w_i^2 : i \in I \text{ is odd}\} \cup \{w_i^3 : i \in I \text{ is even}\} = \{8n + 4i : i \in I\}.$$

We see that the set $W_1 \cup W_2 = \{a, a+d, a+2d, ..., a+(2n-1)d\}$, where $a = 4n+4$ and $d = 4$, is the set of weights of all $4-$*sided* faces of $\mathbb{D}_n$ and it can be seen that the induced mapping $\delta_{g_2} : F(\mathbb{D}_n) \to W_1 \cup W_2$ is bijective.

To characterization of $(a,d)$-face antimagic graphs of $\mathbb{D}_n$ it remains to show that $\mathbb{D}_n$ are $(2n+5, 6)$-face antimagic. We know $(13, 6)$-face antimagic labeling for $\mathbb{D}_4$ (Figure 2) and $(17, 6)$-face antimagic labeling for $\mathbb{D}_6$ (Figure 3). This prompts us to propose the following:

**Conjecture.** *If* $n \equiv 0 \pmod 2, n \geq 4$, *then the plane graph* $\mathbb{D}_n$ *is* $(2n+5, 6)$*-face antimagic.*

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DEPARTMENT OF MATHEMATICS
TECHNICAL UNIVERSITY, LETNÁ 9, 042 00 KOŠICE, SLOVAKIA
*E-mail address*: hollbaca@ccsun.tuke.sk

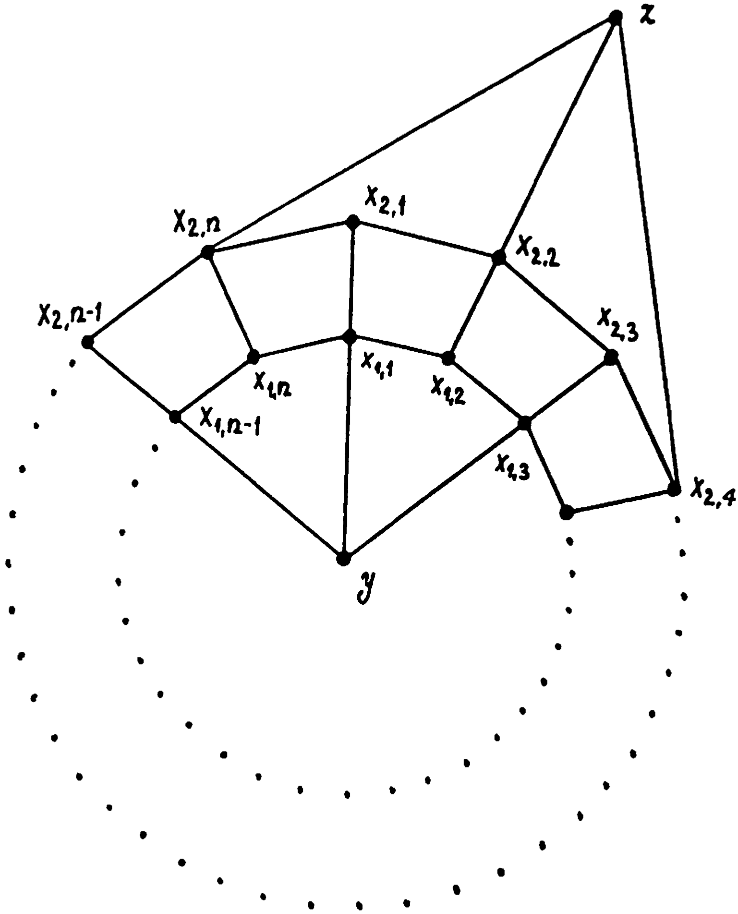


Figure 1

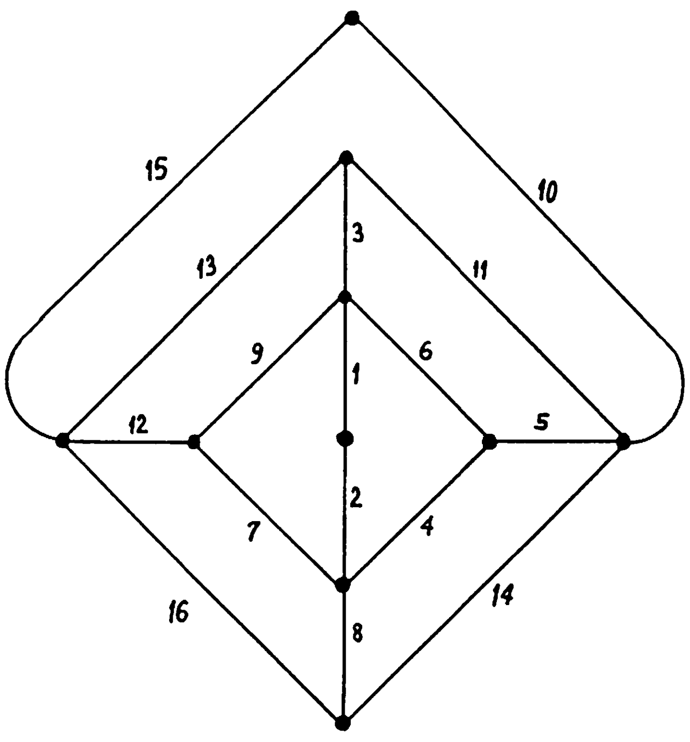


Figure 2

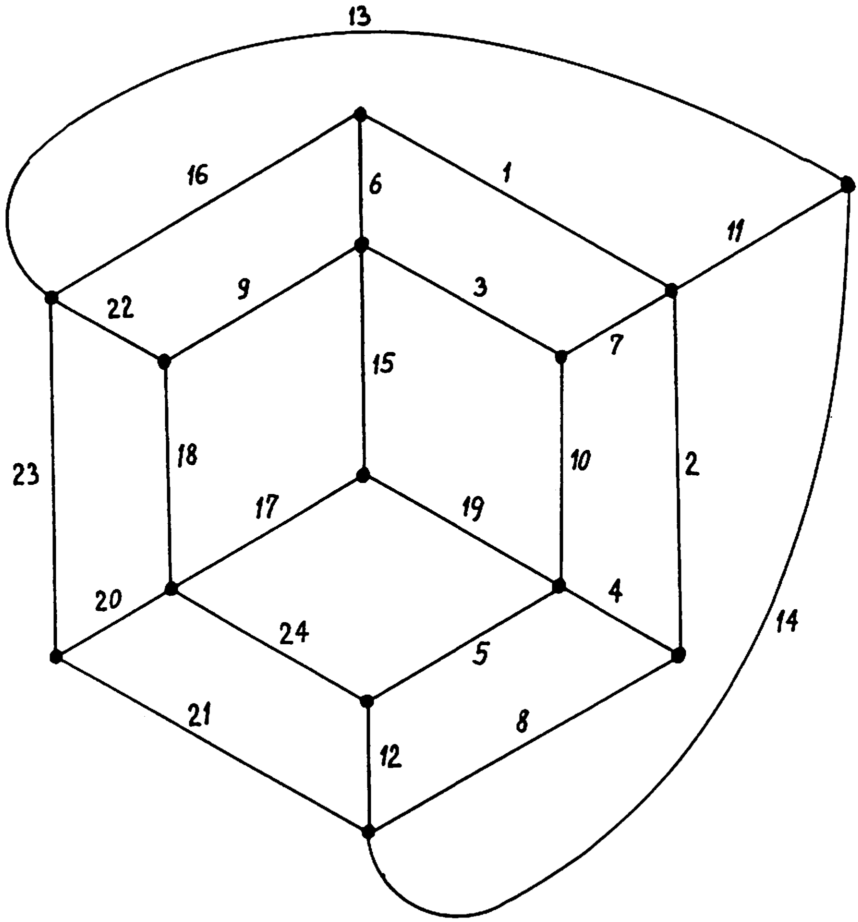


Figure 3