Maximal partial spreads in $PG(3,5)$

Olof Heden
Department of Mathematics
KTH
S-10044 Stockholm
Sweden

ABSTRACT. Maximal partial spreads of the sizes $13, 14, 15, \ldots,$ 22 and 26 are described. They were found by using a computer. The computer also made a complete search for maximal partial spreads of size less then or equal to 12. No such maximal partial spreads were found.

1 Introduction

Maximal partial spreads were first studied by Dale Mesner [9]. He had his children select sets of mutually skew lines of $PG(3,3)$. They first chose one line of $PG(3,3)$. Then they picked a new line, skew to the first chosen line, and so on. The children of Mesner certainly worked randomly and non-sytematically. Mesner observed that either this game terminated in a complete spread with 10 lines or the children were unable to find any more skew lines after seven lines had been choosen. This latter kind of partial spreads nowadays are called maximal partial spreads. Aiden A. Bruen and several others continued the study of maximal partial spreads, see e.g. [8, p. 76-85] or [7].

We had a computer search for maximal partial spreads in $PG(3,5)$. The computer found maximal partial spreads of the sizes $13, 14, 15, \ldots, 22$ and 26. (They are described in section 3.) By results of [9], [3] and [5] there cannot exist maximal partial spreads of a size less than 11 or of the sizes 23, 24 and 25. We also had the computer perform a complete search for maximal partial spreads of size less then or equal to 12. However, no such maximal partial spread was found. Let us also remark that maximal partial spreads in $PG(3,5)$ were already described for the sizes 21, 22 and 26, see e.g. [8, p. 81].

2 Notations and algorithm

For basic geometric facts, see e.g. [2].

Lines of $PG(3,5)$ are considered as 2-dimensional subspaces of the direct product $GF(25) \times GF(25)$. We will always assume that the line $\{0\} \times GF(25)$ is contained in a maximal partial spread. Lines skew to this line may be described as subspaces $[a,b] = \{(x, ax + bx^5) \mid x \in GF(25)\}$ of $GF(25) \times GF(25)$ where a and b are elements of $GF(25)$, see [7]. Let η be a primitive element of $GF(25)$ and let H denote the multiplicative subgroup $H = \{\eta^4, \eta^8, \ldots, \eta^{24} = 1\}$ of $GF(25) \setminus \{0\}$. It was proved in [7] that two lines $[a,b]$ and $[c,d]$ intersect if and only if

$$a - c = (b - d)h \tag{1}$$

for some element $h \in H$. In our search for maximal partial spreads we used the primitive element η of $GF(25)$ satisfying the equation $\eta^2 = 4\eta + 3$. Let ι denote the element $\iota = \eta^4 - \eta^{20}$. As ι belongs to $GF(25) \setminus GF(5)$ we may denote the line $[a,b]$ by a 4-tuple $(\alpha, \beta, \gamma, \delta)$ where $a = \alpha + \iota\beta$ and $b = \gamma + \iota\delta$ and where α, β, γ and δ belongs to $GF(5)$. This will be the notation of lines used below.

As the elements of H are the only elements of $GF(25)$ satisfying $h^6 = 1$ we conclude that equation (1) is equivalent to the equation

$$(a - c)^6 = (b - d)^6 \tag{2}$$

We note that $(a + \iota\beta)^5 = (\alpha - \iota\beta)$ and hence $(\alpha + \iota\beta)^6 = \alpha^2 + \iota^2\beta^2$. As $\iota^2 = 3$ we thus get that two lines $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma', \delta')$ intersect if and only if

$$(\alpha - \alpha')^2 + 3(\beta - \beta')^2 \equiv (\gamma - \gamma')^2 + 3(\delta - \delta')^2 \pmod 5. \tag{3}$$

This condition was used in the algorithm to check whether or not two lines intersect. The advantage of equation (3) is that we do not have to make any calculations in $GF(25)$.

The main idea in the algorithm was, in a systematic way, to perform the game of the children of Mesner, see the introduction. Using automorphisms in $PG(3,5)$ we can always assume that a maximal partial spread contains the three lines $l_0 = \{0\} \times GF(25)$, $l_1 = (0,0,0,0)$ and $l_2 = (0,0,0,1)$. We then determine all lines that are skew to these lines. They are placed in a file f_3. In the second step we choose a line l_3 of f_3 and determine which of the lines of f_3 that is skew to l_3. These new lines are placed in a file f_4 and so on. When you come to a situation where for a line l_k there are no lines of the file f_k skew to l_k then the lines $l_0, l_1, \ldots, l_k$ will constitute a maximal partial spread of size $k+1$. The intention was to use backtracking to get a complete search.

We used a SPARCstation 5 to perform this simple algorithm written in C. The problem was that the computer was not fast enough for making a complete search. We hence had to make some improvements of the algorithm and changes in the ambitions of the search.

We decided only to be interested in the possible number of lines of a maximal partial spread. Some more or less randomly searching on the computer very quickly gave maximal partial spreads of the sizes $14, 15, \ldots, 22$ and 26. We thus were able to concentrate on a search for maximal partial spreads of size less then or equal to 13. This speeded up the random search so that we very quickly produced a maximal partial spread of size 13. With this new knowledge we hence could concentrate on a complete search for maximal partial spreads of size less then or equal to 12.

We get one improvement of the algorithm from the following simple observation: Of the two last lines l_{k-1} and l_k of a maximal partial spread $S = \{l_0, l_1, \ldots, l_k\}$ appearing from the algorithm, at least one must intersect at least half of the lines in the file f_{k-1}. Similarly, of the three last lines l_{k-2}, l_{k-1} and l_k at least one must intersect at least one third of the lines of the file f_{k-2}. By implementing this in the algorithm the computing time was further reduced.

Any line of $PG(3,5)$ is contained in the maximal partial spread or is intersected by a line of the maximal partial spread. Consequently, if the first line l of a file f_j is not contained in the maximal partial spread, then the line l_j of the resulting maximal partial spread must intersect the line l. This condition was also implemented in the algorithm for $k \leq 9$, which reduced the computing time for a complete search.

Let S denote any maximal partial spread of $PG(3,5)$ of size less then or equal to 12. We observed that there must be a regulus $\mathfrak{R}$ of $PG(3,5)$ such that the intersection of $\mathfrak{R}$ and S contains exactly three lines. This can for instance be proved by considering the four equations 17.12 - 17.15 of Lemma 17.6.3 in [8, p. 78]. An assumption that to any three lines of S there is a fourth line of S, such that these four lines are contained in the same regulus of $PG(3,5)$, will contradict these four equations. We may thus, by using automorphisms of $PG(3,5)$, without loss of generality, assume that S contains the lines $l_0 = \{0\} \times GF(25)$, $l_1 = (0,0,0,0)$, $l_2 = (0,0,0,1)$ but none of the lines $l = (0,0,0,2)$, $l' = (0,0,0,3)$ and $l'' = (0,0,0,4)$. These six lines together constitutes a regulus $\mathfrak{R}$. Let $\bar{l}$ and $\bar{l}'$ be any two lines of the opposite regulus. Denote the intersection point of $\bar{l}$ and l by P and the intersection point between $\bar{l}'$ and l' by P'.

The lines l, and l' and l'' must be intersected by lines of S. Again considering those automorphisms of $PG(3,5)$ fixing l_0, l_1 and l_2, we may conclude that one of the following three cases must appear for a maximal partial spread S of size less then or equal to 12:

Case 1: There are lines l_3 and l_4 of S such that the line l_3 meets the points P and P' and the line l_4 meets the lines l' and l''.
Case 2: Equals the first case, but the only line of $\mathfrak{R}$ that the line l_4 intersects is the line l''.
Case 3: There are three lines l_3, l_4 and l_5 of S such that the only point of $\mathfrak{R}$ that l_3 meets is the point P, the only line of $\mathfrak{R}$ that the line l_4 intersects is the line l' and the only line of $\mathfrak{R}$ that the line l_5 intersects is the line l''.

By considering only these three cases and implementing the above improvements in the original simple backtracking algorithm, we were able to perform a complete search for maximal partial spreads of size less then or equal to 12. The computing time on the SPARCstation 5 was only some weeks. The result was that there were not any.

3 Results

We found very many maximal partial spreads of size larger then 12. In this section we present one of each size.

All our maximal partial spreads contain the eight lines in the following set:
$\{l_0, l_1, (0,0,0,1), (1,0,2,3), (1,0,2,4), (1,0,2,0), (0,0,1,1), (0,0,2,0)\}$.
The maximal partial spread of size 13 also contains the five lines in the following set:
$\{(0,0,2,2), (0,0,2,4), (1,0,0,2), (1,0,4,2), (2,0,0,1)\}$.
The maximal partial spread of size 14 also contain the six lines in the set
$\{(0,0,4,2), (1,0,3,3), (1,0,4,3), (1,3,0,4), (2,2,0,4), (4,0,2,4)\}$.
The maximal partial spread of size 15 also contains the following seven lines:
$\{(0,0,2,1), (0,0,2,2), (0,0,2,4), (0,3,4,0), (2,1,2,4), (4,1,4,0), (4,2,1,0)\}$.
The maximal partial spread of size 16 also contains the following eight lines:
$\{(0,0,2,1), (0,0,2,2), (0,0,2,4), (0,0,3,2), (0,0,4,3), (1,0,0,2)$
$(2,1,2,4), (4,4,2,4)\}$.
The maximal partial spread of size 17 also contains the following nine lines:
$\{(0,0,2,1), (0,0,2,2), (0,0,2,3), (0,0,2,4), (0,0,3,2), (0,0,4,2),$
$(0,3,0,0), (4,0,2,4), (4,4,2,4)\}$.
The maximal partial spread of size 18 also contains the following 10 lines:
$\{(0,0,2,1), (0,0,2,2), (0,0,2,3), (0,0,2,4), (0,0,3,2), (0,0,4,2),$
$(0,0,4,3), (2,1,2,0), (2,1,2,3), (2,1,2,4)\}$
The maximal partial spread of size 19 also contains the following 11 lines:
$\{(0,0,2,1), (0,0,2,2), (0,0,2,3), (0,0,2,4), (0,0,3,2), (0,0,4,2),$
$(0,3,0,0), (1,2,0,0), (2,1,2,0), (4,0,2,3), (4,4,2,0)\}$.
The maximal partial spread of size 20 also contains the following 12 lines:
$\{(0,0,2,1), (0,0,2,2), (0,0,2,3), (0,0,2,4), (0,0,3,2), (0,3,4,0),$
$(1,2,4,0), (1,4,4,0), (2,1,2,4), (4,0,2,4), (4,1,4,0), (4,4,2,4)\}$.
The maximal partial spread of size 21 also contained the following 13 lines:

$\{(0,0,2,1),(0,0,2,2),(0,0,2,3),(0,0,2,4),(0,0,3,2),(0,0,4,2),$
$(0,0,4,3),(2,1,2,0),(2,1,2,3),(4,0,2,0),(4,0,2,3),(4,4,2,0),(4,4,2,3)\}.$
The maximal partial spread of size 22 also contains the following 14 lines:
$\{(0,0,2,1),(0,0,2,2),(0,0,2,3),(0,0,2,4),(0,0,3,2),(0,3,0,0),$
$(0,3,0,1),(0,3,4,0),(0,4,1,1),(1,2,0,0),(1,2,0,1),(1,2,4,0),$
$(3,1,1,1),(4,0,2,4)\}.$
The maximal partial spread of size 26 also contains the following 18 lines:
$\{(0,0,2,2),(0,0,2,4),(0,2,3,0),(0,3,0,1),(0,3,4,0),(0,4,1,1),$
$(1,2,0,0),(2,1,2,3),(2,1,4,2),(2,2,0,1),(2,3,1,3),(2,3,2,2),$
$(3,1,1,1),(3,1,2,2),(3,2,1,4),(4,1,3,0),(4,2,1,0),(4,4,2,0)\}.$

4 A remark

A similar algorithm has also been used for performing a search for maximal partial spreads in $PG(3,7)$. The search is not completed yet. As so far only maximal partial spreads of sizes $23, 24, \ldots$ have been found. However the existence of maximal partial spreads of these sizes are already known, see [7] and [6l

References

[1] A.A. Bruen, Partial spreads and replaceable nets, *Canad. J. Math.* **23** (1971), 381–392.

[2] P. Dembowski, *Finite geometries*, Springer, Berlin, 1968.

[3] D.G. Glynn, A lower bound for maximal partial spreads in $PG(3,q)$, *Ars Combin.* **13** (1982), 39–40.

[4] O. Heden, Maximal partial spreads and two-weight codes, *Discrete Math.* **62** (1986), 277–293.

[5] O. Heden, No maximal partial spread of size 10 in $PG(3,5)$, *Ars Combin.* **29** (1990), 297–298.

[6] O. Heden, A greedy search for maximal partial spreads in $PG(3,7)$, *Ars Combin.* **32** (1991), 253–255.

[7] O. Heden, Maximal partial spreads and the modular n-queen problem, *Discrete Math.* **120** (1993), 75–91.

[8] J.W.P. Hirsehfeld, *Finite Projective Spaces of Three dimensions*, Clarendon Press, Oxford, 1985.

[9] D. Mesner, Sets of disjoint lines in $PG(3,q)$, *Canad. J. Math.* **19** (1967), 273–280.