Spanning trails that join given edges in 3-edge-connected graphs

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Abstract

For given edges e_1, e_2 in $E(G)$, a spanning trail of G with e_1 as the first edge and e_2 as the last edge is called a spanning (e_1, e_2)-trail. In this note, we consider best possible degree conditions to assure the existence of these trails for every pair of edges in a 3-edge-connected graph G.

INTRODUCTION

The graphs in this note are finite and loopless. For terms not defined here, see Bondy and Murty [1]. Let G be a graph and let $X \subseteq E(G)$. The contraction G/X is the graph obtained from G by identifying the ends of each edge in X and then deleting the resulting loops. If H is a subgraph of G, then we use G/H for $G/E(H)$. The line graph of G, denoted by $L(G)$, has vertex-set $E(G)$, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent.

Given edges $e_1, e_2 \in E(G)$, a trail T of G with e_1 as the first edge and e_2 as the last edge is called an (e_1, e_2)-trail. An (e_1, e_2)-trail T is a spanning (e_1, e_2)-trail if every vertex of G is either an internal vertex of T or both the origin and the terminus of T. Lesniak and Williamson [15] showed that if G has a spanning (e_1, e_2)-trail, then $L(G)$ has a hamilton (e_1, e_2)-path. Note that if $G = C_4$, the 4-cycle, and if e_1 and e_2 are two nonadjacent edges in C_4, then C_4 has an (e_1, e_2)-trail that is spanning in C_4 but $L(C_4)$ does not have a hamilton (e_1, e_2)-path. This explains why a spanning (e_1, e_2)-trail

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is defined in the way above.

It has been shown ([11], Theorem 3) that the problem to determine if a graph G has a spanning (e_1, e_2)-trail, for given $e_1, e_2 \in E(G)$, is NP-complete. We list some other prior results below.

¿From the proof of Lemma 6 in [16], we easily know that the following theorem holds.

Theorem A (Zhan [16]) If G is 4-edge-connected, then for any pair of distinct edges $e_1, e_2 \in E(G)$, G has a spanning (e_1, e_2)-trail. □

A bond $X \subset E(G)$ is called essential edge-cut if each component of $G - X$ has an edge. It is clear that if $\{e_1, e_2\}$ is an essential edge-cut of G, then G cannot have a spanning (e_1, e_2)-trail.

Since that G is 4-edge-connected implies that G has 2 edge-disjoint spanning trees, (see [10]), Theorem B below improves Theorem A.

Theorem B (Catlin and Lai [6], Theorem 4) If G has two edge-disjoint spanning trees, then for every pair of distinct edges $e_1, e_2 \in E(G)$, exactly one of the following holds:
(i) G has a spanning (e_1, e_2)-trail.
(ii) $\{e_1, e_2\}$ is an essential edge-cut of G. □

Theorem C (Lai and Zhang [11], Theorem 1) Let G be a simple 2-edge-connected graph with $n \geq 27$ vertices. If for every pair of nonadjacent vertices $u, v \in V(G)$,

$$deg(u) + deg(v) > \frac{2n}{3} - 2,$$

then for every pair of distinct edges $e_1, e_2 \in E(G)$, exactly one of the following holds:
(i) G has a spanning (e_1, e_2)-trail.
(ii) $\{e_1, e_2\}$ is an essential edge-cut of G. □

Theorem D (Lai and Zhang [11], Theorem 2) Let G be a simple 2-edge-connected graph with $n \geq 33$ vertices and without 3-cycles. If for every pair of nonadjacent vertices $u, v \in V(G)$,

$$deg(u) + deg(v) > \frac{n}{3},$$

then for every pair of distinct edges $e_1, e_2 \in E(G)$, exactly one of the following holds:

(i) G has a spanning (e_1, e_2)-trail.
(ii) $\{e_1, e_2\}$ is an essential edge-cut of G. □

MAIN RESULTS

In this note we consider 3-edge-connected graphs. In order to exclude essential edge-cut of size 2, a subgraph of the Petersen graph emerges. In the rest of this paper, we let P denote the Petersen graph and let $P - e$ denote the subgraph of P by deleting an edge.

We say that an edge $e \in E(G)$ is subdivided when it is replaced by a path of length 2 whose internal vertex, denoted by $v(e)$, has degree 2 in the resulting graph. This procedure is called subdividing e. For distinct edges $e_1, e_2 \in E(G)$, let $G(e_1, e_2)$ denote the graph obtained from G by subdividing e_1 and e_2. Thus

$$V(G(e_1, e_2)) - V(G) = \{v(e_1), v(e_2)\}.$$

The following lemmas 1 and 2 follow easily from the definitions of collapsibility(See next section) and spanning (e_1, e_2)-trails. Proofs can be found in [11].

Lemma 1 (Lai and Zhang [11], Lemma 3) Let G be a graph and let G' denote the reduction of G. For vertices $u, v \in V(G)$, define u', v' to be the vertices in G' whose preimages contain u and v, respectively. (Note that even $u \neq v$, it is still possible that $u' = v'$). Then G has a spanning (u, v)-trail if and only if G' has a spanning (u', v')-trail. □

Lemma 2 (Lai and Zhang [11], Lemma 1) Let G be a graph with $e_1, e_2 \in E(G)$. G has a spanning (e_1, e_2)-trail if and only if either $G(e_1, e_2)$ has a spanning $(v(e_1), v(e_2))$-trail, or both e_1 and e_2 are incident with a common vertex v such that $G(e_1, e_2) - v$ has a spanning $(v(e_1), v(e_2))$-trail. □

Theorem 1 Let G be a simple 3-edge-connected graph with $n \geq 77$ vertices. If for every pair of vertices $u, v \in V(G)$ with $uv \notin E(G)$,

$$deg(u) + deg(v) \geq \frac{n}{4} - 2, \tag{1}$$

then for every pair of distinct edges $e_1, e_2 \in E(G)$, one of the following holds:
(i) G has a spanning (e_1, e_2)-trail.
(ii) Equality holds in (1) for some vertices $u, v \in V(G)$ with $uv \notin E(G)$ and $G(e_1, e_2)$ can be contracted to $P - e$ with $v(e_1), v(e_2)$ being the two

vertices of degree 2 in $P - e$.

Theorem 2 Let G be a simple 3-edge-connected graph with $n \geq 117$ vertices and without 3-cycles. If for every pair of vertices $u, v \in V(G)$ with $uv \notin E(G)$,

$$deg(u) + deg(v) \geq \frac{n}{8}, \tag{2}$$

then for every pair of distinct edges $e_1, e_2 \in E(G)$, one of the following holds:

(i) G has a spanning (e_1, e_2)-trail.

(ii) Equality holds in (2) for some vertices $u, v \in V(G)$ with $uv \notin E(G)$ and $G(e_1, e_2)$ can be contracted to $P - e$ with $\{v(e_1), v(e_2)\}$ being the only two vertices of degree 2 in $P - e$.

COLLAPSIBLE GRAPHS AND REDUCTIONS

In this section we summarize some mechanism for the proofs. Let G be a graph and let $X \subseteq E(G)$. The contraction G/X is the graph obtained from G by identifying the ends of each edge in X and deleting the resulting loops. If H is a subgraph of G, then we use G/H for $G/E(H)$.

Let $O(G)$ denote the set of vertices of odd degree in G. Let $R \subseteq V(G)$ be a subset with $|R|$ even. If G is connected and if $O(G) = \emptyset$, then G is eulerian. A graph is supereulerian if it has a spanning eulerian subgraph. An R-subgraph Γ of G satisfies that $G - E(\Gamma)$ is connected and that $O(\Gamma) = R$. A graph G is collapsible if for every $R \subseteq V(G)$ with $|R|$ even, G has an R-subgraph. Thus by definition, K_1 is collapsible, and cycles of length less than 4 are collapsible. In [2], Catlin showed that every graph G has a unique collection of pairwise vertex-disjoint maximal collapsible subgraphs $H_1, H_2, \cdots, H_c$ such that $\cup_{i=1}^{c} V(H_i) = V(G)$. The reduction of G, denoted by G', is the graph obtained from G by contracting each of the maximal collapsible subgraph $H_i, (1 \leq i \leq c)$, into a single vertex v_i. A graph G is reduced if it is the reduction of some graph. Each subgraph $H_i, (1 \leq i \leq c)$, is called the preimage of the vertex v_i of G'. A vertex v in the reduction of G is trivial if the preimage in G under the contraction is a K_1 in G.

Let $a(G)$ denote the minimum number of edge-disjoint forests whose union equals G. Nash-Williams [13] showed that

$$a(G) = \max_{H \subseteq G} \left\lceil \frac{|E(H)|}{|V(H)| - 1} \right\rceil .$$

Let $F(G)$ denote the minimum number of extra edges that must be added to G so that the resulting graph has 2 edge-disjoint spanning trees. It was shown (Catlin [3], Theorem 7) that if G is reduced, then $a(G) \leq 2$; and if $a(G) \leq 2$, then

$$F(G) = 2|V(G)| - |E(G)| - 2. \tag{3}$$

Theorem E (Catlin [2]) Let G be a graph.

(i) (Theorem 5 of [2]) G is reduced if and only if G has no nontrivial collapsible subgraphs.

(ii) (Theorem 8 of [2]) If G is reduced, then G is simple and K_3-free and for any $H \subseteq G$, either $H \in \{K_1, K_2\}$ or $|E(H)| \leq 2|V(H)| - 4$.

(iii) (Theorems 2 and 7 of [2]) If $F(G) = 0$, then G is collapsible. If $F(G) = 1$, then then G is collapsible if and only if G is 2-edge-connected.

(iv) (Theorem 3 of [2]) Let H be a collapsible subgraph of G. Then G is collapsible if and only if G/H is collapsible, and G is supereulerian if and only if G/H is supereulerian. □

Theorem F (Catlin, Han and Lai [5], Theorem 1.3) If G is a connected graph with $F(G) \leq 2$, then either G is collapsible, or the reduction of G is a K_2 or a $K_{2,t}$, $(t \geq 1)$. □

Theorem G (Chen [7], Theorem 1) If G is 3-edge-connected graph with at most 11 vertices, then either G is collapsible or G is the Petersen graph. □

ASSOCIATE RESULT

We need the following technique developed by Catlin [3].

Theorem H Let G be a graph, let $wxyzw$ be a 4-cycle in G, and define the partition $\pi = \{w, y\} \cup \{x, z\}$. Denote G/π to be the graph obtained from $G - \{wx, xy, yz, zw\}$ by identifying w and y to form a single vertex u, by identifying x and z to form a single vertex v, and by adding an extra edge uv. Each of the following holds:

(a) (Catlin [3], Corollary 1) If G/π is collapsible, then G is collapsible.

(b) (Catlin [3], Corollary 2) If G/π is supereulerian, then G is supereulerian.

(c) If G is reduced, then $F(G/\pi) = F(G) - 1$.

Proof of (c) of Theorem H Suppose $a(G/\pi) \leq 2$. Since G is reduced, we also have $a(G) \leq 2$. Apply (3) to G/π and G to get

$$F(G/\pi) = 2|V(G/\pi)| - |E(G/\pi)| - 2$$

and

$$F(G) = 2|V(G)| - |E(G)| - 2.$$

By the definition of G/π, we have $|E(G)| - 3 = |E(G/\pi)|$ and $|V(G)| - 2 = |V(G/\pi)|$. Combine these equalities to get $F(G/\pi) = F(G) - 1$.

Suppose $a(G/\pi) > 2$. Since $a(G) = \max_{H \subseteq G} \left\lceil \frac{|E(H)|}{|V(H)|-1} \right\rceil$, G/π has a nontrivial subgraph H_π satisfying $|E(H_\pi)| \geq 2|V(H_\pi)| - 1$, and hence $|E(H_\pi)| \geq 3$. Since G is reduced, (i) and (ii) of Theorem E imply that subgraph H of size at least 3 is also reduced and thus satisfies $|E(H)| \leq 2|V(H)| - 4$.

Case 1 Suppose $|V(H_\pi) \cap \{u, v\}| = k$ for some $k \in \{0, 1\}$. Define $H = G[E(H_\pi)]$. Then $|E(H)| = |E(H_\pi)| \geq 3$, and $|V(H_\pi)| = |V(H)| - k$ and so

$$|E(H)| = |E(H_\pi)| \geq 2|V(H_\pi)| - 1 = 2(|V(H)| - k) - 1 \geq 2|V(H)| - 3,$$

a contradiction.

Case 2 Suppose $\{u, v\} \subseteq V(H_\pi)$. Define

$$H = G[E(H_\pi - uv) \cup \{wx, xy, yz, zw\}].$$

Then a similar contradiction arises: $|E(H)| = |E(H_\pi)| + 3 \geq 2|V(H_\pi)| + 2 = 2|V(H)| - 2$.

Hence $a(G/\pi) > 2$ cann't hold. This proves (c) of Theorem H. □

Corollary I Let G be a graph with a 4-cycle $wxyzw$ and let $u, v, G/\pi$ be defined as in Theorem H. Let v_1, v_2 be two vertices (not necessary distinct) in $V(G)$ and let v_1', v_2' be the corresponding vertices in G/π such that either $v_i = v_i'$, or $v_i \in \{w, y\}$ and $v_i' = u$, or $v_i \in \{x, z\}$ and $v_i' = v$. If G/π has a spanning (v_1', v_2')-trail, then G has a spanning (v_1, v_2)-trail.

Proof: If $v_1 = v_2$, then Corollary I follows from (b) of Theorem H. Hence we assume that $v_1 \neq v_2$. Let v be a vertex not in $V(G)$ and let $G' = G + v_1 v v_2$. thus G has a spanning (v_1, v_2)-trail if and only if G' is supereulerian. Note also that G/π has a spanning (v_1', v_2')-trail implies that G'/π is supereulerian, and so by (b) of Theorem H, G' is supereulerian. Thus G has a spanning (v_1, v_2)-trail. □

For integer $i \geq 1$, let $D_i(G) = \{v \in V(G) : deg(v) = i\}$.

By Theorem 2.4 in [9], an immediate corollary is given as follows:

Corollary J If G is a 2-edge-connected reduced graph with $F(G) \leq 3$, $|D_2(G)| \leq 3$ and $|V(G)| \leq 11$, then G is the Petersen graph or one of the following graphs:

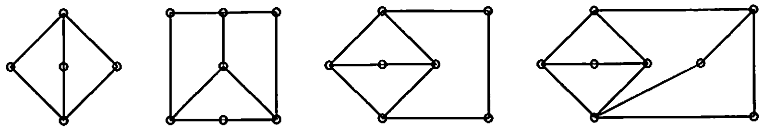

Figure 1

Theorem K ([8], [[9], Theorem 3.2]) Let G be a connected simple graph with $|V(G)| \leq 13$ and $\delta(G) \geq 3$. Then either G is a supereulerian graph with 12 vertices and with an odd cycle, or the reduction of G is in $\{K_1, K_2, K_{1,2}, K_{1,3}, P\}$. □

Theorem L (Chen [7], Lemma 1) Let G be a simple 2-edge-connected graph of order at most 7, with $\delta(G) \geq 2$ and $|D_2(G)| \leq 2$. Then G is collapsible. □

Theorem M (Catlin and Chen [4], Lemma 3) Let G be a simple with $\kappa'(G) \geq 2$, $|V(G)| \leq 8$ and $|D_2(G)| \leq 1$. Then G is collapsible. □

Lemma 3 Let G be a 2-edge-connected graph with at most 9 vertices with $2 \leq \delta(G) \leq \Delta(G) \leq 3$. If $|D_2(G)| \leq 1$, then G is collapsible.

Proof: By Theorem M, we shall consider only $|V(G)| = 9$. If $|D_2(G)| = 0$, then by Theorem K and by $\kappa'(G) \geq 2$, the reduction of G must be K_1 and so G is collapsible. Hence, we assume that $|V(G)| = 9$ and $|D_2(G)| = 1$.

Suppose that G is reduced. Since $|D_2(G)| = 1$ and $|D_3(G)| = 8$, $|E(G)| = 13$. By (3), $F(G) = 2|V(G)| - |E(G)| - 2 = 3$. By Corollary J, G must be one of those graphs in Figure 1, contrary to the assumption that $|V(G)| = 9$. Hence, G has at least one nontrivial subgraph which is collapsible. Let H be a maximal nontrivial collapsible subgraph of G. Since $|V(H)| \geq 3$, $|V(G/H)| \leq 7$. Since G/H is a simple graph with $\kappa'(G/H) \geq 2$ and $|D_2(G/H)| \leq 2$, by Theorem L, G/H is collapsible and so G is collapsible. □

Theorem 3 Suppose that G is a 2-edge-connected reduced graph with at most 10 vertices with $2 \leq \delta(G) \leq \Delta(G) \leq 3$ and with $D_2(G) = \{v_1, v_2\}$. If G does not have a spanning (v_1, v_2)-trail, then v_1v_2 is not an edge in G and $G + v_1v_2$ is the Petersen graph P.

Proof: We shall show first that $v_1v_2 \notin E(G)$. By contradiction, we assume that $v_1v_2 \in E(G)$. Then by Lemma 3, $G/\{v_1v_2\}$ is collapsible and so it is supereulerian too. Since $v_1v_2 \in E(G)$, any spanning eulerian trail in $G/\{v_1v_2\}$ induces a spanning (v_1, v_2)-trail in G, contrary to the assumption that G does not have a spanning (v_1, v_2)-trail.

Consider the graph $G + v_1v_2$. If $G + v_1v_2$ is not supereulerian, then it is not collapsible either. Since G is 2-edge-connected, by Theorem K, $G+v_1v_2$ must be isomorphic to the Petersen graph P. Hence, we only need to prove that $G + v_1v_2$ is not supereulerian.

We may assume that $G + v_1v_2$ is supereulerian. Since G does not have a spanning (v_1, v_2)-trail, G must be supereulerian. Since $\Delta(G) \leq 3$, G is hamiltonian.

Claim 1 If G does not have a spanning (v_1, v_2)-trail, then every edge-cut X with $|X| = 2$ must consist of edges incident with a vertex in $D_2(G)$.

Proof: By contradiction, we assume that G has an edge-cut X with $|X| = 2$ such that each component of $G - X$ has at least 2 vertices. Let G_1 and G_2 be the two components of $G - X$ with $|V(G_1)| \leq |V(G_2)|$. By the following conditions: G is hamiltonian, $|D_2(G)| = 2$, $v_1v_2 \notin E(G)$ and G has no 3-cycles (by (ii) of Theorem E), we can get $|V(G_1)| \geq 5$. Since $|V(G)| \leq 10$, $|V(G_1)| = V(G_2)| = 5$. Hence, G must be the following graph:

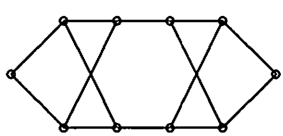

Figure 2

Obviously, G has a spanning (v_1, v_2)-trail, a contradiction. This proves Claim 1.

Claim 2 G has a 4-cycle.

Proof: By contradiction and by (ii) of Theorem E, we may assume that

$$G \text{ has no cycles of length less than 5.} \tag{4}$$

Let $m = |V(G)| \leq 10$ and let $C = u_1u_2...u_mu_1$ be a hamilton cycle of G. Without loss of generality, we assume that $u_1 = v_1$. Since $v_1v_2 \notin E(G)$,

$$u_2, u_m \in D_3(G). \tag{5}$$

Let $u_i, i \notin \{1, m-1\}$, be a neighbor of u_m. By (4), $m-4 \geq i \geq 4$, and so $m \geq 8$. Note that since $|D_2(G)| = 2$, $m \neq 9$

If $m = 8$, then by (4), $u_5 = v_2$ and $u_4u_8 \in E(G)$. Since $|D_2(G)| = 2$, $u_3 \in D_3(G)$, and so by (4), $u_3u_7 \in E(G)$. Thus a 4-cycle $u_3u_4u_8u_7u_3$ exists, contrary to (4).

Hence $m = 10$. Since $u_1 \in D_2(G)$ and since $|D_2(G)| = 2$, not both u_5 and u_7 are in $D_2(G)$, and so we may assume that $u_5 \in D_3(G)$. By (4), either $u_5u_9 \in E(G)$ or $u_5u_{10} \in E(G)$.

Assume first that $u_5u_9 \in E(G)$. Note that by (4), the neighbors of u_{10} must be in $\{u_1, u_4, u_5, u_6, u_9\}$ and so G must have a cycle of length less than 5, contrary to (4).

Thus $u_5u_{10} \in E(G)$. If $u_9 \in D_2(G)$, then $u_4 \in D_3(G)$ and so by (4), $u_4u_8 \in E(G)$. Since $u_9 \in D_2(G)$ now, $u_3 \in D_3(G)$ and so u_3 must be adjacent to u_7, forming a 4-cycle $u_3u_7u_8u_4u_3$, contrary to (4). So we assume that $u_9 \in D_3(G)$. Since $u_5u_{10} \in E(G)$ and by (4), $u_9u_3 \in E(G)$. Since $|D_2(G)| = 2$ and $u_1 \in D_2(G)$, either u_4 or u_8 is in $D_3(G)$. Since by (4). the neighbors of u_4 are in $\{u_3, u_5, u_8\}$, and the neighbors of u_8 are in $\{u_7, u_9, u_4\}$, G must have a cycle of length less than 5, contrary to (4). This proves Claim 2.

By Claim 2, G has a 4-cycle. Let $C = xyzwx$ be a 4-cycle of G such that

$$|V(C)| \text{ contains as many vertices in } D_2(G) \text{ as possible.} \qquad (6)$$

Define G/π, u, v as in Theorem H.

Claim 3: $\kappa'(G/\pi) \geq 2$.

Proof: Since $\kappa'(G) \geq 2$, $\kappa'(G/\pi) \leq 1$ if and only if uv is a cut-edge in G/π, if and only if $E(C)$ is an edge-cut. By contradiction, we assume that $E(C)$ is an edge-cut and G_1 and G_2 are the two sides of $G - E(C)$ with $w, y \in V(G_1)$ and with $x, z \in V(G_2)$. Assume also that $|V(G_1)| \leq |V(G_2)|$. Thus $|V(G_1)| \leq 5$.

If $V(G_1) = \{w, y\}$, then G cannot be hamiltonian, a contradiction. If G_1 has a vertex $v' \in D_3(G)$. Since $|V(G_1)| \leq 5$ and since G has no K_3 ((ii) of Theorem E), v' must be adjacent to some vertex in G_2, contrary to the assumption of $\kappa'(G/\pi) \leq 1$. Thus $V(G_1) - \{w, y\} \subseteq D_2(G)$. But then since $v_1v_2 \notin E(G)$, $|V(G_1) - \{w, y\}| = 1$. We may assume that

$V(G_1) = \{w, y, v_1\}$. If $v_2 \in \{w, y\}$, then G must have a 4-cycle containing both vertices of degree 2 in G, contrary to the choice of C; if $v_2 \notin \{w, y\}$, then G must have a 4-cycle containing at least one vertex of $\{v_1, v_2\}$, contrary to the choice of C. This proves Claim 3.

Since $|D_2(G)| = 2$, $3|V(G)| = 2(|E(G)| + 1)$. Hence it follows by (3) that $F(G) = |E(G)| - |V(G)|$. Since $|V(G)| \leq 10$, $V(G/\pi) \leq 8$ and $F(G) \leq 4$. Let G' denote the reduction of G/π. By (c) of Theorem H, $F(G') \leq F(G/\pi) \leq 3$. Note that G/π has two vertices with degree 2. If G' has at most three vertices with degree 2, by Corollary J, we know that G' must be one of those graphs in Figure 1, namely, G' has three vertices with degree 2. By inspection, G' must have a spanning (v'_1, v'_2)-trail for any $v'_1, v'_2 \in D_2(G')$, and by Lemma 1 and Corollary I, G has a spanning (v_1, v_2)-trail, contrary to the assumption that G does not have any spanning (v_1, v_2)-trails. If G' has four vertices with degree 2, then $G' = C_4$ and G/π has two collapsible subgraphs K_3. And so G/π has four vertices with degree 2, a contradiction. Hence, $G + v_1v_2$ can not be supereulerian. This completes the proof of Theorem 3. □

THE PROOFS OF THE MAIN RESULTS

Lemma 4 If G is reduced, then

$$2F(G) + 4 = \sum_{i=1}^{|V(G)|-1} (4 - i)|D_i(G)|. \tag{7}$$

Proof: This follows by (3) and by counting the incidences of G. □

Proof of Theorem 1: Let G'' be the reduction of $G(e_1, e_2)$. Suppose first that $F(G'') \leq 2$. Since $\kappa'(G) \geq 3$, it follows that $\kappa'(G'') \geq 2$ and so by (iv) of Theorem E, G'' is collapsible if $F(G'') \leq 1$. It follows by Lemma 1 that $G(e_1, e_2)$ has a spanning $(v(e_1), v(e_2))$-trail. Hence by Lemma 2, G has a spanning (e_1, e_2)-trail. If $F(G'') = 2$, then by Theorem F, $G'' = K_{2,t}$ for some $t \geq 2$. Since $\kappa'(G) \geq 3$, the two edges incident with each of $v(e_1), v(e_2)$ are the only edge-cuts of size 2 in $G(e_1, e_2)$ and so $t = 2$. But then it would follow that $\{e_1, e_2\}$ is an edge-cut of G, contrary to $\kappa'(G) \geq 3$. Hence we may assume that $F(G'') \geq 3$.

By Lemma 4 and since $D_1(G'') = \emptyset$, $D_2(G'') \subseteq \{v(e_1), v(e_2)\}$, we have

$$|D_3(G'')| \geq 6, \tag{8}$$

with equality if and only if $D_2(G'') = \{v(e_1), v(e_2)\}$, $\Delta(G'') \leq 4$ and $F(G'') = 3$. By definition, G'' is reduced. By (ii) of Theorem E, G'' is simple and has no 3-cycles. It follows that if $D_3(G'')$ has three trivial vertices, then two of them must be nonadjacent, and so by (1), $n/4 - 2 \leq 6$, which implies that $n \leq 32$, contrary to the assumption that $n \geq 77$. Thus we have

$$D_3(G'') \text{ has at least 4 nontrivial vertices.} \tag{9}$$

Claim 1: For any nontrivial vertex $v \in D_i(G''), 3 \leq i \leq 6$, let H_v denote the preimage of the v. If $\cup_{i=3}^{6} D_i(G'')$ has one vertex ($\neq v$) whose preimage in G includes vertex v_0 with $3 \leq deg(v_0) \leq 6$, then $|V(H_v)| \geq \frac{n}{4} - (1 + deg(v_0))$. In particular, if $D_3(G'')$ has one trivial vertex, then $|V(H_v)| \geq \frac{n}{4} - 4$.

Proof: For any nontrivial vertex $v \in D_i(G''), 3 \leq i \leq 6$, one can choose $w \in V(H_v)$ so that w is incident with at most three edges in $E(G'')$ and is not adjacent to v_0 in G. Thus by (1),

$$|V(H_v)| \geq deg(w) - 3 + 1 = deg(w) + deg(v_0) - deg(v_0) - 2 \geq \frac{n}{4} - 10.$$

Since $n \geq 77$, we may choose $w' \in V(H_v)$ such that w' is not incident with any edge in $E(G'')$ and is not adjacent to v_0 in G. Hence

$$|V(H_v)| \geq deg(w') + 1 = deg(w') + deg(v_0) - deg(v_0) + 1 \geq \frac{n}{4} - deg(v_0) - 1.$$

Claim 2: If $D_3(G'')$ has one trivial vertex v_0, then $\cup_{i=3}^{6} D_i(G'')$ has at most four nontrivial vertices.

Otherwise, by Claim 1, $(n-1) \geq 5(\frac{n}{4} - 4) = n + \frac{n}{4} - 20$. and so $n \leq 76$.

Claim 3: Suppose that $|D_3(G'')| = m$ and all vertices in $D_3(G'')$ are nontrivial. For any $v \in \cup_{i=3}^{6} D_i(G'')$, let H_v denote the preimage of the v.

(i) If $m \geq 6$, $v \in D_3(G'') \cup D_4(G'')$, then H_v has a vertex that incident with no edges in G''.

(ii) If $m \geq 7$, $v \in D_5(G'')$, then H_v has a vertex that incident with no edges in G''.

(iii) If $m \geq 8$, $v \in D_6(G'')$, then H_v has a vertex that incident with no edges in G''.

Proof: (i) Suppose that $v \in D_3(G'')$.

Let v_i, $(1 \leq i \leq m)$ denote the vertices in $D_3(G'')$, and let H_i, $(1 \leq i \leq m)$ denote the preimages of the v_i's, respectively.

Without loss of generality, we may assume that $v = v_1$. If each vertex in H_1 is incident with at least one edge in G'', then since $v_1 \in D_3(G'')$, it follows that $H_1 = K_3$ and so by Claim 1, we have

$$n - 3 \geq \sum_{i=2}^{m} |V(H_i)| \geq 5(\frac{n}{4} - 4) = \frac{5n}{4} - 20.$$

It follows that $n \leq 68$.

Suppose that $v \in D_4(G'')$.

If each vertex in H_v is incident with at least one edge in G'', then since $v \in D_4(G'')$, it follows that we can find $v_0 \in V(H_v)$ such that $deg(v_0) \leq 4$. By Claim 1,

$$n - 1 \geq 6(\frac{n}{4} - 5) = \frac{6n}{4} - 30.$$

It follows that $n \leq 58$.

(ii) Suppose that $m \geq 7$, $v \in D_5(G'')$.

If each vertex in H_v is incident with at least one edge in G'', then since $v \in D_5(G'')$, it follows that we can find $v_0 \in V(H_v)$ such that $deg(v_0) \leq 5$. By Claim 1,

$$n - 1 \geq 7(\frac{n}{4} - 6) = \frac{7n}{4} - 42.$$

It follows that $n \leq 55$.

(iii) Suppose that $m \geq 8$, $v \in D_6(G'')$.

If each vertex in H_v is incident with at least one edge in G'', then since $v \in D_6(G'')$, it follows that we can find $v_0 \in V(H_v)$ such that $deg(v_0) \leq 6$. By Claim 1,

$$n - 1 \geq 8(\frac{n}{4} - 7) = 2n - 56.$$

It follows that $n \leq 55$.

By Claim 2, we only need to consider the following two cases.

Case 1: $D_3(G'')$ has two trivial vertices, say v'_0 and v''_0, and four non-trivial vertices.

By Lemma 4, when $i \geq 5$, $|D_i(G'')| = 0$. By Claim 2, all vertices in $D_4(G'')$ are trivial. If $|D_4(G'')| \neq 0$, let $u \in D_4(G'')$. Since G'' is simple and has no 3-cycle, two vertices in $\{u, v'_0, v''_0\}$ must be nonadjacent, and so by (1), $n/4 - 2 \leq 7$, namely, $n \leq 36$. Hence, $|D_4(G'')| = 0$. Since $|D_3(G'')| = 6$, by lemma 4, $|D_2(G'')| = 2$, namely, $D_2(G'') = \{v(e_1), v(e_2)\}$. Hence, $|V(G'')| = |D_2(G'')| + |D_3(G'')| = 8$. By Theorem 3, G'' has a spanning $(v(e_1), v(e_2))$-trail. By Lemmas 1 and 2, G has a spanning (e_1, e_2)-trail.

Case 2: All vertices in $D_3(G'')$ are nontrivial.

Let v_i, $(1 \leq i \leq m)$ denote the vertices in $D_3(G'')$ and H_i, $(1 \leq i \leq m)$ denote the preimages of the v_i's, respectively. By Claim 3, for each i, $(1 \leq i \leq m)$, H_i has a vertex w_i that is incident with no edges in G''. Then the w_i's form an independent set in G.

If $m \geq 9$, then by (1),

$$2n \geq 2\sum_{i=1}^{m} |V(H_i)| \geq 2\sum_{i=1}^{m}(deg(w_i) + 1) \geq \frac{9n}{4} \tag{10}$$

a contradiction. Note that $m \geq 6$. We consider the following six subcases.

Subcase 1: $F(G'') = 3$, $|D_2(G'')| = 2$, $|D_3(G'')| = 6$ and when $i \geq 5$, $|D_i(G'')| = 0$.

We first show that $|D_4(G'')| \leq 2$. Otherwise, for each $v \in D_3(G'') \cup D_4(G'')$, by Claim 3(i), H_v, the preimage of vertex v, has a vertex w that is incident with no edges in G''. Then the w's form an independent set in G. Hence,

$$2n \geq 2 \sum_{v \in D_3(G'') \cup D_4(G'')} |V(H_v)| \geq 2 \sum_{v \in D_3(G'') \cup D_4(G'')} (deg(w) + 1) \geq \frac{9n}{4},$$

a contradiction. It follows that $|V(G'')| \leq 10$ and $D_2(G'') = \{v(e_1), v(e_2)\}$. By Corollary J, graph G'' does not exist.

Subcase 2: $F(G'') = 3$, $|D_2(G'')| = 2$, $|D_3(G'')| = 7$, $|D_5(G'')| = 1$ and when $i \geq 6$, $|D_i(G'')| = 0$.

Since $|D_3(G'')| = 7$ and $|D_5(G'')| = 1$, by Claim 3 (i), we similarly show that $|D_4(G'')| = 0$. Since $D_2(G'') = \{v(e_1), v(e_2)\}$, by Corollary J, graph G'' does not exist.

Subcase 3: $F(G'') = 4$, $|D_2(G'')| = 2$, $|D_3(G'')| = 8$ and when $i \geq 5$, $|D_i(G'')| = 0$.

Since $|D_3(G'')| = 8$, by Claim 3 (i), we have that $|D_4(G'')| = 0$. Note that $D_2(G'') = \{v(e_1), v(e_2)\}$. Hence, $|V(G'')| = 10$. By Theorem 3, if G does not have a spanning (e_1, e_2)-trail, then (ii) of Theorem 1 must hold.

Subcase 4: $F(G'') = 3$, $|D_2(G'')| = 1$, $|D_3(G'')| = 8$ and when $i \geq 5$, $|D_i(G'')| = 0$.

By Claim 3 (i), we can show that $|D_4(G'')| = 0$. Since $|D_2(G'')| = 1$, by Corollary J, graph G'' does not exist.

Subcase 5: $F(G'') = 3$, $|D_2(G'')| = 2$, $|D_3(G'')| = 8$, $|D_5(G'')| = 2$ and when $i \geq 6$, $|D_i(G'')| = 0$.

For each $v \in D_3(G'') \cup D_5(G'')$, by Claim 3 (i) and (ii), H_v, the preimage of vertex v, has a vertex w that is incident with no edges in G''. Then the w's form an independent set in G. Hence,

$$2n \geq 2 \sum_{v \in D_3(G'') \cup D_5(G'')} |V(H_v)| \geq 2 \sum_{v \in D_3(G'') \cup D_5(G'')} (deg(w) + 1) \geq \frac{10n}{4},$$

a contradiction.

Subcase 6: $F(G'') = 3$, $|D_2(G'')| = 2$, $|D_3(G'')| = 8$, $|D_5(G'')| = 0$, $|D_6(G'')| = 1$ and when $i \geq 7$, $|D_i(G'')| = 0$.

Similarly, we may show that graph G'' does not exist. This completes the proof of Theorem 1. □

Proof of Theorem 2: Let G'' denote the reduction of $G(e_1, e_2)$. Similar to the arguments in the proof of Theorem 1, we may assume that $F(G'') \geq 3$, and so by (4),

$$|D_3(G'')| \geq 6, \tag{11}$$

where equality holds if and only if $D_2(G'') = \{v(e_1), v(e_2)\}$, $\Delta(G'') \leq 4$ and $F(G'') = 3$. Since G'' is reduced, by (ii) of Theorem E, G'' is simple and has no 3-cycles. Thus if $D_3(G'')$ has 3 trivial vertices, then two of them must be nonadjacent and so by (2), $n/8 \leq 6$, contrary to $n \geq 117$. This implies that

$$D_3(G'') \text{ has at least 4 nontrivial vertices.} \tag{12}$$

Claim 1 If v_H is a nontrivial vertices in $\cup_{i=3}^{6} D_i(G'')$ whose preimage in G is H, then

$$|V(H)| \geq \frac{n}{8}.$$

Proof: For any vertex $v \in V(G)$, let $N(v)$ denote its neighborhood in G. Since G is K_3-free, there are a pair of nonadjacent vertices $u, v \in V(H)$. Without loss of generality, we assume that $deg(v) \leq deg(u)$. By (2), we have that $deg(u) \geq n/16$. Since $n \geq 117$, $deg(u) \geq 8$. Hence, we can find two vertices $u', v' \in N(u) \cap V(H)$ such that $deg(v') \leq deg(u')$ (So $deg(u') \geq n/16$). Since G is K_3-free, $u'v' \notin E(G)$ and $|N(u) \cap N(u')| = 0$. Thus by (2), $|V(H)| \geq deg(u) + deg(u') - 6 \geq n/16 + n/16 - 6 = n/8 - 6$. As $n \geq 117$, we may choose u, v nonadjacent in G and not incident with any edges in G''. Since $deg(u) \geq 8$, we may choose u', v' which are not incident with any edges in G''. Hence, $|V(H)| \geq deg(u) + deg(u') \geq n/8$.

Claim 2 (i) $\cup_{i=3}^{6} D_i(G'')$ has at most two trivial vertices.

(ii) Let v_H be a nontrivial vertex in $\cup_{i=3}^{6} D_i(G'')$ whose preimage in G is H. If $\cup_{i=3}^{6} D_i(G'')$ has one trivial vertex with degree j, $(3 \leq j \leq 6)$, then $|V(H)| \geq \frac{n}{4} - 2j$.

Proof: (i) If $\cup_{i=3}^{6} D_i(G'')$ has three trivial vertices, then two of them must be nonadjacent and by (2), $n/8 \leq 12$, namely, $n \leq 96$, contrary to the assumption that $n \geq 117$.

(ii) From the proof of Claim 1, we know that H contains two vertices $u, u' \in V(G)$ which are not adjacent to any edge in G''. Since G is K_3-free and by (2),

$$|V(H)| \geq deg(u) + deg(u') \geq 2(\frac{n}{8} - j) = \frac{n}{4} - 2j.$$

Claim 3 If $D_3(G'')$ has one trivial vertex, then $\cup_{i=3}^{6} D_i(G'')$ has at most four nontrivial vertices.

Otherwise, by Claim 2, $n - 1 \geq 5(n/4 - 6)$, namely, $n \leq 116$.

Hence, similar to the proof of Theorem 2, we need to consider the following two cases.

Case 1: $D_3(G'')$ has two trivial vertices, say v_0' and v_0'', and four nontrivial vertices.

By Lemma 4, when $i \geq 5$, $|D_i(G'')| = 0$. By Claim 3, all vertices in $D_4(G'')$ are trivial. By Claim 2 (i), $|D_4(G'')| = 0$. Since $|D_3(G'')| = 6$, by lemma 4, $|D_2(G'')| = 2$, namely, $D_2(G'') = \{v(e_1), v(e_2)\}$. Hence, $|V(G'')| = |D_2(G'')| + |D_3(G'')| = 8$. By Theorem 3, G'' has a spanning $(v(e_1), v(e_2))$-trail. By Lemmas 1 and 2, G has a spanning (e_1, e_2)-trail.

Case 2: All vertices in $D_3(G'')$ are nontrivial.

Suppose that $|D_3(G'')| = m$. Let $v_i (1 \leq i \leq m)$ denote the vertices in $D_3(G'')$, and let $H_i (1 \leq i \leq m)$ denote the preimages of the v_i's, respectively. By Claim 1, if $m \geq 9$, then $n \geq \sum_{i=1}^{m} |V(H_i)| \geq \frac{9n}{8}$, a contradiction. Note that $m \geq 6$. We consider the following subcases.

Subcase 1: $F(G'') = 3$, $|D_2(G'')| = 2$, $|D_3(G'')| = 6$ and when $i \geq 5$, $|D_i(G'')| = 0$.

If there exists a trivial vertex in $D_4(G'')$, by Claim 2 (ii), $n - 1 \geq 6(n/4 - 8)$, namely, $n \leq 94$, a contradiction. By Claim 1, $D_4(G'')$ has at most two nontrivial vertices. It follows that $|V(G'')| \leq 10$ and $D_2(G'') = \{v(e_1), v(e_2)\}$. By Corollary J, graph G'' does not exist.

Subcase 2: $F(G'') = 3$, $|D_2(G'')| = 2$, $|D_3(G'')| = 7$, $|D_5(G'')| = 1$ and when $i \geq 6$, $|D_i(G'')| = 0$.

If there exists a trivial vertex in $D_4(G'') \cup D_5(G'')$, by Claim 2 (ii), $n - 1 \geq 7(n/4 - 10)$, namely, $n \leq 92$, a contradiction. Hence, by Claim 1, $|D_4(G'')| = 0$ and $D_5(G'')$ has a nontrivial vertex. Since $D_2(G'') = \{v(e_1), v(e_2)\}$, by Corollary J, graph G'' does not exist.

Subcase 3: $F(G'') = 4$, $|D_2(G'')| = 2$, $|D_3(G'')| = 8$ and when $i \geq 5$, $|D_i(G'')| = 0$.

By Claim 1 and Claim 2 (ii), we easily show that $|D_4(G'')| = 0$. Note that $D_2(G'') = \{v(e_1), v(e_2)\}$. Hence, $|V(G'')| = 10$. Let v_i $(1 \leq i \leq 8)$ denote the vertices in $D_3(G'')$ and H_i $(1 \leq i \leq 8)$ denote the preimages of the v_i's, respectively. By Claim 1 and by $n \geq 117$, for each i, $(1 \leq i \leq m)$, H_i contains an edge $u_i u_i' \in E(G)$ that is not adjacent to any edge in G''. Since G is K_3-free and by (2), we have

$$n \geq \sum_{i=1}^{8} |V(H_i)| \geq \sum_{i=1}^{8} (deg(u_i) + deg(u_i')) \geq \frac{8n}{8} = n, \qquad (13)$$

and so for each i, $(1 \leq i \leq 8)$,

$$|V(H_i)| = \frac{n}{8}. \qquad (14)$$

By Theorem 3, if G does not have a spanning (e_1, e_2)-trail, then (ii) of Theorem 2 must hold.

Subcase 4: $F(G'') = 3$, $|D_2(G'')| = 1$, $|D_3(G'')| = 8$ and when $i \geq 5$, $|D_i(G'')| = 0$.

Similarly, we may show that $|D_4(G'')| = 0$. Since $|D_2(G'')| = 1$, by Corollary J, graph G'' does not exist. This completes the proof of Theorem 2. □

(ii) of Theorem 1 and (ii) Theorem 2 can hold. In other words, for every pair of edges e_1, e_2 in a simple 3-edge-connected graph G with $V(G) \geq 77$, the best possible degree conditions to assure the existence of (e_1, e_2)-trail are: for every pair of vertices $u, v \in V(G)$ with $uv \notin E(G)$, $deg(u) + deg(v) > n/4 - 2$; if G is a 3-edge-connected triangle-free simple graph with $V(G) \geq 117$; then the best possible degree conditions are: for every pair of vertices $u, v \in V(G)$ with $uv \notin E(G)$, $deg(u) + deg(v) > n/8$.

Let u_0, v_0 be two vertices of degree 2 and $u_1u_0u_2, v_1v_0v_2$ be two paths with length 2 in $P - e$. Graph G is obtained by replacing two paths $u_1u_0u_2, v_1v_0v_2$ in $P - e$ by two edges e_1, e_2, respectively. Let $s \geq 8$ be an integer and $n = 16s$. Now we construct graph G_1, G_2 by replacing each vertex in G by a complete subgraph K_{2s}, a complete bipartite subgraph $K_{s,s}$, respectively. Then for every pair of nonadjacent $u, v \in V(G_1)$, $deg(u) + deg(v) \geq n/4 - 2$; for every pair of nonadjacent $u, v \in V(G_2)$, $deg(u) + deg(v) \geq n/8$. Note that complete subgraph and complete bipartite subgraph are collapsible. We easily check that for edges $e_1, e_2 \in E(G_1)$, (ii) of Theorem 1 holds; for edges $e_1, e_2 \in E(G_2)$, (ii) of Theorem 2 holds.

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