# Spanning Trees Orthogonal to One-Factorizations of $K_{2n}$

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Abstract: In [3] Brualdi and Hollingsworth conjectured that for any one-factorization $\mathcal{F}$ of $K_{2n}$ there exists a decomposition of $K_{2n}$ into spanning trees orthogonal to $\mathcal{F}$. They also showed that two such spanning trees always existed. We construct three such trees and exhibit an infinite class of complete graphs with an orthogonal decomposition into spanning trees with respect to the one-factorization $GK_{2n}$.

A *one-factorization* $\mathcal{F}$ of the complete graph $K_{2n}$ is a partition of $E(K_{2n})$ into spanning subgraphs $F_1, F_2, \ldots, F_{2n-1}$ called *one-factors* so that each $F_i$ is regular of degree one. See [4] for a survey on one-factorizations of the complete graph. A subgraph $H$ of $K_{2n}$ is said to be *orthogonal* to $\mathcal{F}$ if $|E(H) \cap E(F_i)| = 1$, $1 \leq i \leq 2n-1$. (If, instead of a one-factorization we consider an edge-coloring of $K_{2n}$ so that each color class is a one-factor, then $H$ is said to be *multicolored* as in [3].) A survey of results on orthogonal subgraphs can be found in [1].

Following the notation of [2] we will denote by $GK_{2n}$ the "standard" one-factorization of $K_{2n}$ constructed in the following manner. Label one vertex with $\infty$ and the rest of the vertices with $0, 1, \ldots, 2n-2$. A one-factor consists of the edge $\{\infty, a\}$ (which we call the infinite edge) along with all (finite) edges of the form $\{x, y\}$ where $x + y \equiv 2a \pmod{2n-1}$. Call $a$ the *center* of each of these edges

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and define the *length* of an edge to be $\ell(x,y) = \min\{x-y \pmod{2n-1}, y-x \pmod{2n-1}\}$, where $x$ and $y$ are the finite labels on its endvertices.

In [3] it was shown that for any one-factorization $\mathcal{F}$ of $K_{2n}$ there exist two edge-disjoint spanning trees orthogonal to $\mathcal{F}$. We improve this result to three edge-disjoint spanning trees orthogonal to $\mathcal{F}$ and construct them in the process.

THEOREM 1. *If $\mathcal{F}$ is a one-factorization of $K_{2n}$ $(n > 2)$, then there exist three edge-disjoint spanning trees orthogonal to $\mathcal{F}$.*

*Proof.* First let $n = 3$. It is well-known that there is a unique one-factorization of $K_6$. Without loss of generality, we can consider this factorization to be $GK_6$. Then the three desired spanning trees are

$$\{\{\infty,2\},\{\infty,3\},\{\infty,4\},\{0,2\},\{1,4\}\};$$
$$\{\{\infty,0\},\{0,1\},\{0,4\},\{1,2\},\{3,4\}\};$$
$$\{\{\infty,1\},\{0,3\},\{1,3\},\{2,3\},\{2,4\}\}.$$

Now we assume that $n \geq 4$. Construct $T$ as described in [3], i.e. consider the spanning star $S_0$ on vertex $v_0$ and delete edges $\{v_0,v_1\} \in F_1$ and $\{v_0,v_2\} \in F_2$, forming $S_0'$. Then there is some edge $\{v_1,u_1\} \in F_2$ and an edge $\{v_2,u_2\} \in F_1$. (It is possible that $u_1 = u_2$.) Join these two to $S_0'$ to form $S_0''$, still an orthogonal spanning tree. Now consider the spanning star $S_1$ on $v_1$ and note that $S_1 \cap S_0'' = \{v_1,u_1\}$. To form $S_1'$ we delete from $S_1$ the edge $\{v_1,u_1\}$ and some other edge $\{v_1,v_3\} \in F_3$, $v_3 \neq v_0,v_2,u_2$, chosen so that $\{v_0,u_1\} \notin F_3$. (Say $\{v_0,u_1\} \in F_6$.) This is always possible since there are $2n-1$ one-factors and $n \geq 4$. Then we join to $S_1'$ the edge $\{u_1,w_1\} \in F_3$ and the edge $\{v_3,u_3\} \in F_2$ to form $S_1''$, a second orthogonal spanning tree. Note that it is possible that $w_1 = u_3$ or one or both of $w_1$ and $u_3$ is the same as $u_2$ or $w_1 = v_2$ if $u_2 \neq u_1$.

Now we can construct a third orthogonal spanning tree in the same way. Notice that the only vertices which may have degree greater than 2 in $S_0'' \cup S_1''$ are $v_0,v_1,u_1,u_2,u_3,w_1$. Now $v_3$ can be none of these: $v_3 \neq v_0,u_2$ since it is chosen to be distinct from $v_0,v_2,u_2$; $v_3 \neq v_1,u_3$ since $v_3$ is adjacent to both; $v_3 \neq u_1$ since $\{u_1,v_1\}$ and $\{v_3,v_1\}$ are edges in distinct one-factors; and $v_3 \neq w_1$ since $v_3$ is incident with exactly one edge in $F_3$ but $\{u_1,w_1\}$ and $\{v_1,v_3\}$ are both in $F_3$ and $u_1 \neq v_1$. Therefore the degree of $v_3$ is 2. So now consider the spanning star $S_3$ on $v_3$. Delete from $S_3$ the edge $\{v_3,u_3\} \in S_1'' \cap F_2$ and the edge $\{v_0,v_3\} \in S_0''$ which is also in, say, $F_4$. We now have $S_3'$. Now join to $S_3'$ the edges $\{v_0,v_2\} \in F_2$ and $\{u_3,v_4\} \in F_4$, forming $S_3''$, a third orthogonal spanning tree.

Notice, however, that nothing precludes the edge $\{u_3,v_1\} \in S_1''$ from also being in $F_4$ (i.e. $v_1 = v_4$) in which case the construction of $S_3''$ is prevented. One possible way around this is to let $u_3$ play the role of $v_3$ and construct $S_1'''$

instead of $S_1''$. Then $S_1''' = S_1'' \setminus \{\{v_1, u_3\}, \{u_1, w_1\}\} \cup \{\{v_1, v_3\}, \{u_1, v_6\}\}$, where $\{u_1, v_6\} \in F_4$. We know that $v_6$ can't be either $v_0$ or $v_1$. Let $S_4$ be the star centered at $u_3$. Then, $S_3''' = S_4 \setminus \{\{u_3, v_0\}, \{u_3, v_3\}\} \cup \{\{v_0, v_2\}, \{v_3, v_5\}\}$, where $\{u_3, v_0\}, \{v_3, v_5\} \in F_7$.

This gives the same problem as before if $v_5 = v_1$ and hence $F_7 = F_3$. However, in this case we must have the following sets of edges in the given one-factors: $\{u_3, v_3\} \in F_2, \{v_1, v_3\}, \{v_0, u_3\} \in F_3, \{v_1, u_3\}, \{v_0, v_3\} \in F_4$. Then we can modify the above construction, taking advantage of this symmetry. In constructing $S_0''$ we eliminate from $S_0$ not only $\{v_0, v_1\} \in F_1$ and $\{v_0, v_2\} \in F_2$ but also $\{v_0, v_3\} \in F_4$ and $\{v_0, u_3\} \in F_3$. These are replaced by $\{v_1, u_1\} \in F_2, \{v_2, u_2\} \in F_1, \{v_1, v_3\} \in F_3$ and $\{v_1, u_3\} \in F_4$.

Then the construction of $S_1''$ is modified by deleting from $S_1$ the edges $\{v_1, u_1\} \in F_2, \{v_1, v_3\} \in F_3$, and $\{v_1, u_3\} \in F_4$ as well as $\{v_1, u_4\} \in F_5$, where $F_5$ is chosen arbitrarily as long as it is distinct from $F_1, \ldots, F_4, F_6$. We replace these edges with $\{v_0, v_3\} \in F_4, \{v_3, u_3\} \in F_2, \{u_4, x_1\} \in F_3$, and $\{u_1, x_2\} \in F_5$. Finally, for $S_3''$ we delete from $S_3$ the edges $\{v_3, v_1\} \in F_3, \{v_3, u_3\} \in F_2, \{v_3, v_0\} \in F_4$, and $\{v_3, w_3\} \in F_5$. The construction is completed by replacing these with $\{v_0, v_2\} \in F_2, \{v_0, u_3\} \in F_3, \{v_1, u_4\} \in F_5$, and $\{w_3, y_3\} \in F_4$. ∎

Note that the given spanning trees of $K_6$ can be constructed with the algorithm of the proof if $v_0 = \infty, v_1 = 0, v_2 = 1, u_1 = 2, u_2 = 4, v_3 = 3$, but it is easier to display them than to prove that the construction produces three trees for any $v_0$.

Also in [3] it was conjectured that for any one-factorization $\mathcal{F}$ of $K_{2n}$ there exists a decomposition of the complete graph into spanning trees each orthogonal to $\mathcal{F}$. While this appears to be a difficult problem in general, we have found an infinite class of complete graphs with such an orthogonal decomposition with respect to $GK_{2n}$.

LEMMA 2: If $2n - 1$ is prime and there exists a spanning tree of $K_{2n}$ containing exactly one edge of each length $1, 2, \ldots, n-1$ which is orthogonal to the one-factorization $GK_{2n}$, then there exists a decomposition of $K_{2n}$ into spanning trees orthogonal to $GK_{2n}$.

*Proof*: Let $E_l$ be the set of all edges in $K_{2n}$ of length $l$ and let $e_l \in E_l$. Since each one-factor of $GK_{2n}$ contains exactly one edge of each length, a spanning tree consisting of $E_l - \{e_l\}$ along with the infinite edge to the center of $e_l$ is orthogonal to $GK_{2n}$. Thus the spanning tree described in the statement of the lemma defines which edges $e_l$ will be deleted from $E_l$ for $l = 1, 2, \ldots, n-1$ to form the other spanning trees. ∎

Without the spanning tree described in the statement of the lemma, if $2n-1 = p$ is prime the proof of the lemma describes the construction of $n-1$ spanning trees orthogonal to $GK_{2n}$, namely, $E_l - f_l$ together with the infinite edge centred at $f_l$, $1 \le l \le n-1$, where the $f_l$ are chosen so that no two have the same centre and $f_l$ has length $l$. In general, the construction provides $\frac{\phi(2n-1)}{2}$ spanning trees orthogonal to $GK_{2n}$ because $E_l$ is a tree if and only if $l$ and $2n-1$ are relatively prime.

THEOREM 3: If $2n-1 = p$ is a prime of the form $8m+7$ then there exists an orthogonal decomposition of $K_{2n}$ into spanning trees with respect to the one-factorization $GK_{2n}$.

*Proof*: We will construct a spanning tree of the type described in the lemma. To that end consider the group $G$ whose elements are $\{1, 2, \ldots, n-1\}$ and whose operation is defined as $a \cdot b = \min\{ab, p-ab\} \pmod{p}$. If $a \cdot b = p - ab$ we will say that the result has been reduced by absolute value since the group operation is the same as reducing $ab$ modulo $p$ to one of the values $\{-(n-1), \ldots, 0, 1, \ldots, n-1\}$ and then taking the absolute value of the result. Now consider the set $H = \{2^j | j = 0, 1, \ldots k\}$, where the multiplication performed as in $G$, and $k$ is the smallest value so that $2^{k+1} = 1$. Thus $2^k \equiv \pm\frac{p-1}{2} \pmod{p}$. For the following we calculate $2^j$ recursively, for $j \neq 0$, $2^j = 2 \cdot 2^{j-1}$. Then $2^j$, $j \neq 0$, is an odd number in $H$ iff $2^j$ has been reduced by absolute value. Thus if we were not taking absolute value, but leaving the numbers as positive or negative, each odd number in $H$ would correspond to a change of sign. Hence $2^k \equiv -\frac{p-1}{2} \equiv \frac{p+1}{2}$ iff $H$ contains an even number of odd numbers. Let $K = \{2^j | j = 0, 1, \ldots k\}$ where $2^j$ is reduced modulo $p$ but not by absolute value.

We now construct the spanning tree needed for the application of Lemma 2. We begin with the spanning star centered at $\infty$. From now on, whenever we add an edge of finite length, we also remove the infinite edge containing its center. So add all the edges with one endpoint 0 and the other endpoint in $K$, and remove the appropriate infinite edges. The center of each edge $\{0, 2^j\}$ is $2^{j-1}$ and is thus adjacent to 0 if $j \neq 0$. The center of the edge $\{0, 1\}$ is $\frac{p+1}{2}$ which is adjacent to 0 iff $H$ contains an even number of odd numbers. Thus we still have a connected spanning graph iff $H$ contains an even number of odd numbers.

If $H = G$ then the spanning tree is finished since there is an even number of odd numbers less than $\frac{p+1}{2}$ when $p = 8m+7$. If $H \neq G$ then $G$ can be partitioned by the cosets of $H$. Thus we continue the construction of the spanning tree by letting the elements of the corresponding cosets of $K$ determine the endpoints of new edges incident with 0. Again, we remove the infinite edges containing the centers of these new edges and the centers determined by $cK$ will also be in $cK$ iff there is an even number of odd numbers in $cH$.

Thus it remains to show that $H$ and $cH$ contain an even number of odd numbers. We show first that the parity of the number of odds (evens) in $H$ is the same as the parity of the number of odds (evens) in $cH$. We can assume that $c$ is odd. Otherwise we could write $c = 2^d(2r+1)$ and since $\pm 2^d \in H$, $cH = (2r+1)H$. Now suppose $H = \{1, b_1, \ldots, b_k\}$, where $b_i = 2^i$ reduced to an element of $G$, and consider $cH = \{c, cb_1, \ldots, cb_k\}$ unreduced modulo $p$ or by absolute value. Since $c$ is odd, the parity of $cb_i$ is the same as that of $b_i$. And when we reduce $cb_i$ to an element of $G$, note that the parity of $cb_i$ changes with each reduction by $p$ as well as with reduction by absolute value. Thus we want to show that the total number of reductions in $cH$ is even. We will do this by showing that $cb_i$ is reduced an odd number of times by $p$ iff $cb_{i-1}$ is reduced by absolute value.

Suppose that $lp < cb_i < (l+1)p$ so that $cb_i$ must be reduced $l$ times by $p$. Then consider $cb_{i-1}$ and note that $b_i \equiv 2b_{i-1}$ or $b_i \equiv p - 2b_{i-1}$. If $b_i \equiv 2b_{i-1}$ then $cb_{i-1} = \frac{cb_i}{2}$ so $\frac{lp}{2} < cb_{i-1} < \frac{(l+1)p}{2}$. Thus $cb_{i-1}$ is or is not reduced by absolute value as $l$ is odd or even.

If $b_i = p - 2b_{i-1}$ then $lp < cp - 2cb_{i-1} < (l+1)p$ or $\frac{c-l}{2}p > cb_{i-1} > \frac{c-(l+1)}{2}p$ and again $cb_{i-1}$ is or is not reduced by absolute value as $l$ is odd or even. Taking the same calculations backwards yields the converse. Thus $cb_{i-1}$ is reduced by absolute value iff $cb_i$ is reduced an odd number of times by $p$. Note that $cb_k$ is never reduced by absolute value since $b_k = \frac{p-1}{2}$ so $cb_k = \frac{cp-c}{2}$ and $(\frac{c-1}{2})p < \frac{cp-c}{2} < \frac{cp}{2}$. Thus if there is an even number of odd values in $H$ then the same will be true for all cosets of $H$ and the spanning tree can be completed.

Now if $p = 8m+7$ there are $2m+2$ odd values and $2m+1$ evens among the $4m+3$ finite edge lengths. Since the parity of the number of odd values remains the same among the cosets of $H$, and there is an odd number of cosets of $H$, each coset must contain an even number of odd values. ∎

Since by Dirichlet's Theorem there is an infinite number of primes of the form $8m+7$ we have an infinite family of $K_{2n}$ with an orthogonal decomposition. It should be noted that while it is true that the construction in the proof of Theorem 3 works for some prime numbers of the form $8m+1$, 73 and 89 for instance, it does not work in general since it is possible to have an even number of cosets each containing an odd number of odd values. It is not clear whether it works for infinitely many primes of this form. And, of course, if $p = 8m+3$ or $8m+5$ there is an odd number of odd values so the construction cannot work.

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