A Note on the Unit Interval Number and Proper Interval Number of Graphs

A. Raychaudhuri
The College of Staten Island
Department of Mathematics
2800 Victory Boulevard
Staten Island, New York 10314

Abstract: In this paper, we study intersection assignments of graphs using multiple intervals for each vertex, where each interval is of identical length or in which no interval is properly contained in another. The resulting parameters unit interval number, $i_u(G)$ and proper interval number, $i_p(G)$ are shown to be equal for any graph G. Also, $i_u(G)$ of a triangle-free graph G with maximum degree D is $\lceil (D+1)/2 \rceil$ if G is regular and $\lceil D/2 \rceil$ otherwise.

1. Introduction: This paper addresses two parameters: unit interval number and proper interval number denoted respectively by $i_u(G)$ and $i_p(G)$. The interest in these numbers originates from interval graphs. Interval graphs have been studied extensively in the literature (see Golumbic [1] for a detailed discussion) and have applications in scheduling, genetics, psychology, archaeology, etc. In general, an intersection representation of a graph is a function f that assigns a set to every vertex of G such that $\{x,y\} \in E(G)$ if and only if $f(x) \cap f(y) \neq \emptyset$. A graph is an interval graph if it has an intersection assignment where each assigned set is an interval along the real line (this is an interval representation). Many variations of interval representation have been studied. Since not all graphs are interval graphs, one way of extending the concept of assigning intervals to vertices of all graphs is to use a union of intervals to each vertex. Such a representation is called a multi-interval representation. This leads to the parameter interval number i(G), which is the minimum t such that G has a multi-interval representation f in which each $f(v)$ is a union of at most t intervals. This number was introduced by Trotter and Harary [4] and has been studied extensively.

We shall next consider special cases of interval representation. An interval graph is a unit interval graph if it has an interval representation where each interval has identical length. It is a proper interval graph if it has an interval representation where no interval is properly contained in another. Roberts [3] proved that the class of unit interval graphs is exactly the class of proper interval graphs.

This motivated us to study these special cases for multi-interval representation. In a multi-interval representation, (without loss of generality) the assigned intervals, $f(v)$, for any single vertex v, are pairwise disjoint. A multi-interval representation is unitary if every assigned interval has the same length, and it is proper if no assigned interval is properly contained in another.

The unit interval number $i_u(G)$ of a graph is the minimum integer t such that G has a unitary multi-interval representation with at most t intervals per vertex. The proper interval number $i_p(G)$ of a graph is the minimum t such that G has a proper multi-interval representation with at most t intervals per vertex.

Using Robert's result of the fact that the classes of unit interval graphs and proper interval graphs are the same, we prove that $i_u(G) = i_p(G)$ for any graph G. Griggs and West [2] have proved that $i(G) \leq \lceil (D+1)/2 \rceil$ where D is the maximum degree in G. West [5] has given a shorter proof of the same fact using Eulerian trails and circuits. In that proof every assigned interval has the same length. So the above bound can be restated as $i_u(G) \leq \lceil (D+1)/2 \rceil$. For graphs which are not regular, this bound can be improved to $i_u(G) \leq \lceil D/2 \rceil$. Griggs and West [2] have noted that if G is triangle free and regular, then $i(G) = \lceil (D+1)/2 \rceil$. We have made a similar observation for the parameter $i_u(G)$, for a larger class which includes triangle free graphs.

2. The Results about $i_u(G)$ and $i_p(G)$

Theorem 1. The proper interval number $i_p(G)$ is equal to the unit interval number $i_u(G)$ for any graph G.

Proof. For any multi-interval representation, a unitary representation is a proper representation (since every assigned interval is of the same length, no interval can be properly contained in another); so we only need to show $i_u(G) \leq i_p(G)$. Let f be a proper multi interval representation with at most t pairwise disjoint intervals for every vertex v.

Let us construct a new graph G', whose vertex set V' is the set of all assigned intervals in f taken over all vertices of G. The edge set of G' consists of all pairs of intervals in V' which intersect in the representation f. Let S(v) denote the set of vertices in G' consisting of intervals assigned to the vertex v of G in f. By construction, G' is a proper interval graph. So by Robert's result [3], it is a unit interval graph as well. Let f' be an interval representation of G' where each interval is of identical length. From f' let us define a multi-interval representation g for the graph G in the following way: assign to each vertex $v \in V(G)$ the set g(v) which is the union of intervals assigned to the vertices of S(v) in the representation f'. Now, the assigned intervals in f corresponding to a vertex v of G are pairwise disjoint. So the set of vertices S(v) form an independent set in G'. Therefore, the unit intervals assigned in f' to the set $S(v) (\subseteq V(G'))$, are pairwise disjoint. So in the unitary multi-interval representation g of G, the set g(v) assigned to a vertex v of G, has the same number of intervals as $f(v)$, where f is the proper multi-interval representation of G which uses at most t intervals for each vertex. Hence, $i_u(G) \le i_p(G)$ □

Theorem 2. Let D be the maximum degree in G, then $i_u(G) \le \lceil (D+1)/2 \rceil$. If G is not regular then $i_u(G) \le \lceil D/2 \rceil$. If G is triangle-free and regular then $i_u(G) = \lceil (D+1)/2 \rceil$.

Proof: See West [5] for a proof as mentioned in the introduction. Also, for triangle-free and regular graphs $i_u(G) \ge i(G) = \lceil (D+1)/2 \rceil$. (See Griggs and West [2])

Theorem 3. If G has a vertex whose closed neighborhood induces a graph H requiring at least k cliques to cover the vertices of H, then $i_u(G) = i_p(G) \ge \lceil k/2 \rceil$.

Proof: Let f be a unitary multi-interval representation of G which has at most t intervals for any vertex and let v be any vertex. Each vertex in the neighborhood of v must be assigned an interval I which intersects with at least one interval J assigned to v. But I cannot be properly contained in J. So I must contain one end point of J. All vertices of G whose interval assignment f contain this end point of J must induce a clique in G. If f (v) consists t disjoint intervals then we can obtain at most 2t cliques covering the vertices of H, the graph induced by the closed neighborhood of v. Therefore if we needed at least k cliques to cover the vertices of H, then $2t \ge k$, so, $i_u(G) \ge \lceil k/2 \rceil$ □

For triangle-free regular graphs it is known that $i(G) = \lceil (D+1)/2 \rceil$ (See [2] for a proof).
We can now state a similar result for $i_u(G)$ for a larger class of graphs.

Corollary 3.1: If G has a vertex of maximum degree D whose neighborhood is an independent set, then $i_u(G) = i_p(G) = \lceil D/2 \rceil$ except when G is regular, in which case it is $\lceil (D+1)/2 \rceil$.

Acknowledgement:

The author is grateful to the reviewer for his guidance in rewriting the final manuscript of this paper.

References

[1] M.C.Golumbic, *Algorithmic Graph Theory and Perfect Graphs,* (Acad. Press 1980).
[2] J.R. Griggs and D.B. West, Extremal values of the interval number of a graph, *SIAM J. Algeb. Disc. Meth.* 1(1980), 1-7.
[3] F.S. Roberts, Indifference graphs, in *Proof Techniques in Graph Theory* (F. Harary, ed.), (Academic Press, 1969), 139-146.
[4] W.T. Trotter and F. Harary, On double and multiple interval graphs, *J.Graph Theory* 2(1978), 137-142.
[5] D.B. West, A short proof of the degree bound on interval number, *Discrete Math 73(1989), 309-310.*