On $(351, 126, 45)$-Difference Sets[1]

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Abstract

In this paper, it is proved that an abelian $(351, 126, 45)$–difference set only exists in the groups with exponent 39. This fills two missing entries in Lopez and Sanchez's table with answer "no". Furthermore, if a Spence difference set D has Character Divisibility Property, then D is one of the difference sets constructed by Spence.

1. Introduction

Let G be a multiplicative group of order v, a k-element subset D of G is called a (v, k, λ)-difference set if each nonidentity element of G can be written as $d_i d_j^{-1}$, with $d_i, d_j \in D$ and $d_i \neq d_j$, in exactly λ different ways. Using the notation of the group ring $\mathbf{Z}[G]$, D is a difference set iff $D = \sum_{d \in D} d$ as an element of $\mathbf{Z}[G]$, satisfies the equation:

$$DD^{(-1)} = n + \lambda G$$

where $D^{(-1)} = \sum_{d \in D} d^{-1}$ and $n = k - \lambda$ is the order of D. we refer the reader to Jungnickel[1] and Jungnickel & Schmidt[2] for more about difference sets.

The question that was mostly considered in the study of difference sets is: given a parameter family seires (v, k, λ), which group can contain difference sets with these parameters. Most results on this question were got by using character theory and algebric number theory, and self-conjugacy condition plays an important role in the study. An integer m is self-conjugate modulo w if for each prime divisor p of m there exists a nonnegative integer j, such that $p^j \equiv -1 \pmod{w_p}$, where w_p is the largest divisor of w such that w_p primes to p .

Spence[6] has given a construction for a family of difference set with the parameters (v, k, λ, n) equal to $(3^{d+1}(3^{d+1} - 1)/2, 3^d(3^{d+1} + 1)/2, 3^d(3^d +$

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$1)/2, 3^{2d}$). The construction is given for the abelian groups with elementary abelian Sylow 3-subgroup. For an abelian group G, when the condition "3 is self-conjugate modulo v^*, where v^* is the exponent of G" holds, based on a theorem of Turyn[7], it is easy to know that if G admits a Spence difference set in it, the Sylow 3-subgroup of G must be elementary abelian. But for the cases where the self-conjugacy condition does not hold, there is still no result on it.

Recently, Ma[4] and Schmidt[8] have developed some tools to deal with the cases where the self-conjugacy condition does not hold. Following the method of Ma[4], in this paper, we shall investigate the groups which contain $(351, 126, 45)$-difference set. Our main result is that such a group must have exponent 39. We thus fill two missing entries in Lopez and Sanchez's table with answer "no". Furthermore, if a Spence difference set D in G has Character Divisibility Property(if D is a difference set with the order $n = u^2$, D is said to have the Character Divisibility Property if for each nontrivial character χ, $\chi(D)$ is divisible by u), then the Sylow 3–subgroup of G must be an elementary abelian group, and the difference set just is one of the difference sets constructed by Spence.

2. Preliminaries

In this section, we will state some lemmas about the 'structure' of an algebric integer with given modulo and how we can determine the structure of $y \in \mathbf{Z}[G]$ from the values of $\chi(y)$ for some character of G.

Throughout this paper, we use ξ_w to denote the complex wth root of unity $e^{2\pi i/w}$ and use D_w to denote the ring of algebric integers $\mathbf{Z}[\xi_w]$. αD_w denotes the ideal that α generates in D_w.

Lemma 2.1 *Let α be an algebric integer in D_q, such that $\alpha\overline{\alpha} = p^s$, where p and q are distinct odd primes. If f is the order of p modulo q, and f is odd, then*

$$\alpha = \xi_q^m \sum_{i=1}^{e} a_i S_i,$$

where m is a nonnegative integer, e is the number of cosets of $\langle p\rangle$ in $\mathbf{Z}_q^*$, $S_i = \sum_{t\in T_i} \xi_q^t$, $(i = 1, 2, \cdots, e)$, where T_i is the cosets of $\langle p\rangle$ in $\mathbf{Z}_q^*$.

proof: Since $\alpha\overline{\alpha} = p^s$, then $(\alpha D_q)(\overline{\alpha} D_q) = (pD_q)^s$. Let σ be the automorphism of D_q such that $\sigma(\xi_q) = \xi_q^p$, then σ fixes all the prime ideals over p, thus we have $\sigma(\alpha)D_q = \alpha D_q$, and then $\sigma(\alpha) = \alpha\delta\xi_q^j$, where $\delta = \pm 1$ and j is an integer. Because

$$\sigma^f(\alpha) = \sigma^{f-1}(\alpha\delta\xi_q^j) = \cdots = \delta^f \cdot \alpha \cdot \xi_q^{\frac{p^f-1}{p-1}j}$$

and q is odd, we have $\delta^f = 1$, hence $\delta = 1$. Since $(j, q) = 1$ or $j = 0$, we can choose a suitable m such that $\sigma(\xi_q^{-m}\alpha) = \xi_q^{-m}\alpha$. Since $\xi_q, \xi_q^2, \cdots, \xi_q^{q-1}$

is a basis of $\mathbf{Z}[\xi_q]$ over $\mathbf{Z}$, $\xi_q^{-m}\alpha$ can be written uniquely as $\sum_{i=1}^{q-1} b_i\xi_q^i$, so we have $b_i = b_j$, when i, j in the same coset of $\langle p\rangle$ in $\mathbf{Z}_q^*$. Collect the terms ξ_q^j, where j runs over a coset T_i of $\langle q\rangle$ in $\mathbf{Z}_q^*$, together, denote the sum as S_i, then we get the equation. □

As to the case we will use, $q = 13$ and $p = 3$ in the lemma, $\langle 3\rangle$ has four cosets in $\mathbf{Z}_q^*$, and we denote $\xi_{13} + \xi_{13}^3 + \xi_{13}^9$ by A, and $B = A^{(2)}$, $C = A^{(4)}$, $D = A^{(8)}$. We will also use A, B, C, D for the elements in group ring that we get by replacing ξ_{13} by an element β of order 13.

The next two lemmas were proved in Ma[4], we list them here without proof.

Lemma 2.2(Ma[4]) *Let G be an abelian group and χ be a character of G of order w and $K = Ker(\chi)$. If $y \in \mathbf{Z}[G]$ such that $\chi(y) = f(\xi_w)$, where $f(X)$ is a polynomial in $\mathbf{Z}[X]$, then*

$$Ky = f(g)K + \sum\langle K, g^{w/q_i}\rangle x_i$$

where $q_1, q_2, \ldots, q_r$ are all the prime divisors of w, $x_1, x_2, \ldots, x_r \in \mathbf{Z}[G]$, and g is an element of G such that $\chi(g) = \xi_w$.

Corollary 2.3 *Let $\langle\beta\rangle$ be a cyclic group of order q, and $f(X)$ be a polynomial in $\mathbf{Z}[X]$ such that $f(\xi_q)f(\xi_q^{-1}) = p^s$ then $f(\beta)f(\beta)^{(-1)} = p^s + \mu\langle\beta\rangle$. Furthermore, if $f(1)^2 = p^s$, then $\mu = 0$.*

proof: It is a straight application of lemma 2.2. □

Lemma 2.4(ma[4]) *Let $G = \langle\alpha\rangle \times \langle\beta\rangle$ be a cyclic group of order $v = p^t w$, where $o(\alpha) = p^t, o(\beta) = w, t \geq 1$, p is an odd prime, and $(p(p-1), w) = 1$. If $y \in \mathbf{Z}[G]$ satisfies $\chi(y)\chi(y) = p^s$ for a character χ of G such that $\chi(\alpha) = \xi_{p^t}$ and $\chi(\beta) = \xi_w$. then*

$$y = \left[\sum_{i=1}^{p}(\frac{i}{p})\alpha^{ip^{t-1}}\right]^{\epsilon} f(\beta)\alpha^c + \sum_{i=1}^{r}\langle\beta^{w/q_i}\rangle x_i + \langle\alpha^{p^{t-1}}\rangle x_{r+1}$$

where $(\frac{i}{p})$ is the Legendre symbol, $\epsilon = 0$ or 1, c is an integer, $x_1, x_2, \ldots, x_{r+1} \in \mathbf{Z}[G]$, $q_1, q_2, \ldots, q_r$ are all the prime divisors of w, and $f(X)$ is a polynomial in $\mathbf{Z}[X]$ such that $f(\xi_w)f(\xi_w^{-1}) = p^{s-\epsilon}$.

When w is a prime, the structure of y in Lemma 2.4 can be determined more precisely.

Lemma 2.5 *Let $G = \langle\alpha\rangle \times \langle\beta\rangle$, where $\alpha^{p^t} = 1$, $\beta^q = 1$, and p and q are odd primes such that $(p(p-1), q) = 1$. Let $y \in \mathbf{Z}[G]$ and $yy^{(-1)} = p^{2s} + \lambda G$, then*

$$y = f(\beta)\alpha^c + \langle\alpha^{p^{t-1}}\rangle x_1,$$

where $f(X) \in \mathbf{Z}[X]$ such that $f(\beta)f(\beta)^{(-1)} = p^{2s}$ and $x_1 \in \mathbf{Z}[G]$.

proof: Let χ be the character such that $\chi(\alpha) = \xi_{p^t}$ and $\chi(\beta) = \xi_q$, then $\chi(y)\overline{\chi(y)} = p^{2s}$. By lemma 2.4, we have

$$y = \left[\sum_{i=1}^{p-1}(\frac{i}{p})\alpha^{ip^{t-1}}\right]^{\epsilon} g(\beta)\alpha^c + \langle\alpha^{p^{t-1}}\rangle x_1 + \langle\beta\rangle x_2 \qquad (1)$$

where $g(\xi_q)g(\xi_q^{-1}) = p^{2s-\epsilon}$, and $x_1, x_2 \in \mathbf{Z}[G]$. Let χ_1 be the character such that $\chi_1(\alpha) = \xi_p^t$ and $\chi_1(\beta) = 1$, then we have $\chi_1(y)\overline{\chi_1(y)} = p^{2s}$ and $\chi_1(y) \in \mathbf{Z}[\xi_{p^t}]$, thus

$$\chi_1(y) = \delta p^s \xi_{p^t}^d, \qquad (2)$$

where $\delta = 1$ or -1.

If $\epsilon = 1$ in (1), we have $\chi_1(y) = [\sum_{i=1}^{p-1}(\frac{i}{p})\xi_p^i]g(1)\xi_{p^t}^c + q\chi_1(x_2)$, together with (2), we have

$$\chi_1(x_2) = \frac{1}{q}\left[\delta p^s \xi_{p^t}^d - \sum_{i=1}^{p-1}(\frac{i}{p})g(1)\xi_{p^t}^{c+ip^{t-1}}\right]$$

since $p^s \not\equiv 0 \pmod q$, and $\{\xi_{p^t}^{c+ip^{p-1}} : i = 1, 2, \ldots, p\}$ is a linear independent set in $\mathbf{Z}[\xi_{p^t}]$ over $\mathbf{Z}$, whatever the value of d is, the coefficients of them in the right side of the equation can not be integer simultaneously, it is in contradiction to that $\chi_1(x_2)$ is an algebric integer. So $\epsilon \neq 1$ in (1) .

If $\epsilon = 0$ in (1), applying χ_1 to (1), we get $\chi_1(y) = g(1)\xi_{p^t}^c + q\chi_1(x_2)$, together with (2), we have

$$\chi_1(x_2) = \frac{1}{q}\left[\delta p^s \xi_{p^t}^d - g(1)\xi_{p^t}^c\right]$$

since $p^s \not\equiv 0 \pmod q$, to make $\chi_1(x_2)$ an algebric integer, there must be $d = c$ and $\chi_1(x_2) = l\xi_{p^t}^c$. By lemma 2.2,

$$\begin{aligned}\langle\beta\rangle x_2 &= l\langle\beta\rangle\alpha^c + \langle\beta, \alpha^{p^{t-1}}\rangle x_2' \\ &= l\langle\beta\rangle\alpha^c + \langle\alpha^{p^{i-1}}\rangle x_2''\end{aligned}$$

Rewrite $x_2'' + x_1$ as x_1, and $g(\beta) + l\langle\beta\rangle$ as $f(\beta)$, we get that

$$y = f(\beta)\alpha^c + \langle\alpha^{p^{t-1}}\rangle x_1.$$

Applying χ_1 on it, we have $\chi_1(y) = f(1)\xi_{p^s}^c$, so $f(1)^2 = \chi_1(y)\overline{\chi_2(y)} = p^{2s}$. By corollary 2.3, we have $f(\beta)f(\beta)^{(-1)} = p^{2s}$. □

3. Main Results

In this section, we will study the structure of the abelian groups which contain a Spence difference set of order 81.

Theorem 3.1 *If $G = \langle\alpha\rangle \times \langle\beta\rangle$, where $\alpha^{27} = \beta^{13} = 1$, then there is no $(351, 126, 45)$-difference set in G.*

proof: If not, let D be such a difference set. Then $DD^{(-1)} = 81 + 45G$, by lemma 2.5,

$$D = f(\beta)\alpha^c + \langle\alpha^9\rangle x_1$$

where $f(\beta)f(\beta)^{(-1)} = 81$, c is an integer, and $x_1 \in \mathbf{Z}[G]$. Because the coefficients of the elements in D are 0 or 1, we know that the coefficients in x_1 must be 0 or 1, the coefficients of $f(X)$ can be $-1, 0$ or 1. Count the number C of identity in $f(\beta)f(\beta)^{(-1)}$, then $C \leq |\langle\beta\rangle| = 13 < 81$, so $f(\beta)f(\beta)^{(-1)}$ can not be equal to 81. So the difference set does not exist. □

Theorem 3.2 *There is no abelian $(351, 126, 45)$-difference set in the group of exponent 117.*

proof: If not, let $G = \langle\gamma\rangle \times \langle\alpha\rangle \times \langle\beta\rangle$ be a group of order 351, with $\gamma^3 = \alpha^9 = \beta^{13} = 1$, and D be a difference set in G, $DD^{(-1)} = 81 + 45G$.

Let $\rho : G \to G/\langle\gamma\rangle$ be the canonical hormomorphism and we use $\overline{x}$ for $\rho(x)$, then $\rho(D) = \sum_{d\in D} \overline{d} \in \mathbf{Z}[G/\langle\gamma\rangle]$ satisfies

$$\rho(D)\rho(D)^{(-1)} = 81 + 135G/\langle\gamma\rangle \tag{3}$$

By lemma 2.5, $\rho(D) = f(\overline{\beta})\overline{\alpha}^c + \langle\overline{\alpha}^3\rangle x_1$, where $f(X) = \sum_{i=0}^{13} a_i X^i$ is a polynomial satisfying $f(\overline{\beta})f(\overline{\beta}^{-1}) = 81$ and $x_1 \in \mathbf{Z}[G/\langle\gamma\rangle]$. We can write $x_1 = \sum_{k\in R} l_k k$, where R is a complete coset representation system of $\langle\overline{\alpha}^3\rangle$ in $G/\langle\gamma\rangle$. For an element g of $G/\langle\gamma\rangle$, $l_{(g)}$ refers to l_k such that $k \in R$ and $k\langle\overline{\alpha}^3\rangle = g\langle\overline{\alpha}^3\rangle$.

Since the coefficients of the elements in $\rho(D)$ are $0, 1, 2, 3$, we know that $0 \leq l_k \leq 3$ for $k \in R$ and $|a_i| \leq 3$. So $f(\xi_{13}) = \xi_{13}^m(b_0 + b_1 A + b_2 B + b_3 C + b_4 D)$, with $|b_i| \leq 3$, $i = 1, 2, 3, 4$, and $f(\xi_{13})f(\xi_{13}^{(-1)}) = 81$, so $f(\xi_{13}) = \delta\xi_{13}^m(3W + 3W^{(-1)} - 3V)$, where $\delta = \pm 1$ and $W \in \{A, B, C, D\}$, $V \neq W$ and $V \neq W^{(-1)}$. Then we have $f(\overline{\beta}) = \delta\overline{\beta}^m(3W + 3W^{(-1)} - 3V)$, thus

$$\rho(D) = \delta\overline{\beta}^m(3W + 3W^{(-1)} - 3V)\overline{\alpha}^c + \langle\overline{\alpha}^3\rangle x_1 \tag{4}$$

When $\delta = 1$, since the coefficients in $\rho(D)$ are nonnegative and not greater than 3, $l_{(x\overline{\alpha}^c\overline{\beta}^m)} = 0$ for $x \in W \cup W^{(-1)}$, and $l_{(y\overline{\alpha}^c\overline{\beta}^m)} = 3$ for $y \in V$. Now we have known the values of 9 l_ks, denote the set of the others l_k as N. Count the coefficient of identity in each side of equation (3), we get $\sum_N l_k = 30$ and $\sum_N l_k^2 = 36$. There are two cases:$(i)N = \{3, 0, 0, 1, 1, \ldots, 1\}$ and $(ii)N = \{2, 2, 2, 1, 1, \ldots, 1, 0, 0, 0\}$.

When $\delta = -1$, we have $l_{(x\overline{\alpha}^c\overline{\beta}^m)} = 3$ for $x \in W \cup W^{(-1)}$, and $l_{(y\overline{\alpha}^c\overline{\beta}^m)} = 0$ for $y \in V$. Use the same notation as above, we get $\sum_N l_k = 27$ and $\sum_N l_k^2 = 27$, hence $N = \{1, 1, \cdots, 1\}$.

So whatever the value of δ is, there are at most 3 of l_k whose value is 2. Then in $\rho(D)$ there are at most 9 elements with coefficient 2.

On the other hand, by (4) we know that whatever δ is , there is at least $O \in \{A, B, C, D\}$ such that $O\langle\gamma\rangle\alpha^{3+c}\beta^m, O\langle\gamma\rangle\alpha^{6+c}\beta^m \subset D$, and $O\langle\gamma\rangle\alpha^c\beta^m \bigcap D = \phi$.

Let $\psi : G \to G/\langle\gamma\alpha^3\rangle$ be the canonical homomorphism, then

$$\psi(D) = \delta'(3W' + 3W'^{(-1)} - 3V')\overline{\beta}^n\overline{\gamma\alpha}^d + \langle\overline{\gamma^3\alpha^3}\rangle x_2,$$

by the same argument as previous, we can choose $O' \in \{A, B, C, D\}$ such that $O'\beta^n(\gamma\alpha)^d\langle\gamma\alpha^3\rangle\alpha^3, O'\beta^n(\gamma\alpha)^d\langle\gamma\alpha^3\rangle\alpha^6 \subset D$ and $O'\beta^n(\gamma\alpha)^d\langle\gamma\alpha^3\rangle \bigcap D = \phi$. Since $O\langle\gamma\rangle\alpha^{3+c}\beta^m$, $O\langle\gamma\rangle\alpha^{6+c}\beta^m$ are mapped to elements with coefficients 2 in $\psi(D)$ by ψ, they are disjoint with $O'\beta^n(\gamma\alpha)^d\langle\gamma\alpha^3\rangle\alpha^3$, $O'\beta^n(\gamma\alpha)^d\langle\gamma\alpha^3\rangle\alpha^6$.

Now consider the third canonical homomorphism $\mu : G \to G/\langle\gamma\alpha^6\rangle$, μ maps $O\langle\gamma\rangle\alpha^{3+c}\beta^m, O\langle\gamma\rangle\alpha^{6+c}\beta^m$, $O'\beta^n(\gamma\alpha)^d\langle\gamma\alpha^3\rangle\alpha^3, O'\beta^n(\gamma\alpha)^d\langle\gamma\alpha^3\rangle\alpha^6$ to 18 elements with coefficients 2 in $\mu(D)$, it is in contradiction to that there are at most 9 elements with coefficient 2 in $\mu(D)$. Thus D can not exist. □

Now we know that $(351, 126, 45)$−difference sets only exist in $\mathbf{Z}_3^3 \times \mathbf{Z}_{13}$. In [2] a new condition *Character Divisibility Property(CDP)* was posed to weaken the self-conjugacy condition. In the next theorem, we consider the Spence difference sets with CDP in $H \times P$, where $|P| = 3^{d+1}$ and $(|H|, 3) = 1$. It is proved that P must be elementary abelian group, and such difference sets must be the difference sets which Spence has constructed.

we cite a lemma from [5] without proof.

Lemma([5]) *Let p be a prime and let $G = H \times P$ be an abelian group with a cyclic Sylow p−subgroup P of order p^s. Let P_i denote the unique subgroup of order p^i in G. If $Y \in \mathbf{Z}[G]$ satisfies*

$$\chi(Y) \equiv 0 \bmod p^a$$

for all characters χ of order divisiable by p^{s-r} where r is some fixed number $r \leq min\{a, s\}$, then there are elements $X_0, X_1, \cdots, X_r, X$ in $\mathbf{Z}[G]$ such that

$$Y = p^a X_0 + p^{a-1} P_1 X_1 + \cdots + p^{a-r} P_r X_r + P_{r+1} X.$$

(if $r = s$ we delete the last term $P_{r+1}X$)

Moreover, if Y has non-negative coefficients then we can choose the $X_i, i = 1, 2, \ldots, r$ such that they have non-negative coefficients.

Theorem 3.3 *If D is a $(\frac{3^{d+1}(3^{d+1}-1)}{2}, \frac{3^d(3^{d+1}+1)}{2}, \frac{3^d(3^d+1)}{2})-$difference set in $G = P \times H$ where $|P| = 3^{d+1}$ and $|H| = \frac{3^{d+1}-1}{2}$, then*

1) P is an elementary abelian group.

2) Let $H_i, i = 1, 2, \ldots, \frac{3^{d+1}-1}{2}$ be the hyperplans of P, then there exists integer i_0, and $h_i \in H, k_i \in P$ such that

$$D = (P - H_{i_0}h_{i_0})k_{i_0} + \sum_{i \neq i_0} H_i h_i k_i.$$

proof: 1) Suppose the exponent of P is 3^s and U is subgroup of P such that $G/U = A \times H$, where A is a cyclic group of order 3^s. Let ρ be the canonical homomorphism. For each character χ of G/U, there exists a character ϕ of G, such that $\phi = \chi \circ \rho$, then $\chi(\rho(D)) = \phi(D)$ and $3^d | \chi(\rho(D))$. By the lemma, we have

$$\rho(D) = 3^d X_0 + 3^{d-1} P_1 X_1 + \cdots + 3^{d-s} P_s X_s$$

where $X_i \in \mathbf{Z}[G/U]$ with non-negative coefficients.

Let χ_1 be a character of G/U such that $\chi_1|_{P_1} \neq 1$, then $\chi_1(\rho(D)) = 3^d \chi_1(X_0) \neq 0$, so $X_0 \neq 0$. For each element in X_0, its coefficient in $\rho(D)$ is at least 3^d, but all the coefficients in $\rho(D)$ cannot be greater than $|U|$, so

$$3^{d+1-s} \geq 3^d.$$

Then $s = 1$, which means that P is an elementary abelian group.

Furthermore, for each element in X_0, its coefficient in $\rho(D)$ is 3^d.

2) Given a hyperplan H_i of P, let ρ_i be the canonical homomorphism from G to G/H_i. Then as in 1), we have

$$\rho_i(D) = 3^d X_i + 3^{d-1} P_1 Y_i.$$

where $X_i \neq 0, Y_i \in \mathbf{Z}[G/H_i]$. The elements in X_i are different from the elements in $P_1 X_1$, so they have different $H-$component.

Let $U_i = \rho_i^{-1}(X_i) \bigcap D$, then if $Ker(\chi) = H_i$,

$$\chi(D - U_i) = 0. \tag{5}$$

Since any two elements from U_i and $U_j (i \neq j)$, always have different $H-$component, U_i's are pairwisely disjoint.

Then

$$D = \sum_{i=1}^{\frac{3^{d+1}-1}{2}} H_i U_i + L.$$

By (5), for each nonprincipal character χ, $\chi(L) = 0$, then $L = PL'$, where $L' \in \mathbf{Z}[G]$. On the other hand, since $|U_i| \geq 1$, then $|L| \leq 3^d$, so

$L = 0$, $|U_i| = 2$ for exactly one i, and $|U_j| = 1$ for $j \neq i$. Thus complete the proof.

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