# Isoperimetric numbers and bisection widths of double coverings of a complete graph*

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**Abstract**

The aim of this paper is to study the isoperimetric numbers of double coverings of a complete graph. It turns out that these numbers are very closely related to the bisection widths of the double coverings and the degrees of unbalance of the signed graphs which derive the double coverings. For example, the bisection width of a double covering of a complete graph $K_m$ is equal to $m$ times its isoperimetric number. We determine which numbers can be the isoperimetric numbers of double coverings of a complete graph.

# 1 Introduction

Let $G$ be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$, and let $|X|$ denote the cardinality of a set $X$. A graph means a finite simple graph throughout this paper. For a nonempty proper subset $X$ of $V(G)$, we denote by $\partial X$ the set of edges of $G$ having one end in $X$ and the other

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end in $V(G) - X$, and call it the *boundary* of $X$ in $G$. The quantity

$$\begin{aligned} i(G) &= \min\left\{ \frac{|\partial X|}{\min\{\,|X|, |V(G) - X|\,\}} : X \subset V(G) \right\} \\ &= \min\left\{ \frac{|\partial X|}{|X|} : X \subset V(G) \text{ and } 1 \leq |X| \leq \frac{1}{2}|V(G)| \right\} \end{aligned}$$

is called the *isoperimetric number* of $G$. A subset $X$ of $V(G)$ is called an *isoperimetric set* of $G$ if $i(G) = \frac{|\partial X|}{|X|}$. From the definition of $i(G)$, we can see that $i(G)$ is a measure of the connectivity of the graph $G$. The *bisection width* of a graph $G$ is the quantity $\min\left\{ |\partial X| : |X| = \left\lfloor \frac{|V(G)|}{2} \right\rfloor \right\}$. In [16], Kwak *et al.* got some upper and lower bounds for the isoperimetric numbers of graph coverings and graph bundles, with exact values in some special cases.

A *signed graph* is a pair $G_\phi = (G, \phi)$ of a graph $G$ and a function $\phi : E(G) \to \mathbb{Z}_2$, $\mathbb{Z}_2 = \{1, -1\}$. We call $G$ the *underlying graph* of $G_\phi$ and $\phi$ the *signing* of $G$. A signing $\phi$ is in fact a $\mathbb{Z}_2$-voltage assignment of $G$, which was defined by Gross and Tucker [9].

For a graph $G$, we denote the set of vertices adjacent to $v \in V(G)$ by $N(v)$ and call it the *neighborhood* of a vertex $v$. A graph $\widetilde{G}$ is called a *covering* of $G$ with projection $p : \widetilde{G} \to G$ if there is a surjection $p : V(\widetilde{G}) \to V(G)$ such that $p|_{N(\tilde{v})} : N(\tilde{v}) \to N(v)$ is a bijection for any vertex $v \in V(G)$ and $\tilde{v} \in p^{-1}(v)$. We also say that the projection $p : \widetilde{G} \to G$ is a *double covering* of $G$ if $p$ is two-to-one.

It is known ([3], [9]) that every double covering of a graph $G$ can be constructed as follows: Let $\phi$ be a signing of $G$. The double covering $G^\phi$ of $G$ derived from $\phi$ has as its vertex set $V(G) \times \{1, -1\}$ and, for all $uv \in E(G)$ and $g, g' \in \mathbb{Z}_2$, the two vertices $(u, g)$ and $(v, g')$ are joined by an edge in $G^\phi$ if and only if $g' = \phi(uv)g$. Notice that the first coordinate projection $p^\phi : G^\phi \to G$ is a covering projection.

In a signed graph, the edges which are assigned the value 1 are said to be *positive* and the others *negative*. A cycle $C = e_1\ e_2\ \cdots e_n$ in a signed graph $G_\phi$ is *negative* if $C$ has an odd number of negative edges, *i.e.*, $\prod_{i=1}^n \phi(e_i) = -1$ in $\mathbb{Z}_2$. A signed graph $G_\phi$ is *balanced* if no cycle in it is negative, or equivalently, its vertex set may be partitioned into two disjoint classes in such a way that an edge is negative if and only if its two endpoints belong to distinct classes. Equivalently, $G_\phi$ is *balanced* if and only if a lifting of any cycle in $G$ is also a cycle in $G^\phi$. Given a signed graph $G_\phi$, the *degree of unbalance* $d(G_\phi)$ of $G_\phi$ is the smallest number $d$ such that there exists a balanced signed graph $G_{\tilde{\phi}}$ having the property that

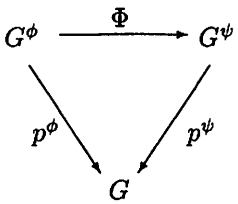
$|\{ e \in E(G) : \phi(e)\tilde{\phi}(e) = -1 \}| = d$, that is, $d(G_\phi)$ is the smallest number $d$ such that $G_\phi$ may be converted into a balanced one by changing the values of $d$ edges. Hansen [12] proposed algorithms for finding $d(G_\phi)$, and Akiyama *et al.* [1] studied the largest $d(G_\phi)$ among all signings $\phi : E(G) \to \mathbb{Z}_2$.

In section 2, we give an upper bound for the isoperimetric number of a double covering of a graph. We also show that the isoperimetric number of a double covering $K_m^\phi$ of a complete graph $K_m$ is equal to $\frac{2}{m}$ times the degree of unbalance of the signed graph $(K_m, \phi)$. Some more interesting relations between isoperimetric numbers and the bisection widths will be shown. A lower and another upper bound for the isoperimetric number of the double covering $K_m^\phi$ is given in terms of the largest eigenvalue of a corresponding signed graph in section 3. In the final section 4, we determine the range of isoperimetric numbers of double coverings of a complete graph.

# 2 Isoperimetric numbers of double coverings

Let $G$ be a graph and let $C^1(G;\mathbb{Z}_2)$ denote the set of all signings of $G$. For any signing $\phi \in C^1(G;\mathbb{Z}_2)$ and for any nonempty proper subset $X$ of $V(G^\phi) = V(G) \times \{1, -1\}$, let $\partial_\phi X$ denote the boundary of $X$ in the double covering $G^\phi$ of $G$.

Two double coverings $p^\phi : G^\phi \to G$ and $p^\psi : G^\psi \to G$ are *isomorphic* if there exists a graph isomorphism $\Phi : G^\phi \to G^\psi$ such that the diagram




commutes. Such a $\Phi$ is called a *covering isomorphism.* We simply write $\phi \sim \psi$ if the coverings $G^\phi$ and $G^\psi$ are *isomorphic.*

Kwak and Lee [14] showed an algebraic characterization for two graph bundles to be isomorphic; it can be rephrased for double coverings as follows:

**Theorem 1** *Let $\phi$ and $\psi$ be two signings in $C^1(G;\mathbb{Z}_2)$. Then the two double coverings $G^\phi$ and $G^\psi$ are isomorphic if and only if there exists a function $f : V(G) \to \mathbb{Z}_2$ such that $\psi(uv) = f(v)\phi(uv)f(u)$ for any $uv \in E(G)$.*

Notice that a function $f : V(G) \to \mathbb{Z}_2$ can be described as a characteristic function, that is, there exists a subset $X$ of the vertex set $V(G)$ such that for any $v \in V(G)$

$$f(v) = \begin{cases} 1 & \text{if } v \in X \text{ ,} \\ -1 & \text{if } v \notin X. \end{cases}$$

Hence, we have the following corollary.

**Corollary 1** *Two double coverings $G^{\phi}$ and $G^{\psi}$ are isomorphic, that is, $\phi \sim \psi$, if and only if there exists a signing $\alpha \in C^1(G;\mathbb{Z}_2)$ such that the vertex set $V(G)$ may be partitioned into two disjoint classes in such a way that $\alpha(e) = -1$ for an edge $e \in E(G)$ if and only if two end points of the edge $e$ belong to distinct classes, and $\psi(e) = \alpha(e)\phi(e)$ for every $e \in E(G)$.*

For a signing $\phi : E(G) \to \mathbb{Z}_2$ and for any $X \subset V(G)$, let $\phi_X$ be the signing obtained from $\phi$ by reversing the sign of each edge having exactly one end point in $X$. If $\psi = \phi_X$ for some $X \subset V(G)$ then $\phi$ and $\psi$ are said to be *switching equivalent* (See [7]). Now, Corollary 1 can be rephrased as follows:

**Corollary 2** *For any two signings $\phi$ and $\psi$ in $C^1(G;\mathbb{Z}_2)$, $\phi \sim \psi$ if and only if they are switching equivalent. The equivalence class $[\phi]$ of $\phi$ is $\{\phi_X : X \subset V(G)\}$.*

**Corollary 3** *The number of isomorphism classes of double coverings of $G$ is equal to the number of switching equivalence classes of signings of $G$.*

Hofmeister [13] showed that the number of isomorphism classes of double coverings of a connected graph $G$ is equal to $2^{\beta(G)}$, where $\beta(G) = |E(G)| - |V(G)| + 1$ is the betti number of $G$.

For a subset $X \subset V(G)$, let $G_X$ denote a new graph with $V(G_X) = V(G)$. Two vertices in $X$ or in $V(G) - X$ are adjacent in $G_X$ if they are adjacent in $G$, while a vertex in $X$ and a vertex in $V(G) - X$ are adjacent in $G_X$ if they are not adjacent in $G$. Two graphs $G$ and $H$ are *Seidel switching equivalent* if there exists a subset $X \subset V(G)$ such that $G_X$ is isomorphic to $H$. For its properties and applications the reader is suggested to refer [6] or [7]. Clearly, the Seidel switching equivalence is an equivalence relation on the set of graphs, and the equivalence class $[G]$ of a graph $G$ is $\{G_X : X \subset V(G)\}$. In a complete graph $K_m$, the signings of $K_m$ are one-to-one correspondence with the spanning subgraphs of $K_m$ by corresponding a signing $\phi$ to the spanning subgraph having $\phi^{-1}(-1)$ as its edge set. Such a spanning subgraph is call the *support* of $\phi$, and denoted by *spt*$(\phi)$. We note that any graph $G$ can be described as a support *spt*$(\phi)$ of a signing $\phi \in C^1(K_{|V(G)|};\mathbb{Z}_2)$.

**Corollary 4** *Let $\phi$ and $\psi$ be two signings in $C^1(K_m;\mathbb{Z}_2)$, $G = spt(\phi)$ and $H = spt(\psi)$. Then the following statements are equivalent.*

(1) *Two graphs $G$ and $H$ are Seidel switching equivalent.*

(2) *Two signings $\phi$ and $\psi$ are switching equivalent.*

(3) *Two double coverings $K_m^{\phi}$ and $K_m^{\psi}$ of $K_m$ are isomorphic as coverings, i.e., $\phi \sim \psi$.*

First, we derive an upper bound for the isoperimetric number of a double covering of $G$. To do this, we start with the following lemma.

**Lemma 1** (1) *For any two signings $\phi$ and $\psi$ in $C^1(G;\mathbb{Z}_2)$, $d(G_\phi) = d(G_\psi)$ if $\phi \sim \psi$.*

(2) *For a signing $\phi$ in $C^1(G;\mathbb{Z}_2)$, $G_\phi$ is balanced if and only if $\phi \sim t$, where $t$ is the trivial signing, that is, $t(e) = 1$ for any $e \in E(G)$.*

(3) *For any signing $\phi \in C^1(G;\mathbb{Z}_2)$,*

$$\min\left\{|\psi^{-1}(-1)| : \phi \sim \psi\right\} = \min\left\{|\phi_X^{-1}(-1)| : X \subset V(G)\right\} = d(G_\phi).$$

Proof: (1) Suppose that $\phi \sim \psi$. Then there exists a function $f : V(G) \to \mathbb{Z}_2$ such that $\psi(uv) = f(v)\phi(uv)f(u)$ for any $uv \in E(G)$. Let $\tilde{\phi}$ be a signing in $C^1(G;\mathbb{Z}_2)$ such that $G_{\tilde{\phi}}$ is balanced and

$$\left|\left\{e \in E(G) : \phi(e)\tilde{\phi}(e) = -1\right\}\right| = d(G_\phi).$$

Now, we define a new signing $\tilde{\psi} : E(G) \to \mathbb{Z}_2$ by $\tilde{\psi}(uv) = f(v)\tilde{\phi}(uv)f(u)$ for $uv \in E(G)$. Then $G_{\tilde{\psi}}$ is balanced and $\psi(uv)\tilde{\psi}(uv) = \phi(uv)\tilde{\phi}(uv)$ for $uv \in E(G)$. Thus, it follows

$$d(G_\psi) \le \left|\left\{e \in E(G) : \psi(e)\tilde{\psi}(e) = -1\right\}\right| = d(G_\phi).$$

Similarly, we can show $d(G_\phi) \le d(G_\psi)$. This completes the proof of (1).

(2) Suppose that $G_\phi$ is balanced. Then, there exists a signing $\psi$ which has the value 1 on the edges of a spanning tree of $G$ and $\phi \sim \psi$ (see [14] Corollary 1). By (1), $G_\psi$ is also balanced, and then $\psi$ must be the trivial signing. Conversely, suppose that $\phi \sim t$. Then, by (1), $d(G_\phi) = d(G_t) = 0$ and $G_\phi$ is clearly balanced.

(3) For a given signing $\phi \in C^1(G;\mathbb{Z}_2)$, let $\varphi$ be a signing in $C^1(G;\mathbb{Z}_2)$ such that $\phi \sim \varphi$ and $|\varphi^{-1}(-1)| = \min\left\{|\psi^{-1}(-1)| : \phi \sim \psi\right\}$. By (1),

$d(G_\phi) = d(G_\varphi) \le \left|\varphi^{-1}(-1)\right|$. On the other hand, by the definition of $d(G_\phi)$, there exists a signing $\tilde{\phi} \in C^1(G;\mathbb{Z}_2)$ such that $G_{\tilde{\phi}}$ is balanced and $d(G_\phi) = |\{e \in E(G) : \phi(e)\tilde{\phi}(e) = -1\}|$. By (2) and Theorem 1, there is a function $f : V(G) \to \mathbb{Z}_2$ such that $f(v)\tilde{\phi}(uv)f(u) = 1 = t(uv)$ for any $uv \in E(G)$. This implies that $f(u)f(v) = \tilde{\phi}(uv)$ for any $uv \in E(G)$. Now, we define a new signing $\tilde{\psi} : E(G) \to \mathbb{Z}_2$ by $\tilde{\psi}(uv) = f(v)\phi(uv)f(u)$ for $uv \in E(G)$. Then, we have

$$\min\left\{|\psi^{-1}(-1)| : \phi \sim \psi\right\} = \min\left\{|\psi^{-1}(-1)| : \tilde{\psi} \sim \psi\right\}$$

and $\tilde{\psi}(e) = \phi(e)\tilde{\phi}(e)$ for any $e \in E(G)$. Since

$$\begin{aligned}\min\left\{|\psi^{-1}(-1)| : \tilde{\psi} \sim \psi\right\} &\le |\tilde{\psi}^{-1}(-1)| \\ &= \left|\left\{e \in E(G) : \phi(e)\tilde{\phi}(e) = -1\right\}\right| = d(G_\phi),\end{aligned}$$

we have $\min\left\{|\psi^{-1}(-1)| : \phi \sim \psi\right\} = d(G_\phi)$, which completes the proof. □

**Theorem 2** *For any signing $\phi$ in $C^1(G;\mathbb{Z}_2)$, we have*

$$i(G^\phi) \le \min\left\{i(G), \frac{2}{|V(G)|}d(G_\phi)\right\}.$$

Proof: Let $X \subset V(G)$ be an isoperimetric set of $G$. Then, $|\partial_\phi(X \times \{1,-1\})| = 2|\partial X|$ and hence

$$i(G^\phi) \le \frac{|\partial_\phi(X \times \{1,-1\})|}{|X \times \{1,-1\}|} = \frac{2|\partial X|}{2|X|} = i(G).$$

Note that $i(G^\phi) = i(G^\psi)$ if $\phi \sim \psi$. Let $\psi$ be any signing in $C^1(G;\mathbb{Z}_2)$ such that $\phi \sim \psi$. Then, by using Lemma 1 (3) with the inequality

$$i(G^\phi) = i(G^\psi) \le \frac{|\partial_\psi(V(G) \times \{1\})|}{|V(G)|} = \frac{2}{|V(G)|}|\psi^{-1}(-1)|,$$

we have $i(G^\phi) \le \frac{2}{|V(G)|}d(G_\phi)$. □

Now, we aim to estimate the isoperimetric number of a double covering of a complete graph $K_m$. Note that $V(K_m^\phi) = V(K_m) \times \{1,-1\}$. For any $X \subset V(K_m) \times \{1,-1\}$, let $X_1$ be the set of vertices $v$ of $K_m$ such that at least one of $(v,1)$ and $(v,-1)$ is contained in $X$, and let $X_2$ be the set of vertices $v$ of $K_m$ such that both $(v,1)$ and $(v,-1)$ are contained in $X$. Note that $X_2 \subset X_1 \subset V(K_m)$ and $|X| = |X_1| + |X_2|$.

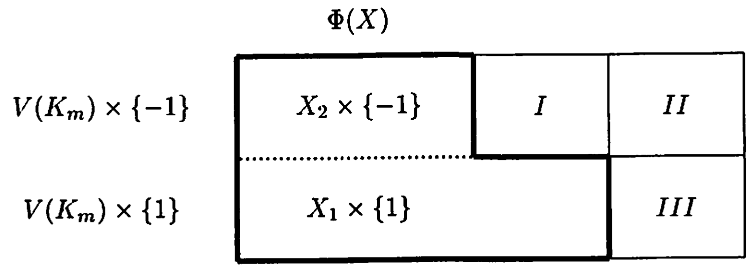



Figure 1: The set $V(K_m) \times \{1, -1\}$

**Lemma 2** *Let $\phi$ be a signing in $C^1(K_m; \mathbb{Z}_2)$ and let $X \subset V(K_m) \times \{1, -1\}$ with $|X| \leq m$. If $X_2$ is not empty, then there exists a subset $Y$ of $V(K_m) \times \{1, -1\}$ such that $|Y| = |X|$, $|Y_1| = |X_1| + 1$, $|Y_2| = |X_2| - 1$ and $|\partial_\phi X| > |\partial_\phi Y|$.*

Proof: Let $X$ be any nonempty subset of $V(K_m^\phi)$ such that $|X| \leq \frac{1}{2}|V(K_m^\phi)| = m$ and $X_2$ is not empty. Define a function $f : V(K_m) \to \mathbb{Z}_2$ by

$$f(v) = \begin{cases} -1 & \text{if } v \in X_1 - X_2 \text{ and } (v, -1) \in X, \\ 1 & \text{otherwise.} \end{cases}$$

Define a new signing $\psi$ of $K_m$ by $\psi(uv) = f(v)\phi(uv)f(u)^{-1}$ for any $uv \in E(K_m)$. Then, the map $\Phi : K_m^\phi \to K_m^\psi$ defined by $\Phi((v, i)) = (v, f(v)i)$ for $i = 1, -1$ is a covering isomorphism and $\Phi(X) = (X_1 \times \{1\}) \cup (X_2 \times \{-1\})$, disjoint union. Moreover, $\partial_\phi X = \partial_\psi ((X_1 \times \{1\}) \cup (X_2 \times \{-1\}))$.

To compute $|\partial_\psi ((X_1 \times \{1\}) \cup (X_2 \times \{-1\}))|$, consider Figure 1. The number of boundary edges in $\partial_\psi ((X_1 \times \{1\}) \cup (X_2 \times \{-1\}))$ having one end in $X_1 \times \{1\}$ and the other end in $II \cup III$ is equal to $|X_1|(m - |X_1|)$. The number of boundary edges in $\partial_\psi ((X_1 \times \{1\}) \cup (X_2 \times \{-1\}))$ having one end in $(X_1 - X_2) \times \{1\}$ and the other end in $I$ is equal to $2|\{e \in E(X_1 - X_2) : \psi(e) = -1\}| = 2|\psi^{-1}(-1) \cap E(X_1 - X_2)|$. The number of boundary edges in $\partial_\psi ((X_1 \times \{1\}) \cup (X_2 \times \{-1\}))$ which are not contained in the above two cases, that is, the edges either having one end in $X_2 \times \{-1\}$ and the other end in $I \cup II \cup III$, or one end in $X_2 \times \{1\}$ and the other end in $I$, is equal to $|X_2|(m - |X_2|)$. Thus we have

$$|\partial_\psi \Phi(X)| = |X_1|(m - |X_1|) + |X_2|(m - |X_2|) + 2|\psi^{-1}(-1) \cap E(X_1 - X_2)|.$$

Since $|X| \leq m$ and $X_2$ is not empty, the set $X_1$ is a proper subset of $V(K_m)$. Now choose two vertices $v_0 \in X_2$ and $v_\infty \notin X_1$, and let

$$W = ((X_1 \cup \{v_\infty\}) \times \{1\}) \cup ((X_2 - \{v_0\}) \times \{-1\}).$$

Then it is clear that $W_1 = X_1 \cup \{v_\infty\}$, $W_2 = X_2 - \{v_0\}$ and $W_1 - W_2 = (X_1 - X_2) \cup \{v_0, v_\infty\}$. Also, it can be shown that

$$|\partial_\psi W| = |W_1|(m - |W_1|) + |W_2|(m - |W_2|) + 2|\psi^{-1}(-1) \cap E(W_1 - W_2)|.$$

Define a function $g : V(K_m) \to \mathbb{Z}_2$ as follows: For any $v$ not in $\{v_0, v_\infty\}$, we define $g(v) = 1$. For $v = v_0$ or $v_\infty$, let

$$I_v = |\{w \in X_1 - X_2 : \psi(vw) = -1\}| .$$

Now, we divide the discussion into the following three cases.

Case 1. If $I_{v_0} + I_{v_\infty} < |X_1| - |X_2|$, we define $g(v_0) = g(v_\infty) = 1$.

Case 2. If $I_{v_0} + I_{v_\infty} = |X_1| - |X_2|$, we define $g(v_0)$ and $g(v_\infty)$ as follows:

$$\begin{array}{lll} g(v_0) = 1, & g(v_\infty) = 1 & \text{if } \psi(v_0v_\infty) = 1 \\ g(v_0) = -1, & g(v_\infty) = 1 & \text{if } \psi(v_0v_\infty) = -1 \text{ and } I_{v_0} \geq \frac{|X_1|-|X_2|}{2} \\ g(v_0) = 1, & g(v_\infty) = -1 & \text{if } \psi(v_0v_\infty) = -1 \text{ and } I_{v_0} < \frac{|X_1|-|X_2|}{2}. \end{array}$$

Case 3. If $I_{v_0} + I_{v_\infty} > |X_1| - |X_2|$, we define $g(v_0) = g(v_\infty) = -1$.

Now, we define another signing $\tilde{\psi}$ of $K_m$ by $\tilde{\psi}(uv) = g(v)\psi(uv)g(u)^{-1}$ for $uv \in E(K_m)$. Then, we have

$$2|\tilde{\psi}^{-1}(-1) \cap E(W_1 - W_2)| \leq 2|\psi^{-1}(-1) \cap E(X_1 - X_2)| + 2\left(|X_1| - |X_2|\right)$$

from the definition of $\tilde{\psi}$ and $g$. Define a map $\Psi : K_m^{\psi} \to K_m^{\tilde{\psi}}$ by $\Psi\left((v, i)\right) = (v, g(v)i)$. Then $\Psi$ is a covering isomorphism and

$$\begin{aligned} |\partial_{\tilde{\psi}} W| &= |W_1|(m - |W_1|) + |W_2|(m - |W_2|) + 2|\tilde{\psi}^{-1}(-1) \cap E(W_1 - W_2)| \\ &\leq |W_1|(m - |W_1|) + |W_2|(m - |W_2|) + 2|\psi^{-1}(-1) \cap E(X_1 - X_2)| \\ &\quad +2(|X_1| - |X_2|) \\ &= (|X_1| + 1)(m - |X_1| - 1) + (|X_2| - 1)(m - |X_2| + 1) \\ &\quad +2\left|\{e \in E(X_1 - X_2) : \psi(e) = -1\}\right| + 2(|X_1| - |X_2|) \\ &= |\partial_\phi X| - 2. \end{aligned}$$

Take $Y = \Phi^{-1}(\Psi^{-1}(W))$. Then $|Y_1| = |X_1| + 1$, $|Y_2| = |X_2| - 1$, and

$$|\partial_\phi X| - |\partial_\phi Y| = |\partial_\phi X| - |\partial_\psi \Phi(Y)| = |\partial_\phi X| - |\partial_{\tilde{\psi}} W|.$$

Hence, $|\partial_\phi X| - |\partial_\phi Y| > 0$. This completes the proof. □

It follows from Lemma 2 that every isoperimetric set $X$ of a double covering $K_m^\phi$ of $K_m$ must satisfy the equation $|X| = |X_1|$.

**Lemma 3** *Let $\phi$ be a signing in $C^1(K_m; \mathbb{Z}_2)$. If $X \subset V(K_m) \times \{1, -1\}$ and $|X| = |X_1| < m$, then there exists a subset $Y$ of $V(K_m) \times \{1, -1\}$ such that $|Y| = |Y_1|$, $|Y| = |X| + 1$ and $\frac{|\partial_\phi X|}{|X|} > \frac{|\partial_\phi Y|}{|Y|}$.*

Proof: By the same method as in the proof of Lemma 2, we can take a signing $\psi \in C^1(K_m;\mathbb{Z}_2)$ and a covering isomorphism $\Phi : K_m^\phi \to K_m^\psi$ such that $\Phi(X) = X_1 \times \{1\}$ and

$$|\partial_\phi X| = |\partial_\psi \Phi(X)| = |X_1|(m - |X_1|) + 2\,|\{e \in E(X_1) : \psi(e) = -1\}|\,.$$

Let $W = (X_1 \cup \{v_\infty\}) \times \{1\}$ with a fixed vertex $v_\infty \notin X_1$. Then $|W| = |W_1|$. Define a function $g : V(K_m) \to \mathbb{Z}_2$ by $g(v) = 1$ for $v \neq v_\infty$ and

$$g(v_\infty) = \begin{cases} -1 & \text{if } \ |\{u \in X_1 : \psi(uv_\infty) = -1\}| \geq \frac{|X_1|+1}{2}, \\ 1 & \text{otherwise.} \end{cases}$$

Define another signing $\tilde{\psi}$ of $K_m$ by $\tilde{\psi}(uv) = g(v)\psi(uv)g(u)^{-1}$ and a map $\Psi : K_m^\psi \to K_m^{\tilde{\psi}}$ by $\Psi(v,i) = (v, g(v)i)$. Then $\Psi$ is a covering isomorphism. Note that

$$\begin{aligned} |\partial_{\tilde{\psi}} W| &= |W_1|\,(m - |W_1|) + 2\left|\left\{e \in E(W_1) : \tilde{\psi}(e) = -1\right\}\right| \\ &\leq (|X_1| + 1)\,(m - |X_1| - 1) + 2\,|\{e \in E(X_1) : \psi(e) = -1\}| + |X_1|. \end{aligned}$$

Take $Y = \Phi^{-1}(\Psi^{-1}(W))$. Then $|Y| = |X| + 1$, $|Y| = |Y_1|$ and

$$\begin{aligned} \frac{|\partial_\phi Y|}{|Y|} &= \frac{|\partial_\psi \Phi(Y)|}{|X|+1} = \frac{|\partial_{\tilde{\psi}} W|}{|X|+1} \\ &\leq m - |X| - 1 + \frac{2}{|X|+1}\,|\{e \in E(X_1) : \psi(e) = -1\}| + \frac{|X|}{|X|+1}. \end{aligned}$$

Since

$$\frac{|\partial_\phi X|}{|X|} = m - |X| + \frac{2}{|X|}\,|\{e \in E(X_1) : \psi(e) = -1\}|\,,$$

we have

$$\frac{|\partial_\phi X|}{|X|} - \frac{|\partial_\phi Y|}{|Y|} \geq 1 - \frac{|X|}{|X|+1} > 0.$$

This completes the proof. □

Lemmas 2 and 3 say that an isoperimetric set of a double covering $K_m^\phi$ of $K_m$ is a set of $m$ vertices of $K_m^\phi$ each of which is taken from each fibre of $K_m$. Therefore, we obtain

**Theorem 3** *For any signing $\phi$ in $C^1(K_m;\mathbb{Z}_2)$, the bisection width of the double covering $K_m^\phi$ of $K_m$ equals $m$ times its isoperimetric number $i(K_m^\phi)$.*

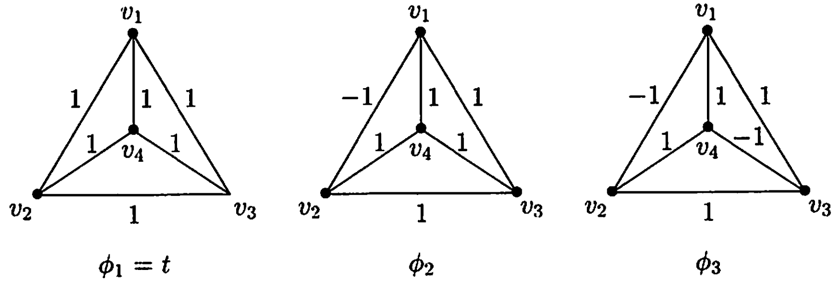



Figure 2: Three non-equivalent signings

Now, we give a computation for $i(K_m^\phi)$ in terms of the degree of unbalance $d(K_{m_\phi})$ of $K_{m_\phi}$.

**Theorem 4** *For any signing $\phi$ in $C^1(K_m;\mathbb{Z}_2)$, we have*

$$i(K_m^\phi) = \frac{2}{m} d(K_{m_\phi}).$$

Proof: By Lemmas 2 and 3, we get

$$i(K_m^\phi) = \frac{1}{m} \min \{|\partial_\phi X| : |X| = m = |X_1|\}.$$

Let $X \subset V(K_m) \times \{1, -1\}$ be a subset satisfying $|X| = m = |X_1|$. Then there exists a signing $\psi \in C^1(K_m;\mathbb{Z}_2)$ as shown in the proof of Lemma 2 such that $\phi \sim \psi$ and

$$|\partial_\phi X| = |\partial_\psi (V(K_m) \times \{1\})| = 2|\psi^{-1}(-1)|.$$

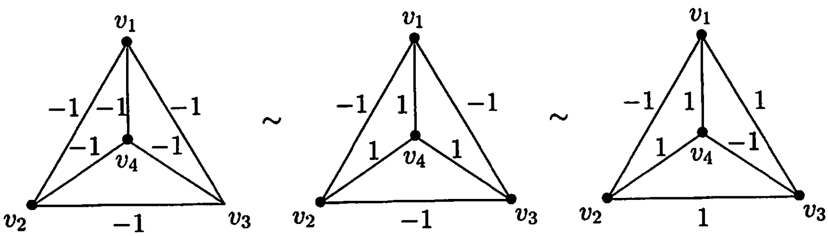
Since $|V(K_m) \times \{1\}| = m$,

$$i(K_m^\phi) = \frac{1}{m} \min \{|\partial_\phi X| : |X| = m = |X_1|\} = \frac{2}{m} \min \left\{|\psi^{-1}(-1)| : \phi \sim \psi\right\}.$$

Now, the theorem comes from Lemma 1 (3). □

**Example 1** Consider three signings $\phi_i$ on $K_4$ as illustrated in Figure 2. For each $i = 1, 2, 3$, $|\phi_i^{-1}(-1)| \le |\phi_{i_X}^{-1}(-1)|$ for any subset $X$ of $V(K_4)$. This implies that $d(K_{4_{\phi_1}}) = 0$, $d(K_{4_{\phi_2}}) = 1$ and $d(K_{4_{\phi_3}}) = 2$, so that

$$i(K_4^{\phi_i}) = \frac{1}{2} d(K_{4_{\phi_i}}) = \begin{cases} 0 & \text{if } i = 1, \\ 1/2 & \text{if } i = 2, \\ 1 & \text{if } i = 3. \end{cases}$$




$\phi$ $\quad$ $\phi_{\{v_4\}}$ $\quad$ $\phi_{\{v_3,v_4\}}$

Figure 3: Equivalence by switching the signs

Note that these isoperimetric numbers for $i = 1, 2, 3$ are all distinct, and hence the three double coverings $K_4^{\phi_i}$, $i = 1, 2, 3$, of $K_4$ are not isomorphic to one another.

Let $\phi$ be the signing on $K_4$ such that $\phi(uv) = -1$ for all $uv \in D(K_4)$. The steps in Figure 3 show that $\phi$ is equivalent to $\phi_3$ given in Figure 2. Note that $\phi_{\{v_4\}}$ is obtained from $\phi$ by switching the signs of all values of the edges incident with $v_4$, and it gives the first equivalence relation $\sim$ in Figure 4. In fact, if we take $f : V(K_4) \to \mathbb{Z}_2$ such that $f(v_4) = -1$ and $f(v_i) = 1$ for $i = 1, 2, 3$, then $\phi_{\{v_4\}}(v_iv_j) = f(v_j)\phi(v_iv_j)f(v_i)^{-1}$ for all $v_iv_j \in E(K_4)$. Similarly, we can get the second $\sim$ in Figure 3 by switching the signs of all values of the edges incident with $v_3$. This implies that $\phi$ and $\phi_{\{v_3,v_4\}}$ are equivalent and $d(K_{4_\phi}) = 2$. In fact, $K_4^\phi$ is the 3-dimensional hypercube $Q_3$. □

Akiyama *et al.* gave an upper bound of the number $d(G_\phi)$ as follows:

**Theorem 5** [1] *Let $G$ be a graph and let $G_1, G_2, \ldots, G_\ell$ be vertex disjoint subgraphs of $G$ so that $\sum_{i=1}^{\ell} |V(G_i)| = |V(G)|$. Let $\phi \in C^1(G; \mathbb{Z}_2)$ be a signing. If $(G_i, \phi|_{E(G_i)})$ is balanced for each $i = 1, 2, \ldots, \ell$, then*

$$d(G_\phi) \le \frac{1}{2}\left(|E(G)| - \sum_{i=1}^{\ell} |E(G_i)|\right).$$

□

By using Theorem 5, they showed that for any $\phi \in C^1(K_m; \mathbb{Z}_2)$,

$$d(K_{m_\phi}) \le \left\lfloor \frac{1}{4}(m-1)^2 \right\rfloor,$$

with the equality whenever the vertex set $V(K_m)$ may be partitioned into two classes in such a way that an edge is positive if and only if its two endpoints belong to distinct classes.

## 3 Spectral estimation

Given a signed graph $G_\phi$, its adjacency matrix $A(G_\phi) = (a_{ij})$ is a square matrix of order $|V(G)|$ defined by

$$a_{ij} = \begin{cases} 1 & \text{if } v_iv_j \text{ is positive} \\ -1 & \text{if } v_iv_j \text{ is negative} \\ 0 & \text{if } v_iv_j \text{ is not an edge} \end{cases}$$

for $1 \le i, j \le |V(G)|$, and its characteristic polynomial is $\det(\lambda I - A(G_\phi))$. By $P(G_\phi; \lambda)$, we denote the characteristic polynomial of $G_\phi$.

From now on, let $\lambda_1(A)$ ($\lambda_M(A)$ resp.) denote the second smallest (the largest resp.) eigenvalue of a matrix $A$. Let $L(G) = \mathrm{diag}(\deg(v)) - A(G)$ be the *Laplacian matrix* of a graph $G$. It is well-known that every eigenvalue of the Laplacian matrix $L(G)$ of a graph $G$ is nonnegative and its smallest eigenvalue is zero. Note that $A(G) = A(G_t)$ and hence $P(G;\lambda) = P(G_t;\lambda)$. Mohar proved the following theorem.

**Theorem 6** [18] *Let $G$ be a graph with $|V(G)| \ge 4$. Then*

$$\frac{\lambda_1(L(G))}{2} \le i(G) \le \sqrt{\lambda_1(L(G))\,(2\Delta(G) - \lambda_1(L(G)))}.$$

Chae *et al.* computed the characteristic polynomial of a double covering of a given graph $G$ as follows:

**Theorem 7** [5] $P(G^\phi;\lambda) = P(G;\lambda)P(G_\phi;\lambda)$ *for any* $\phi \in C^1(G;\mathbb{Z}_2)$.

Let $G$ be a regular graph of degree $k$. Then the characteristic polynomial of the Laplacian matrix $L(G)$ of $G$ is given by

$$\begin{aligned} \det(\lambda I - L(G)) &= \det(\lambda I - \mathrm{diag}(\deg(v)) + A(G)) \\ &= \det(\lambda I - kI + A(G)) \\ &= (-1)^{|V(G)|} P(G; -\lambda + k). \end{aligned}$$

Furthermore, any double covering of a regular graph of degree $k$ is also a regular graph of the same degree. So the characteristic polynomial of the Laplacian matrix $L(G^\phi)$ is given by

$$\begin{aligned} \det(\lambda I - L(G^\phi)) &= (-1)^{|V(G)|} P(G^\phi; -\lambda + k) \\ &= (-1)^{|V(G)|} P(G; -\lambda + k) P(G_\phi; -\lambda + k), \end{aligned}$$

by Theorem 7. Hence, we have

$$\lambda_1(L(G^\phi)) = \min\{\lambda_1(L(G)),\, k - \lambda_M(A(G_\phi))\}$$

for any regular graph $G$ of degree $k$.

**Theorem 8** *Let $G$ be a graph having more than two vertices. Then we have*

$$\frac{|V(G)|}{4}\lambda_1(L(G^\phi)) \le d(G_\phi).$$

*In particular, if $G$ is a regular graph of degree $k$, then we have*

$$\frac{|V(G)|}{4}\min\{\lambda_1(L(G)),\, k - \lambda_M(A(G_\phi))\} \le d(G_\phi).$$

Proof: By Theorems 2 and 6, we have

$$\frac{1}{2}\lambda_1(L(G^\phi)) \le i(G^\phi) \le \frac{2}{|V(G)|}d(G_\phi),$$

which gives the proof. □

In particular, if $G$ is the complete graph $K_m$, then we have

$$\begin{aligned}\lambda_1(L(K_m^\phi)) &= \min\{\lambda_1(L(K_m)),\, m-1-\lambda_M(A(K_{m_\phi}))\}\\ &= m-1-\lambda_M(A(K_{m_\phi}))\end{aligned}$$

for any signing $\phi \in C^1(K_m;\mathbb{Z}_2)$, because $\lambda_1(L(K_m)) = m$ and

$$\frac{m}{4}\lambda_1(L(K_m^\phi)) \le d(K_{m_\phi}) \le \left\lfloor \frac{1}{4}(m-1)^2 \right\rfloor$$

for any $\phi \in C^1(K_m;\mathbb{Z}_2)$.

By using Theorems 6 and 8, we can get new bounds of the degree $d(K_{m_\phi})$ of unbalance and the isoperimetric number $i(K_m^\phi)$ in terms of the largest eigenvalue $\lambda_M(A(K_{m_\phi}))$ of $A(K_{m_\phi})$.

**Theorem 9** *For any signing $\phi$ in $C^1(K_m;\mathbb{Z}_2)$, we have*

$$\frac{m}{4}\left(m-1-\lambda_M(A(K_{m_\phi}))\right) \le d(K_{m_\phi}) \le \frac{m}{2}\sqrt{(m-1)^2 - \lambda_M(A(K_{m_\phi}))^2},$$

*and*

$$\frac{1}{2}\left(m-1-\lambda_M(A(K_{m_\phi}))\right) \le i(K_m^\phi) \le \sqrt{(m-1)^2 - \lambda_M(A(K_{m_\phi}))^2}.$$

Since $d(K_{m_\phi})$ is an integer for any $\phi \in C^1(K_m;\mathbb{Z}_2)$, we have a lower bound of $i(K_m^\phi)$ as follows. Note that it is sharper than that given in Theorem 6.

**Corollary 5** *For any signing $\phi$ in $C^1(K_m;\mathbb{Z}_2)$, we have*

$$\frac{2}{m}\left\lceil \frac{m}{4}\left(m-1-\lambda_M(A(K_{m_\phi}))\right)\right\rceil \le i(K_m^\phi).$$

# 4 Range of the isoperimetric numbers

A graph $G$ is said to be *minimal* if it has the minimal number of edges among the graphs in the Seidel switching equivalence class $\{G_X : X \subset V(G)\}$ of $G$. The following comes from Corollary 2 and Lemma 1 (3).

**Theorem 10** *For a signing $\phi$ in $C^1(K_m;\mathbb{Z}_2)$, the support $spt(\phi)$ of $\phi$ is minimal if and only if $d(K_{m_\phi}) = |E(spt(\phi))|$.*

**Lemma 4** (1) *A graph $G$ is minimal if and only if $2|\partial X| \le |X|\,|V(G)-X|$ for any nonempty proper subset $X$ of $V(G)$.*

(2) *Every spanning subgraph of a minimal graph is also minimal.*

Proof: The statement (2) can be proved easily by using (1). To prove (1), let $X$ be a nonempty subset of $V(G)$ and let $G[X]$ be the subgraph of $G$ induced by $X$. Then

$$|E(G)| = |E(G[X])| + |E(G[V(G)-X])| + |\partial X|$$

and

$$|E(G_X)| = |E(G[X])| + |E(G[V(G)-X])| + |X|\,|V(G)-X| - |\partial X|.$$

This implies that

$$|E(G)| - |E(G_X)| = 2|\partial X| - |X|\,|V(G)-X|.$$

Thus, $G$ is minimal if and only if $2|\partial X| \le |X|\,|V(G)-X|$ for any nonempty proper subset $X$ of $V(G)$. □

As the final part of this paper, we determine which numbers can be the isoperimetric numbers of double coverings of a complete graph $K_m$. Or, equivalently, we determine which numbers can be the degree of unbalance of a signed graph $K_{m_\phi}$.

First, recall that an upper bound of $d(K_{m_\phi})$ is given at the end of section 3: for any $\phi \in C^1(K_m;\mathbb{Z}_2)$,

$$d(K_{m_\phi}) \leq \left\lfloor \frac{1}{4}(m-1)^2 \right\rfloor,$$

with the equality whenever the vertex set $V(K_m)$ may be partitioned into two classes in such a way that an edge is positive if and only if its two endpoints belong to distinct classes. Let $G = K_{\lfloor \frac{m}{2} \rfloor} \cup K_{\lceil \frac{m}{2} \rceil}$, disjoint union, as a subgraph of $K_m$, and let $\phi \in C^1(K_m;\mathbb{Z}_2)$ be a signing such that $\phi(e)$ is negative if and only if $e$ is an edge of $G$. Then $G = spt(\phi)$ and it is minimal, because we can show that $2|\partial X| \leq |X|\,|V(G) - X|$ for any nonempty proper subset $X$ of $V(G)$. By Theorem 10, we have

$$d(K_{m_\phi}) = |E(G)| = \left\lfloor \frac{1}{4}(m-1)^2 \right\rfloor.$$

Now, by taking all spanning subgraphs of $G$, which are also minimal, we have

**Corollary 6** *For any* $0 \leq d \leq \lfloor \frac{1}{4}(m-1)^2 \rfloor$, *there exists a signing* $\phi$ *in* $C^1(K_m;\mathbb{Z}_2)$ *such that* $d(K_{m_\phi}) = d$ *or* $i(K_m^\phi) = \frac{2d}{m}$.

The last corollary says that the isoperimetric number of a double covering of the complete graph $K_m$ is one of $\frac{2d}{m}$ for $d = 0, 1, \ldots, \lfloor \frac{1}{4}(m-1)^2 \rfloor$.

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