IDENTITIES AND GENERATING FUNCTIONS FOR CERTAIN CLASSES OF *F*- PARTITIONS

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ABSTRACT: Four generalized theorems involving *F*–partitions and (n+t) – color partitions are proved combinatorially. Each of these theorems gives us infinitely many partition identities. We obtain new generating functions for *F*–partitions and discuss some particular cases which provide elegant Rogers–Ramanujan type identities for *F*–partition.

1. INTRODUCTION, DEFINITIONS AND THE MAIN RESULTS

Using a technique of [3] we establish four generalized identities between *F*–partitions and (n+t) – color partitions. These identities help us in finding generating functions for *F*–partitions from the known generating functions for (n+t) – color partitions. These generating functions in some particular cases provide us elegant Rogers–Ramanujan type identities for *F*–partitions. We first recall following definitions :

Definition 1. [4, p.1]. A two rowed arrays of non–negative integers

$$\begin{pmatrix} a_1\ a_2 \ldots a_r \\ b_1\ b_2 \ldots b_r \end{pmatrix}$$

with each row arranged in non increasing order is called a generalized Frobenius partition or more simply an *F* – partition of ν if.

$$\nu = r + \sum_{i=1}^{r} a_i + \sum_{i=1}^{r} b_i.$$

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Definition 2. [3]. An (n+t) – color partition, $t \geq 0$ is a partition in which a part of size n, $n \geq 0$ can come in (n+t) different colors denoted by subscripts : n_1, n_2 .... n_{n+t}. In the part n_i, n can be zero if and only if $i \geq 1$. But in no partition are zeros permitted to repeat.

Definition 3. [3]. The weighted difference of two parts m_i, n_j $m \geq n$ is defined by $m - n - i - j$ and is denoted by $((m_i - n_j))$.

The Rogers – Ramanujan identities were stated combinatorially by P.A. Mac Mohan as follows [5, Theorems. 364, 365, p.291].

1. The number of partitions of n into parts with minimal difference 2 equals the number of partitions of n into parts which are congruent to ± 1 (mod 5).

2. The number of partitions of n with minimal part 2 and minimal difference 2 equals the number of partitions of n into parts which are congruent to ± 2 (mod 5).

For k = { — 1, 1, 3, 5, ...} we shall prove in this paper the following four theorems :

Theorem 1. *Let A_1^k (v) denote the number of F–partitions of v such that*

(1.a) $a_i \geq b_i$, and
(1.b) $b_i \geq (k+3)/2 + a_{i+1}$

Let B_1^k (ν) denote the number of n–color partitions of ν such that

(1.c) even parts appear with even subscripts and odd with odd, and

(1.d) The weighted difference of each pair of parts is greater than k.

Then $A_1^k(\nu) = B_1^k(\nu)$ for all ν.

Theorem 2. Let A_2^k (ν) denote the number of *F*–partitions of ν such that

(2.a) $a_i \geq b_i$, and
(2.b) $b_i \geq (k+1)/2 + a_{i+1}$
Let B_2^k (ν) denote the number of n–color partitions of ν such that

(2.c) even parts appear with even subscripts and odd with odd subscripts greater than 1, and

(2.d) The weighted difference of each pair of parts is greater than or equal to k–1.

Then $A_2^k(\nu) = B_2^k(\nu)$ for all ν.

Theorem 3. Let $A_3^k(\nu)$ denote the number of F–partitions of ν such that

(3.a) $a_i \neq 1$.

(3.b) $a_i \leq 2 + b_i$.

(3.c) $a_i > (k+7)/2+b_{i+1}$

(3.d) if $a_r > 0$ then $a_r > (k+3)/2$.

Let $B_3^k(\nu)$ denote the number of (n+2) –color partitions of ν such that

(3.e) if m_i is not the smallest part, the m–i > 0.

(3.f) even parts appear with even subscripts and odd with odd.

(3.g) the weighted difference of each pair of parts is greater than k.

(3.h) the smallest part is of the form i_{i+2}.

Then $A_3^k(\nu) = B_3^k(\nu)$ for all ν.

Theorem 4. Let $A_4^k(\nu)$ denote the number of F–partitions of ν such that

(4.a) $a_i \neq 1$.

(4.b) $a_i < 2 + b_i$.

(4.c) $a_i \geq (k+5)/2+b_{i+1}$

(4.d) if $a_r > 0$ then $a_r > (k+1)/2$.

Let $B_4^k(\nu)$ denote the number of (n+2) –color partitions of ν such that

(4.e) if m_i is not the smallest part, the m–i > 0.

(4.f) even parts appear with even subscripts and odd with odd greater than 1.

(4.g) the weighted difference of each pair of parts is greater than or equal to k–1.

(4.h) the smallest part is of the form i_{i+2}.

Then $A_4^k(\nu) = B_4^k(\nu)$ for all ν.

Remark : In Theorems 1 – 4 the conditions on $\mathcal{F}$–partitions clearly imply $a_i \geq (k+3)/2 + a_{i+1}$ and $b_i \geq (k+3)/2 + b_{i+1}$.

In the next section we give a detail proof of Theorem 3 to make all ideas clear and in Section 3 we sketch the changes required to prove the other theorems. Section 4 is devoted to a study of generating functions for $\mathcal{F}$–partitions while in Section 5 we discuss some particular cases. We conclude in Section 6 by posing two open problems.

2. PROOF OF THEOREM 3

We establish a 1–1 correspondence between the $\mathcal{F}$–partitions enumerated by $A_3^k(\nu)$ and the (n+2)– color partitions enumerated by $B_3^k(\nu)$. We do this by mapping each column $\begin{smallmatrix} a \\ b \end{smallmatrix}$ of the F–partition to a single part m_i of a partition enumerated by $B_3^k(\nu)$. The mapping ϕ is

$$\phi \begin{pmatrix} a \\ b \end{pmatrix} = (a+b+1)_{b-a+1} \qquad (2.1)$$

and the inverse mapping ϕ^{-1} is given by

$$\phi^{-1}(m_i) = \begin{pmatrix} (m-i+2)/2 \\ (m+i-4)/2 \end{pmatrix} \qquad (2.2)$$

Now for any two adjacent columns $\begin{matrix} a & c \\ b & d \end{matrix}$

in the F–partition enumerated by $A_3^k(\nu)$

with $\phi \begin{pmatrix} a \\ b \end{pmatrix} = m_i$ and $\phi \begin{pmatrix} c \\ d \end{pmatrix} = n_j$ (defined by (2.1), we have

$$m-i = 2a-2 \qquad (2.3)$$

and $((m_i-n_j)) = 2a-2d-6$ (2.4)

Clearly (3.a) and (2.3) imply (3.e). In the definition (2.1) the fact that a+b+1 and b–a+3 have the same parity implies (3.f). (3.c) and (2.4) imply (3.g). (3.d) guarantees (3.h). For if, $a_r = 0$ then $\begin{pmatrix} a_r \\ b_r \end{pmatrix}$ corresponds to $(b_r+1)_{b_r+3}$ which is of the form i_{i+2} and if $a_r > 0$ then we attach the "phantom" column $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ which corresponds to the part 0_2 which is allowed as a part by the definition of (n+2)– color partitions. Now (3.d) ensures that (3.g) is satisfied in this case by the two parts which correspond to $\binom{a_r}{b_r}$ and $\binom{0}{-1}$.

To see the reverse implication, we note that

$$\phi^{-1}\ (m_i) = \begin{pmatrix} (m-i+2)/2 \\ (m+i-4)/2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

and $$\phi^{-1}\ (n_j) = \begin{pmatrix} (n-j+2)/2 \\ (n+j-4)/2 \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$$

that is,

$$a = (m-i+2)/2 \quad (2.5)$$

$$b \quad (m+i-4)/2 \quad (2.6)$$

$$c = (n-j+2)/2 \quad (2.7)$$

$$d = (n+j-4)/2 \quad (2.8)$$

and so,

$$b-a = i-3 \quad (2.9)$$

$$d-c = j-3 \quad (2.10)$$

$$a-d = \frac{1}{2}((m_i-n_j))+3 \quad (2.11)$$

Now (3.e), (2.5) and (2.7) imply (3.a). (2.9) and (2.10) imply (3.h). (3.c) follows from (3.g) and (2.11). (3.h) implies that there is a column of the form $\binom{0}{i-1}$ Such a column has to be last in the F-partition and i_{i+2} must be the smallest part of its partition, since if $i_{i+2} > n_j$ then

$$(i_{i+2}-n_j) = -2-n-j$$

which can not be greater than k since $k \in \{-1, 1, 3, 5, \ldots\}$.

Now 0_2 is allowed to be a part in partitions enumerated by $B_3^k(\nu)$. 0_2

corresponds to the "phantom" column $\binom{0}{-1}$, which is dropped from the corresponding F–partition. In fact 0_2 occurs as a part only to make it sure that (3.d) is satisfied. This completes the proof of Theorm 3.

To illustrate the bijection we have constructed we close this section with the example for k= −1, ν = 8 shown in the following table:

F–partitions enumerated by A_3^{-1} (8)	Image under ϕ, that is (n+2)–color partitions enumerated by B_3^{-1} (8)
$\binom{0}{7}$	8_{10}
$\binom{4}{3}$	8_2+0_2
$\binom{3}{4}$	8_4+0_2
$\binom{2}{5}$	8_6+0_2
$\begin{pmatrix} 4 & 0 \\ 2 & 0 \end{pmatrix}$	7_1+1_3
$\begin{pmatrix} 3 & 0 \\ 3 & 0 \end{pmatrix}$	7_3+1_3
$\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$	$8_1+3_{1+}0_2$

3. SKETCH OF THE PROOFS OF THEOREMS 1, 2 AND 4

Proof of Theorem 1.

The map ϕ is

$$\phi : \binom{a}{b} \to (a+b+1)_{a-b+1}, \quad a \geq b$$

and ϕ^{-1} is given by

$$\phi^{-1} : m_i \to \binom{(m+i-2)/2}{(m-i)/2}$$

Proof of theorem 2.

The map ϕ is

$$\phi : \binom{a}{b} \rightarrow (a+b+1)_{a-b+1}, \qquad a > b$$

and ϕ^{-1} is given by

$$\phi^{-1} : m_i \rightarrow \binom{(m+i-2)/2}{(m-i)/2}, \qquad m \neq i,\ i \neq 1$$

Proof of Theorem 4.

The map ϕ is

$$\phi : \binom{a}{b} \rightarrow (a+b+1)_{b-a+1}, \qquad a < b+2,\ a \neq 1$$

and ϕ^{-1} is given by

$$\phi^{-1} : m_i \rightarrow \binom{(m-i+2)/2}{(m+i-4)/2}, \qquad i \neq 1$$

4. GENERATING FUNCTIONS FOR *F*–PARTITIONS

Using the generating functions for $B_i^k(\nu)$, $1 \leq i \leq 4$, obtained in [2], we get the following generating functions for $A_i^k(\nu)$, $1 \leq i \leq 4$:

$$\sum_{\nu=0}^{\infty} A_1^k(\nu)\, q^{\nu} = \sum_{n=0}^{\infty} \frac{q^{n[1+(k+3)(n-1)/2]}}{(q\,;\,q)_{2n}} \tag{4.1}$$

$$\sum_{\nu=0}^{\infty} A_2^k(\nu)\, q^{\nu} = \sum_{n=0}^{\infty} \frac{q^{n[2+(k+3)(n-1)/2]}}{(q;\,q)_{2n}} \tag{4.2}$$

$$\sum_{\nu=0}^{\infty} A_3^k(\nu)\, q^{\nu} = \sum_{n=0}^{\infty} \frac{q^{n[1+(n+1)(k+3)/2]}}{(q;\,q)_{2n+1}} \tag{4.3}$$

$$\sum_{\nu=0}^{\infty} A_4^k(\nu)\, q^{\nu} = \sum_{n=0}^{\infty} \frac{q^{n[(n+1)(k+3)/2]}}{(q;\,q)_{2n+1}}, \tag{4.4}$$

$$\text{where} \quad (a;q)_n = \prod_{i=0}^{\infty} \frac{(1-aq^i)}{(1-aq^{n+i})}.$$

5. PARTICULAR CASES

Clearly, for every value of k ∈ [−1, 1, 3, 5,....] each of our four theorems gives us one partition identify. Thus each theorem gives us infinitely many partition identities. But here we shall discuss only those particular cases which yield Rogers–Ramanujan Type Identities (abbreviated as RRTI in the sequel) for *F*–partitions. We divide this section into four subsections and discuss the particular cases of each theorem separately.

5.a. Particular cases of Theorem 1

Case 1. When k = − 1

In view of the identity [6, (79) – (98)]

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_{2n}} \quad \frac{1}{(q;q)_\infty} \prod_{n=1}^{\infty} (1-q^{10n-2})(1-q^{10n-8})(1-q^{20n-4}) \times (1-q^{20n-6})(1-q^{10n}) \qquad (5.a.1)$$

we get the following RRTI for *F*–partitions :

Corollary (1.a). The number of *F*–partitions of ν with $a_i \geq b_i$ and $b_i \geq 1+a_{i+1}$ equals the number of partitions of ν into parts ≡ 0, ± 2 ± 6, ± 8, 10 (mod 20).

Case 2. when k = 1

In view of the identity [6, (84) – (85)]

$$\sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(q;q)_{2n}} = \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(q;q)_{2n+1}} = \prod_{n=0}^{\infty} \frac{1}{1-q^{2n+1}}, \qquad (5.a.2)$$

we are led to the following RRTI for *F*–partitions:

Corollary (1.b). The number of *F*–partitions of ν with $a_i \geq b_i$ and $b_i \geq 2+a_{i+1}$ equals the number of partitions of ν into odd parts.

5.b Particular cases of Theorem 2

Case 1. When $k = -1$

In view of the identify [6, (99)]

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_{2n}} = \frac{1}{(q;q)_\infty} \prod_{n=1}^{\infty} (1-q^{10n-1})(1-q^{10n-9})(1-q^{20n-8}) \times (1-q^{20n-12})(1-q^{10n}) \qquad (5.b.1)$$

We obtain the following RRTI for F–partitions :

Corollary (2.a). The number of F–partitions of ν with $a_i > b_i$ and $b_i \geq a_{i+1}$ equals the number of partitions of ν into parts $\equiv 0, \pm 1, \pm 8, \pm 9, 10$ (mod 20).

Case 2. When $k = 1$

In view of the identify [6, (39) – (83)]

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q)_{2n}} = \frac{1}{(q;q)_\infty} \prod_{n=1}^{\infty} (1-q^{8n-1})(1-q^{8n-7})(1-q^{16n-10}) \times (1-q^{16n-6})(1-q^{8n}), \qquad (5.b.2)$$

we get the following RRTI for F–partitions :

Corollary (2.b). The number of F–partitions of ν with $a_i > b_i$ and $b_i \geq 1+a_{i+1}$ equals the number of partitions of ν into parts $\equiv 0, \pm 1, \pm 6, \pm 7, 8$ (mod 16).

5.c. Particular cases of Theorem 3

Case 1, when $k = -1$

In view of the identify [6, (96)]

$$\sum_{n=0}^{\infty} \frac{q^{n(n+2)}}{(q;q)_{2n+1}} = \frac{1}{(q;q)_\infty} \prod_{n=1}^{\infty} (1-q^{10n-4})(1-q^{10n-6}) \times (1-q^{20n-18})(1-q^{20n-2}) \times (1-q^{10n}), \qquad (5.c.1)$$

we are led to the following RRTI for F–partitions.

Corollary 3.a. The number of F–partitions of ν with $1 \neq a_i \leq 2+b_i$ and $a_i \geq 3+b_{i+1}$ equals the number of partitions of ν into parts $\not\equiv 0, \pm 2, \pm 4, \pm 6, 10$ (mod 20).

5.d. Particular cases of Theorem 4

Case 1. When k = − 1

In view of the identify [6, (94)]

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_{2n+1}} = \frac{1}{(q;q)_\infty} \prod_{n=1}^{\infty} (1-q^{10n-3})(1-q^{10n-7}) \times (1-q^{20n-16})(1-q^{20n-4}) \times (1-q^{10n}), \qquad (5.d.1)$$

we get the following RRTI for *F*–partitions :

Corollary 4 (a). The number of *F*-partitions of ν with $1 \neq a_i < 2+b_i$ and $a_i \geq 3+b_{i+1}$ equals the number of partitions of ν into parts $\equiv 0, \pm 3, \pm 4, \pm 7, 10 \pmod{20}$.

Case 2, when k = 1

In view of the identify [6, (38) – (86)]

$$\sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q)_{2n+1}} = \frac{1}{(q;q)_\infty} \prod_{n=1}^{\infty} (1-q^{8n-3})(1-q^{8n-5}) \times (1-q^{16n-14})(1-q^{16n-2}) \times (1-q^{8n}), \qquad (5.d.2)$$

we obtain the following RRTI for *F*–partitions :

Corollary 4 (b). The number of *F*-partitions of ν with $1 \neq a_i < 2 + b_i$ and $a_i \geq 3 + b_{i+1}$ equals the number of partitions of ν into parts $\equiv 0, \pm 2, \pm 3, \pm 5, 8 \pmod{16}$.

6. CONCLUSION

We remark that in the above corollaries if we replace *F*–partitions by corresponding (n+t)– color partitions we get all results of [1]. Thus the work done here generalizes the results of [1]. From this work many questions arise. Most obvious among them are :

(1) Do Theorems 1–4 have nice analytic counter parts?

(2) We have found Rogers–Ramanujan Type Identities by using Theorems 1–4 for some particular values of k, is it possible to find them for general value of k?

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