RE-ORIENTING TOURNAMENTS BY PUSHING VERTICES

Gary MacGillivray and Kathryn L.B. Wood

Department of Mathematics and Statistics
University of Victoria
Victoria, British Columbia
Canada V8W 3P4

gmacgill@math.uvic.ca   kathryn@math.uvic.ca

Abstract

A digraph operation called *pushing* a set of vertices is studied with respect to tournaments. When a set X of vertices is pushed, the orientation of every arc with exactly one end in X is reversed. We discuss the problems of which tournaments can be made transitive and which can be made isomorphic to their converse using this operation.

1. Introduction.

Let D be a digraph and $X \subseteq V(D)$. We define D^X to be the digraph obtained from D by reversing the orientation of all arcs with exactly one end in X, and say that D^X is the result of *pushing* X.

This operation has been studied by Mosesian [10] (also see [11]), Pretzel (see [11]), Fisher and Ryan [4], Klostermeyer [5], and Klostermeyer, et al. [6,7]. Mosesian studied the operation in the context of ordered sets, starting from the observation that if one takes the digraph of an ordered set and reverses all arcs incident with a vertex of out-degree zero, one obtains the digraph of a different ordered set. Pretzel studied an invariant that characterizes when two orientations can be obtained from each other by reversing all arcs incident with a vertex of out-degree zero, and used this approach to produce classes of graphs which can not be oriented as the Hasse diagram of an ordered set. Fisher and Ryan used the equivalence relation, $\approx$, on the set of labelled tournaments, defined by $T_1 \approx T_2$ if and only if there exists $X \subseteq V(T_1)$ such that $T_1^X = T_2$ to find the number of *positive* tournaments (a tournament T is *positive* if there is a positive vector $\mathbf{x}$ such that $K\mathbf{x} = \mathbf{1}$, where K is the $\pm 1, 0$ adjacency matrix of T). Klostermeyer [5] proved that the problems of deciding whether a given digraph can be made strongly connected, or Hamiltonian, or semi-connected, or acyclic using the push operation are NP-complete. By contrast, he showed that every tournament on at least three vertices, with two exceptions, can be transformed into a Hamiltonian (equivalently, strong) tournament by pushing some subset of vertices. In [7] it was shown that if the tournament has at least seven vertices, then the set X can be chosen to be either empty or a singleton. It is also proved that almost any *balanced* bipartite tournament can be made Hamiltonian by pushing some subset of vertices. An $O(n^2)$ algorithm that decides whether a given balanced bipartite tournament can be made Hamiltonian using the push operation is described. The authors also investigate which complete multipartite tournaments can be made Hamiltonian. Results concerning which powers of digraphs can be made Hamiltonian can be found in [6].

As an aside, we observe that a balanced bipartite tournament B can be made Hamiltonian unless $|V| \equiv 2 \pmod 4$ and there is a homomorphism of B to C_4. The existence of such a mapping can be tested in $O(|V|^2)$ time [8].

This paper is mostly concerned with pushing vertices of tournaments. Preliminaries are treated in Section two. In Section three we characterize the tournaments that can be made transitive using the push operation. Finally we turn our attention to converses in Section four and characterize the tournaments that can be transformed into one isomorphic to their converse.

2. Preliminaries.

In this section we introduce some definitions, notation, and preliminary results. Any terms not defined here can be found in [3] or [9].

Let D be a digraph and $X, Y \subseteq V(D)$. The subdigraph of T induced by X is denoted $<X>_D$. The subscript will be omitted if the digraph D is clear from the context. If all arcs with one end in X and the other end in Y have their tail in X, we write $X \rightarrow Y$.

We use $\cong$ to denote isomorphism, and when we want to emphasize that this is accomplished under a particular function f, we write $\cong_f$.

A tournament is *strong* if, for every pair of distinct vertices u and v, there exists both a directed (u, v) path and a directed (v, u) path. The vertex set of any tournament T can be partitioned into $\{V_1, V_2, \ldots, V_c\}$, where each set V_i induces a maximal strongly connected subtournament and, if $i < j$, then $V_i \rightarrow V_j$. We refer to each of $V_1, V_2, \ldots, V_c$ as a strong component of T and the list $V_1, V_2, \ldots, V_c$ as the *ordered list* of strong components of T.

Every Hamiltonian tournament is clearly strong. The converse, and much more, is established in the following classic result of Moon (see [3] or [9]).

Theorem 2.1. *(Moon, 1966) Each vertex of a strong tournament T with $m \geq 3$ vertices is contained in a directed k-cycle, $3 \leq k \leq m$.*

Let T_{in} (respectively T_{out}) be the four vertex tournament obtained from a directed 3-cycle by orienting all remaining arcs towards (respectively away from) the fourth vertex. The tournaments T_{in} and T_{out} are the only ones on three or more vertices that can not be made Hamiltonian using the push operation. It turns out (see Corollary 3.2) that a tournament can be made transitive using the push operation if and only if it contains no subtournament isomorphic to T_{in} nor T_{out}.

We observe that for any digraph D and any $X \subseteq V(D)$, $D^X = D^{\overline{X}}$, and for each positive integer m, the relation $\equiv$, on the set of m-vertex tournaments, defined by $T_1 \equiv T_2$ if and only if there exists a set $X \subseteq V(T_1)$ such that ${T_1}^X \cong T_2$, is an equivalence relation.

According to [9] there are four non-isomorphic tournaments on four vertices: the transitive tournament T_t, the unique strong tournament T_s, and T_{in} and T_{out}. The equivalence classes under $\equiv$ are $A = \{T_t, T_s\}$ and $B = \{T_{out}, T_{in}\}$. Thus, for a tournament T, if $X, S \subseteq V(T)$, with $|S| = 4$, then $<S>_T \in A$ if and only if $<S>_{T^X} \in A$.

Fisher and Ryan [4] proved that for n-vertex labelled tournaments, the relation $\approx$ has $2^{\binom{n-1}{2}}$ equivalence classes. Some of these are isomorphic to each other. The number of equivalence classes under $\equiv$ is the number of non-isomorphic equivalence classes of $\approx$. Fisher and Ryan used Burnside's Lemma to derive a formula for this quantity.

Let T be a tournament. We define $n_{in}(T)$ (respectively $n_{out}(T)$) to be the number of subtournaments of T that are isomorphic to T_{in} (respectively T_{out}). We define $n(T) = n_{in}(T) + n_{out}(T)$. The following result shows that $n(T)$ is the same for all tournaments equivalent to T.

Proposition 2.2. *Let T be a tournament. For all $X \subseteq V$, $n(T^X) = n(T)$.*

Proof. Let $S \subseteq V(T)$, $|S| = 4$. Define

$$f(S) = \begin{cases} 1 & \text{if } <S> \text{ is either } T_{out} \text{ or } T_{in} \\ 0 & \text{otherwise} \end{cases}.$$

Then, using the comment above,

$$n(T) = \sum_{S \subseteq V(T), |S|=4} f(S) = \sum_{S \subseteq V(T^X), |S|=4} f(S) = n(T^X)$$

∎

Corollary 2.3. *If $T_1 \equiv T_2$, then $n(T_1) = n(T_2)$.*

In general, the implication $n(T_1) = n(T_2) \Rightarrow T_1 \equiv T_2$ is not true. In a personal communication, Klostermeyer provided an example of two such tournaments on eight vertices. It follows from the results in the next section that the implication is true when $n(T) = 0$. It is also true when $n(T) = 1$, since the only tournaments with $n(T) = 1$ are T_{in} and T_{out}.

We conclude this section with a proposition we will use several times in the next two sections.

Proposition 2.4. *If $d^-_{T^X}(v) = 0$, then $X = N^-_T(v)$, and if $d^+_{T^X}(v) = 0$, then $X = N^+_T(v)$.*

3. Transitive Tournaments

In this section we determine which tournaments can be made transitive by using the push operation. We begin by observing that neither T_{in} nor T_{out} can be made transitive. Therefore, no tournament that contains T_{in} or T_{out} can be made transitive. Our immediate goal is to prove that these two forbidden subtournaments characterize the tournaments that can be made transitive.

Theorem 3.1. *A tournament T can be made transitive using the push operation if and only if for every $v \in V(T)$ both $<N^+(v)>$ and $<N^-(v)>$ are transitive.*

Proof. ($\Leftarrow$) By Proposition 2.4, $X = N^-_T(v)$ for some vertex $v \in V(T)$. Suppose T^X is not transitive. Then, by Moon's Theorem, T^X contains a 3-cycle $C : x, y, z, x$. Since both $<N^+_T(v)>_T$ and $<N^-_T(v)>_T$ are transitive, neither contains C. Also, $v \notin V(C)$ as $d^-_{T^X}(v) = 0$. Thus, without loss of generality, either $x, y \in N^-_T(v)$ and $z \in N^+_T(v)$, or $x, y \in N^+_T(v)$ and $z \in N^-_T(v)$. Both cases lead to a contradiction: in the former case, z, y, v, z

is a 3-cycle in $<N_T^+(x)>_T$, and in the latter case x, z, v, x is a 3-cycle in $<N_T^-(y)>_T$. Therefore, T^X is transitive.

($\Rightarrow$) Choose $v \in V(T)$. If $<N_T^+(v)>_T$ (respectively $<N_T^-(v)>_T$) is not transitive, then, by Moon's Theorem, it contains a 3-cycle, and T contains T_{out} (respectively T_{in}). Thus, T can not be made transitive. ∎

Theorem 3.1 yields an easy $O(|V|^2)$ algorithm for making a tournament transitive by pushing a subset of vertices, or certifying that this is not possible.

Tournaments in which every neighbourhood induces a transitive subtournament have been previously studied by Alspach and Tabib [1].

Corollary 3.2. *A tournament T can be made transitive using the push operation if and only if T contains neither T_{in} nor T_{out} as a subtournament.*

Corollary 3.3. *A tournament T can be made transitive using the push operation if and only if $n(T) = 0$.*

Corollary 3.4. *Let T_1 and T_2 be tournaments on m vertices. If $n(T_1) = n(T_2) = 0$, then $T_1 \equiv T_2$.*

Proof. By Corollary 3.3 there exists X so that T_1^X is transitive, and there exists Y so that T_2^Y is transitive. Hence, $(T_1^X)^Y$ is isomorphic to T_2. The result follows. ∎

Corollary 3.5. *A tournament T with $m \geq 5$ vertices can be made transitive using the push operation if and only if every proper subtournament of T can be made transitive.*

Proof. ($\Rightarrow$) If T can be made transitive, then for every $v \in V(T)$ both $<N^+(v)>$ and $<N^-(v)>$ are transitive. This property is inherited by any subtournament of T. Hence, by Theorem 3.1, every proper subtournament of T can be made transitive.

($\Leftarrow$) Suppose every proper subtournament of T can be made transitive. Then, for every $v \in V(T)$ neither $<N^-(v)>$ nor $<N^+(v)>$ contains a 3-cycle, so T contains neither T_{in} nor T_{out} as a proper subtournament. Since $m \geq 5$, T is not isomorphic to T_{in} nor T_{out}. Therefore, T can be made transitive. ∎

4. Converses of Tournaments

The converse of a tournament T, denoted $conv(T)$, is the tournament obtained from T by reversing the orientation of every arc. If there exists $X \subseteq V(T)$ such that $T^X \cong conv(T)$, then we call the tournament T *conversible*. In this section we investigate conversible tournaments. We begin by showing that either all elements of an equivalence class of $\equiv$ are conversible, or none are. Then, we give a characterization of conversible non-strong tournaments. Since each equivalence class contains a tournament that is not strong, this result characterizes the equivalence classes that contain conversible tournaments.

Proposition 4.1. *Let T be a tournament and $X \subseteq V(T)$. Then, $conv(T^X) = conv(T)^X$.*

Proof. The tournament resulting from pushing X and then taking the converse of T^X has arcs in $<X>_T$ and $<\overline{X}>_T$ reversed, and arcs between X and $\overline{X}$ oriented as in T. The same tournament results when the converse of T is taken (i.e., each arc of T is reversed) and then X is pushed. ∎

Corollary 4.2. *Let T be a tournament and $X \subseteq V(T)$. Then, $T \equiv conv(T)$ if and only if $T^X \equiv conv(T^X)$.*

Proof. ($\Rightarrow$) Suppose $T \equiv conv(T)$. Then, $T^X \equiv (T^X)^X = T \equiv conv(T) \equiv conv(T)^X = conv(T^X)$.

($\Leftarrow$) Suppose $T^X \equiv conv(T^X)$. Then, $T \equiv T^X \equiv conv(T^X) = conv(T)^X \equiv conv(T)$. ∎

Corollary 4.3. *Let T be a tournament and $[T]$ be the equivalence class of T with respect to $\equiv$. Then either every element of $[T]$ is conversible, or no element of $[T]$ is conversible.*

We now investigate which tournaments are conversible. We consider tournaments that are not strong and first study the structure of the set X.

Proposition 4.4. *Let T be a tournament with ordered list of strong components $S_1, S_2, \ldots, S_t$, where $t \geq 2$, and $X \subseteq V(T)$ is such that $X \cap S_i \neq \emptyset$, and $S_i \setminus X \neq \emptyset$ for every $i \in \{1, 2, \ldots, t\}$, then T^X is strong.*

Proof. Suppose T and X are as in the statement of the Proposition. Consider the $2t$ classes $X \cap S_1$, $S_1 \setminus X$, $X \cap S_2$, $S_2 \setminus X$, $\ldots$, $X \cap S_t$, $S_t \setminus X$. In T^X, if $i < j$, $X \cap S_i \to X \cap S_j$, $S_i \setminus X \to S_j \setminus X$, $S_j \setminus X \to X \cap S_i$, and $X \cap S_j \to S_i \setminus X$. Thus, there exists a directed closed walk containing all vertices of T^X, so T^X is strong. ∎

Theorem 4.5. *Let T be a tournament with ordered list of strong components S_1, S_2, $\ldots$, S_t, where $t \geq 2$. If there exists $X \subseteq V(T)$ such that $T^X \cong conv(T)$ (i.e., T is conversible) then, without loss of generality, $X = S_1 \cup \ldots \cup S_{k-1} \cup Y$ where $Y \subseteq S_k$, $k \leq t$.*

Proof. If $X = V(T)$ or $X = \emptyset$, the result follows, so suppose $\emptyset \subset X \subset V(T)$. Partition $\{S_1, S_2, \ldots, S_t\}$ into three classes $\mathcal{A} = \{S_i : S_i \subseteq X\}$, $\mathcal{B} = \{S_i : (S_i \cap X \neq \emptyset) \text{ and } (S_i \setminus X \neq \emptyset)\}$, and $\mathcal{C} = \{S_i : S_i \cap X = \emptyset\}$.

Following the argument in the proof of Proposition 4.4, there is a strong component of T^X that contains $\cup_{S_i \in \mathcal{B}} S_i$. Further, each element of $\mathcal{A}$ is contained in a strong component of T^X, as is each element of $\mathcal{C}$. Thus, if $|\mathcal{B}| > 1$, T^X has fewer than t strong components and so $T^X \ncong conv(T)$. Hence, we may assume $|\mathcal{B}| \leq 1$.

For $j = 1, 2, \ldots, t$, let $B_j = S_j \cap X$ and $W_j = S_j \cap \overline{X}$. By the above argument, there is at most one $i \in \{1, 2, \ldots, t\}$ such that both $B_i \neq \emptyset$ and

$W_i \neq \emptyset$.

By an *interval* with respect to X we mean a maximal sequence $\mathcal{B}_{ik} = B_i, B_{i+1}, \ldots, B_k$ (respectively $\mathcal{W}_{ik} = W_i, W_{i+1}, \ldots, W_k$) each element of which is non-empty. Intervals $\mathcal{B}_{ij}$ and $\mathcal{W}_{kl}$ (respectively $\mathcal{W}_{ij}$ and $\mathcal{B}_{kl}$) are *consecutive* if $j \leq k \leq j+1$. Let $\mathcal{B}_{ij}$, $\mathcal{W}_{kl}$, and $\mathcal{B}_{rs}$ (respectively $\mathcal{W}_{ij}$, $\mathcal{B}_{kl}$, and $\mathcal{W}_{rs}$) be consecutive intervals. For any vertices $b_p \in B_p$ and $w_q \in W_q$ there is a directed cycle in T^X: $b_i, b_{i+1}, \ldots, b_j, b_r, b_{r+1}, \ldots, b_s, w_k, w_{k+1}, \ldots, w_l, b_i$. Thus, there is a directed closed walk in T^X containing all vertices of $(\cup_{\alpha=i}^{j} B_\alpha) \cup (\cup_{\alpha=k}^{l} W_\alpha) \cup (\cup_{\alpha=r}^{s} B_\alpha)$.

Therefore, if there are more than two intervals with respect to X, then T^X has fewer than t strong components, a contradiction.

Hence there are at most two intervals with respect to X. Since pushing X is the same as pushing $\overline{X}$, we may assume without loss of generality that $X \cap S_1 \neq \emptyset$, the result follows. ∎

Thus, without loss of generality, X can be taken to equal the union of the first $k-1$ strong components of T together with some vertices from the k-th strong component.

The main result of this section gives necessary and sufficient conditions for a tournament which is neither strong nor self-converse (see [9]; there called self-complimentary) to be conversible. Since every equivalence class under $\equiv$ contains a tournament which is not strong (push the in-neighbourhood of some vertex), this also yields a criterion for whether a strong tournament is conversible.

Theorem 4.6. *Suppose T is neither strong nor self-converse. Then there exists $X \subseteq V(T)$ such that $T^X \cong conv(T)$ if and only if*

(i) $V(T) = V_1 \cup V_2 \cup V_3 \cup V_4 \cup V_5 \cup V_6$ (disjoint union), where $V_i \rightarrow V_j$ if $i < j$, except possibly $i = 2, j = 5$,

(ii) $<V_1> \cong conv(<V_3>)$, $<V_4> \cong conv(<V_6>)$, and

(iii) $<V_1 \cup V_2 \cup V_5 \cup V_6>^{V_1 \cup V_2 \cup V_3} \cong conv(<V_2 \cup V_3 \cup V_4 \cup V_5>)$,

or

(iv) T has exactly two strong components A and B, and

(v) $<A>$ *and* $<B>$ *are both self-converse.*

Proof. ($\Rightarrow$) Assume there exists an X such that $T^X \cong_f conv(T)$. Without loss of generality, the set X has the structure described in Theorem 4.5. Let the ordered list of strong components of $<X>$ (respectively $<\overline{X}>$) be $A_1, A_2, \ldots, A_k$ (respectively $B_1, B_2, \ldots, B_t$).

Suppose first that X is a union of strong components of T, so that the ordered list of strong components of T^X is $A_1, A_2, \ldots, A_k, B_1, B_2, \ldots, B_t$. If $k = t = 1$, then it is easy to see that (iv), (v) and (vi) hold. Assume then, that at least one of k and t is greater than one. If $k = 1$, set $V_1 = V_3 = \emptyset$ and $V_2 = A_1$, otherwise set $V_1 = A_1, V_2 = A_2 \cup A_3 \cup \cdots \cup A_{k-1}$, and $V_3 = A_k$. Analogously, if $t = 1$, set $V_4 = V_6 = \emptyset$ and $V_5 = B_1$, otherwise set $V_4 = B_1, V_5 = B_2 \cup B_3 \cup \cdots \cup B_{t-1}$, and $V_6 = B_t$. Then (i) holds by construction. The ordered list of strong components of T^X is $B_1, B_2, \ldots, B_t, A_1, A_2, \ldots, A_k$. Since an isomorphism of tournaments maps strong components to strong components in the same order as they occur in the ordered lists, we have B_1 maps to B_t, B_2 maps to $B_{t-1}, \ldots, B_t$ maps to B_1, A_1 maps to A_k, A_2 maps to $A_{k-1}, \ldots, A_k$ maps to A_1, and (ii) and (iii) hold.

Now suppose that X is not a union of strong components of T. Without loss of generality, X has the structure described in Theorem 4.5, and there is a strong component S such that both $S \cap X$ and $S \cap \overline{X}$ are non-empty. Thus, there is at least one arc from $\overline{X}$ to X. Let r (respectively s) be the smallest (respectively largest) index i such that there is an arc from $\overline{X}$ to A_i, and let p (respectively q) be the smallest (respectively largest) index j such that there is an arc from B_j to X. Define $V_1 = A_1 \cup A_2 \cup \cdots \cup A_{r-1}$, $V_2 =$

$A_r \cup A_{r+1} \cup \cdots \cup A_s$, $V_3 = A_{s+1} \cup A_{s+2} \cup \cdots \cup A_k$, $V_4 = B_1 \cup B_2 \cup \cdots \cup B_{p-1}$, $V_5 = B_p \cup B_{p+1} \cup \cdots \cup B_q$, $V_6 = B_{q+1} \cup B_{q+2} \cup \cdots \cup B_t$. Then, by construction, $V(T)$ is the disjoint union $V_1 \cup V_2 \cup \cdots \cup V_6$.

The ordered list of strong components of T^X is $B_1, B_2, \ldots, B_{p-1}$, Y, $A_{s+1}, A_{s+2}, \ldots$, A_k, where $Y = V_1 \cup V_2 \cup V_5 \cup V_6$. The ordered list of strong components of $conv(T)$ is $B_t, B_{t-1}, \ldots, B_{q+1}$, $W, A_{r-1}, A_{r-2}, \ldots$, A_1, where $W = V_2 \cup V_3 \cup V_4 \cup V_5$. Since strong components of T^X must map under f to strong components of $conv(T)$, it follows that B_1 maps to B_t, B_2 maps to $B_{t-1}, \ldots, A_k$ maps to A_1. The implication will follow if we can show f maps Y to W.

Suppose $f(Y) = A_i, 1 \leq i \leq r-1$. Then, since $A_i \subseteq Y$, $i = 1$. Therefore $V_2 = \emptyset$ (since $V_2 \subseteq Y$) and, consequently, $V_5 = \emptyset$. Hence X is a union of strong components of T, and the result follows from the earlier argument.

Suppose $f(Y) = A_i$, $r \leq i \leq s$. Thus $V_2 \neq \emptyset$. Consequently, $V_5 \neq \emptyset$, and $A_i \subset Y$, so $|A_i| < |Y|$, a contradiction since f is an isomorphism.

The cases $f(Y) = B_j$, $p \leq j \leq q$, and $f(Y) = B_j$, $q+1 \leq j \leq t$ are similar to the above.

Therefore, $f(Y) = W$ and the implication follows.

($\Leftarrow$) If T satisfies $(iv), (v), (vi)$ in the statement of the Theorem, then the result is clear, so suppose it satisfies $(i), (ii), (iii)$, with $X = V_1 \cup V_2 \cup V_3$, $<V_3>_T \cong_{f_1} <V_1>_{conv(T)}$, $<V_4>_T \cong_{f_2} <V_6>_{conv(T)}$, and $<V_1 \cup V_2 \cup V_5 \cup V_6>_T \cong_{f_3} <V_2 \cup V_3 \cup V_4 \cup V_5>_{conv(T)}$. Define $f : V(T^X) \to V(conv(T))$ by

$$f(v) = \begin{cases} f_1(v) & \text{if } v \in V_3 \\ f_2(v) & \text{if } v \in V_4 \\ f_3(v) & \text{otherwise} \end{cases} .$$

By its definition, f is a bijection. Thus, it remains to show $uv \in A(T^X)$ if and only if $f(u)f(v) \in A(conv(T))$. In fact, since T^X and $conv(T)$ are tournaments, it is enough to prove only if. The implication is clear if u and

v both belong to V_3, V_4, or $V_1 \cup V_2 \cup V_5 \cup V_6$. By the structure of $V(T)$, there are two cases to consider.

CASE 1: $u \in V_4$.

If $v \in V_3$, then $f(u) \in V_6$ and $f(v) \in V_1$. Since, in $conv(T)$, $V_6 \rightarrow V_1$, we are done. If $v \in V_1 \cup V_2 \cup V_5 \cup V_6$, then $f(u) \in V_6$ and $f(v) \in V_2 \cup V_3 \cup V_4 \cup V_5$. Since, in $conv(T)$, $V_6 \rightarrow V_2 \cup V_3 \cup V_4 \cup V_5$, we are done.

CASE 2: $u \in V_1 \cup V_2 \cup V_5 \cup V_6$.

Then, in fact, $v \in V_3$. In this case we have $f(u) \in V_2 \cup V_3 \cup V_4 \cup V_5$ and $f(v) \in V_1$, and since, in $conv(T)$, $V_2 \cup V_3 \cup V_4 \cup V_5 \rightarrow V_1$, we are done.

This completes the proof. ∎

While Theorem 4.6 does not suggest an efficient algorithm for deciding if a given tournament is conversible, it does yield a method of constructing conversible tournaments. As examples of conversible tournaments, let V_1, V_3, V_4, and V_6 be singletons, V_2 and V_5 induce 3-cycles, and the arcs between V_2 and V_5 be oriented arbitrarily. Other examples can be constructed similarly.

All of the non-strong tournaments that can be transformed into their converse using the push operation seem to have the following additional properties: $<V_2>\cong conv(<V_2>)$, $<V_5>\cong conv(<V_5>)$, and $<V_2 \cup V_5>^{V_2} \cong conv(<V_2 \cup V_5>)$.

With regard to the complexity of deciding if a given tournament T is conversible, we now show that it is at least as hard as Tournament Isomorphism. The latter problem can be solved in time $O(|V|^{\log(|V|)})$ [2].

Proposition 4.7. *Tournament Isomorphism polynomially transforms to the problem of deciding if a given tournament is self-converse, and the latter problem polynomially transforms to the problem of deciding if a given tournament is conversible.*

Proof. Since an isomorphism between tournaments A and B maps strong components to strong components in the same sequence they appear in the ordered lists, and since the ordered lists of strong components can be found in polynomial time, isomorphism of strong tournaments is polynomially equivalent to isomorphism of general tournaments. We show first that this problem polynomially transforms to the problem of of deciding if a given tournament is self-converse.

Given two strong tournaments A and B, construct a tournament T from the disjoint union of A and $conv(B)$ by adding arcs from each vertex of A to each vertex of $conv(B)$. This can clearly be done in polynomial time. It is not hard to see that T is self-converse if and only if A and B are isomorphic.

We now show that, in turn, the problem of deciding if a given tournament is self-converse polynomially transforms to the problem of deciding if a given tournament is conversible. Given a tournament A, construct a tournament T from two disjoint copies, A_1 and A_2, of A by adding arcs from each vertex in A_1 to each vertex in A_2. The construction clearly takes polynomial time.

We claim that A is self converse if and only if T is conversible. If A is self-converse, then so is T, and thus T is conversible (push the empty set). Now suppose T is conversible, and let $X \subseteq V(T)$ be such that $T^X \cong conv(T)$. If X is empty, or equals $V(T)$, or $V(A)$, then clearly A is self-converse. We show these are the only possibilities. By Proposition 4.4, if all four sets $(X \cap V(A_1))$, $(X - V(A_1))$, $(X \cap V(A_2))$, and $(X - V(A_2))$ are non-empty, then T^X is strong, a contradiction. Thus at least one of these sets is empty. We may assume without loss of generality that $X \cap V(A_1) \neq \emptyset$. Suppose $(X - V(A_1)) \neq \emptyset$ while $(X \cap V(A_2)) = \emptyset$. Then, in T^X, every vertex in $V(A_2)$ is adjacent from every vertex of $V(A_1) - X$, and adjacent to every vertex of $V(A_1) \cap X$. Since A_1 is strong, there is an arc from a vertex of $V(A_1) \cap X$ to a vertex of $V(A_1) - X$. Thus, some strong component of T^X properly contains A_2, a contradiction since $T^X \cong conv(T)$, so it has exactly two strong components, both of which are isomorphic to $conv(A)$, and therefore each have $|V(A)|$ vertices. The case where X contains A_1 and part of A_2 is similar. This proves the claim, and completes the proof. ∎

Proposition 4.7 has also been proved by Jing Huang (private communication).

In conclusion we ask:

Question 4.8. *Is there a characterization for strong tournaments that can be transformed into their converse?*

Question 4.9. *Does every non-strong conversible tournament have the additional properties mentioned above?*

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