Two families of graphs that are not CCE-orientable

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Abstract

Let D be a digraph. The competition-common enemy graph of D has the same set of vertices as D and an edge between vertices u and v if and only if there are vertices w and x in D such that (w,u), (w,v), (u,x), and (v,x) are arcs of D. We call a graph a CCE-graph if it is the competition-common enemy graph of some digraph. We also call a graph $G = (V, E)$ CCE-orientable if we can give an orientation F of G so that whenever (w,u), (w,v), (u,x), and (v,x) are in F, either (u,v) or (v,u) is in F. Bak *et al.* [1997] found a large class of graphs that are CCE-orientable and proposed an open question of finding graphs that are not CCE-orientable. In this paper, we answer their question by presenting two families of graphs that are not CCE-orientable. We also give a CCE-graph that is not CCE-orientable, which answers another question proposed by Bak *et al.* [1997]. Finally we find a new family of graphs that are CCE-orientable.

Key words: competition graphs, CCE-graphs, CCE-orientable graphs.

*The authors thank the Research Institute of Basic Sciences of Kyung Hee University for its support and the Korean Ministry of Education for its support under grant BSRI-97-1432.

†This work is supported by Faculty Research Fund of Ewha granted in 1998

1 Introduction

The *competition-common enemy graph* (*CCE-graph*) of a digraph D has the same set of vertices as D and an edge between vertices u and v if and only if there are vertices w and x in D such that (w, u), (w, v), (u, x), (v, x) are arcs of D (for all undefined graph theory terminology, see [2, 6]). We will call a graph a *CCE-graph* if it is the CCE-graph of some digraph. The notion of competition-common enemy graph was introduced by Scott [15] in 1987 as one of the variants of *competition graph* which has been studied by many authors. The literature of competition graph is summarized in [9, 10, 13, 16].

There have been efforts to characterize competition graphs of various digraphs. Dutton and Brigham [3] gave a necessary and sufficient condition for a graph to be a competition graph. Roberts and Steif [14] characterized the competition graph of a loopless digraph. Most recently, Fraughnaugh *et al.* [4] characterized the competition graph of a strongly connected digraph. Regarding CCE-graphs, Scott [15] gave a necessary and sufficient condition for a graph to be a CCE-graph. From the view point that the condition is too complicated to be of practical use in checking whether or not a given graph is a CCE-graph, Bak *et al.* [1] sought a better characterization of CCE-graphs and found interesting classes of CCE-graphs by introducing a new notion called CCE-orientable graph. An *orientation* F of a graph G is a digraph on its vertices such that uv is an edge of G if and only if exactly one of (u, v) or (v, u) is an arc of F. Then F is a *CCE-orientation* of G if whenever (w, u), (w, v), (u, x), and (v, x) are arcs, uv is an edge of G. We say G is *CCE-orientable* if it has a CCE-orientation. They showed that any CCE-orientable graph is a CCE-graph, and bipartite graphs and chordal graphs are CCE-orientable. They also presented an algorithm for constructing another CCE-orientable graph out of a CCE-orientable graph. In the same paper, they proposed an open question of finding graphs that are not CCE-orientable. In Section 2, we answer their question by giving two families of graphs that are not CCE-orientable. We also give an example of a CCE-graph that is not CCE-orientable. This answers another question proposed in the same paper. In Section 3, we give another class of graphs that are CCE-orientable graphs and therefore CCE-graphs. In Section 4, we propose some interesting open questions.

2 Two families of graphs that are not CCE-orientable

Lemma 1 gives conditions on a CCE-orientation of a complete bipartite graph.

Lemma 1 *Let $K_{m,n}$ have vertices $u_1, \ldots, u_m, v_1, \ldots, v_n$ and edges u_iv_j for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$ (see Figure 1(a)). Let F be a CCE-orientation of $K_{m,n}$. If $m \geq 2$ and $n \geq 4$, then $u_1, \ldots, u_m$ either all have at most one out-neighbor, or all have at most one in-neighbor in F.*

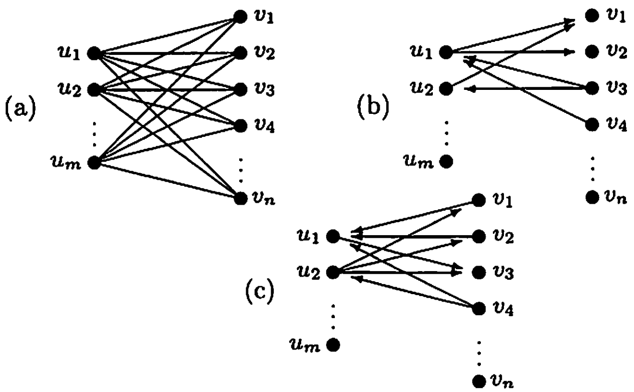


Figure 1 – Constructions from Lemma 1. Graph (a) shows $K_{m,n}$. In (b), we assume a CCE-orientation has a vertex u_1 with at least two in-neighbors and at least two out-neighbors, which leads to a contradiction if $m \geq 2$. In (c), we assume a CCE-orientation has a vertex u_1 with at most one out-neighbor, and a vertex u_2 with at most one in-neighbor. This also gives a contradiction if $n \geq 4$.

Proof. Suppose a vertex, say u_1, has two out-neighbors in F, say v_1 and v_2, and two in-neighbors, say v_3 and v_4. Then one of v_1 or v_2, say v_1, is an out-neighbor of u_2; for otherwise, arcs (u_1, v_1), (u_1, v_2), (v_1, u_2) and (v_2, u_2) would force edge v_1v_2 (see Figure 1b). Similarly, one of v_3 or v_4, say v_3, is an in-neighbor of u_2. Then arcs (v_3, u_1), (v_3, u_2), (u_1, v_1) and (u_2, v_1) force edge u_1u_2, a contradiction. Thus each of $u_1, \ldots, u_m$ has at most one out-neighbor, or at most one in-neighbor in F.

Now suppose a vertex, say u_1, has at most one out-neighbor, while another vertex, say u_2, has at most one in-neighbor (see Figure 1(c)). Since $n \geq 4$, at least two vertices, say v_1 and v_2, are in-neighbors of u_1 and out-neighbors of u_2. Then arcs (u_2, v_1), (u_2, v_2), (v_1, u_1) and (v_2, u_1) force edge v_1v_2, a contradiction. Therefore $u_1, \ldots, u_m$ either all have at most one out-neighbor, or all have at most one in-neighbor. □

Let H be a graph on vertices $v_1, \ldots, v_n$. Let $G_1, \ldots, G_n$ be pairwise vertex-disjoint graphs. Then the *composition graph* $H[G_1, \ldots, G_n]$ is the union of $G_1, \ldots, G_n$ with all edges between vertices of G_i and G_j whenever v_iv_j is an edge in H (see Figure 2). Let I_n be the edgeless graph on n

vertices. We then have the following theorem:

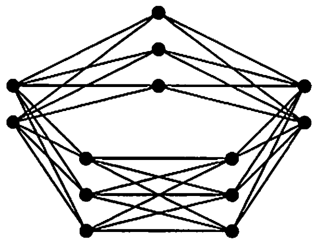
$C_5[I_3, I_2, I_3, I_2, I_3] =$ 

Figure 2. Theorem 2 shows this is not CCE-orientable. It is the smallest known graph that is not CCE-orientable.

Theorem 2 *Let $m \geq 5$ be an odd integer. Let $n_1, \ldots, n_m \geq 2$ be integers with either $n_i > 2$ or $n_{i+1} > 2$ for all $i = 1, \ldots, m-1$, and either $n_m > 2$ or $n_1 > 2$ (i.e., when viewed circularly, the sequence $n_1, \ldots, n_m$ does not have consecutive two's). Then $C_m[I_{n_1}, \ldots, I_{n_m}]$ is not CCE-orientable.*

Proof. Suppose $C_m[I_{n_1}, \ldots, I_{n_m}]$ has a CCE-orientation F. Since $m > 3$, the vertices of $I_{n_{i-1}}$, I_{n_i} and $I_{n_{i+1}}$ (identify n_0 as n_m, and n_{m+1} as n_1) induce a complete bipartite graph. Since $n_i \geq 2$ and $n_{i-1} + n_{i+1} \geq 4$, Lemma 1 shows the vertices of I_{n_i} either all have at most one out-neighbor, or all have at most one in-neighbor.

Classify $i = 1, \ldots, m$ as "Type A" if every vertex of I_{n_i} has at most one out-neighbor, and "Type B" otherwise. Since m is odd, there must be i and j of the same type where either $j = i + 1$, or $i = n$ and $j = 1$. If both i and j are Type A, then F has at most n_i arcs from I_{n_i} to I_{n_j}, and at most n_j arcs from I_{n_j} to I_{n_i}. There are $n_i n_j$ edges between I_{n_i} and I_{n_j} in $C_m[I_{n_1}, \ldots, I_{n_m}]$. Since $n_i, n_j \geq 2$ and either $n_i > 2$ or $n_j > 2$, we have that $n_i + n_j < n_i n_j$. Thus not every edge between I_{n_i} and I_{n_j} can be oriented in F, a contradiction. A similar contradiction results if i and j are both Type B. □

Now we give another family of graphs that are not CCE-orientable. Let the *Cartesian product* of graphs G and H be the graph $G \times H$ on vertices (g, h) for all vertices $g \in G$ and $h \in H$ with edges $(g, h_1)(g, h_2)$ for all edges $h_1 h_2$ in H, and $(g_1, h)(g_2, h)$ for all edges $g_1 g_2$ in G (see Figure 3). A *coloring* assigns labels to vertices so that neighbors have different labels. The *chromatic number* $\chi(G)$ is the minimum number of labels used in a coloring of G.

Theorem 3 *If $G \times C_3$ is CCE-orientable, then $\chi(G) \leq 8$.*

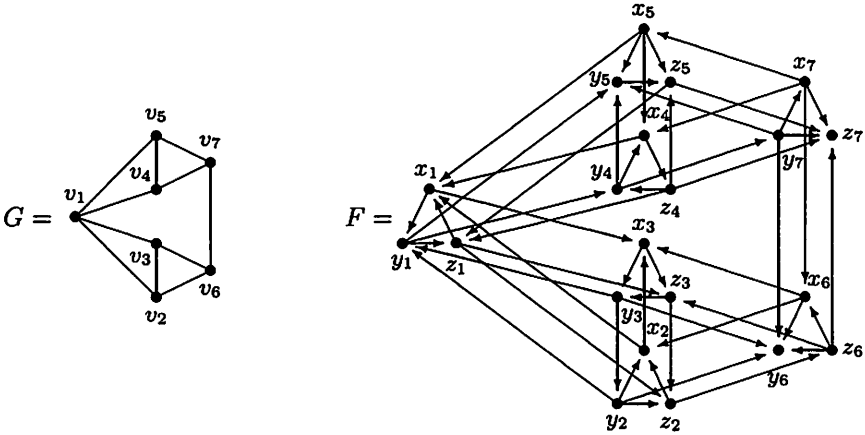


Figure 3 – A CCE-orientation F of $G \times C_3$.

Proof. Let $v_1, \ldots, v_n$ be the vertices of G. Let $G \times C_3$ have vertices $x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n$, with edges x_iy_i, x_iz_i and y_iz_i for all $i = 1, \ldots, n$; and edges x_ix_j, y_iy_j and z_iz_j for all edges v_iv_j in G (see Figure 3).

Let F be a CCE-orientation of $G \times C_3$. Let G_1, G_2 and G_3 be spanning subgraphs of G where v_iv_j is an edge of G_1, G_2, or G_3, if x_ix_j and y_iy_j, x_ix_j and z_iz_j, or y_iy_j and z_iz_j, respectively, have the same orientation in F (see Figure 4). Since at least two of x_ix_j, y_iy_j, and z_iz_j have the same orientation, every edge of G is in either G_1, G_2, or G_3.

Let v_iv_j be an edge of G_1. Then x_ix_j and y_iy_j have the same orientation in F. Since x_iy_j and y_ix_j are not edges of $G \times C_3$, we must have that x_iy_i and x_jy_j have opposite orientations in F. The vertices of G_1 may be partitioned into two classes dependent on the orientation of x_iy_i in F. These form a coloring showing that G_1 is bipartite. Similarly G_2 and G_3 are also bipartite.

For $j = 1, 2, 3$, let $\ell_j(v)$ be a 2-coloring of G_j. For all v_i, let $\ell(v_i) = (\ell_1(v_i), \ell_2(v_i), \ell_3(v_i))$ (see Figure 5). Then ℓ uses eight or fewer labels. Since any edge of G is an edge in either G_1, G_2, or G_3, we have $\ell(v_i) \neq \ell(v_j)$ for all edges v_iv_j in G. So ℓ is a coloring of G, and $\chi(G) \leq 8$. □

Corollary 4 *If $\chi(G) \geq 9$, then $G \times C_3$ is not CCE-orientable.*

Bak *et al.* [1] proposed a problem of finding a CCE-graph that is not CCE-orientable. The following theorem answers their question:

Theorem 5 *Let G be $K_9 \cup I_4$. Then $G \times C_3$ is a CCE-graph.*

Proof. Let $H = G \times C_3$. Let H have vertices $x_{i1}, \ldots, x_{i9}, y_{i1}, \ldots, y_{i4}$ for each i = 1, 2, 3 and edges $x_{ik}x_{il}$ for i = 1, 2, 3 and distinct k and l in

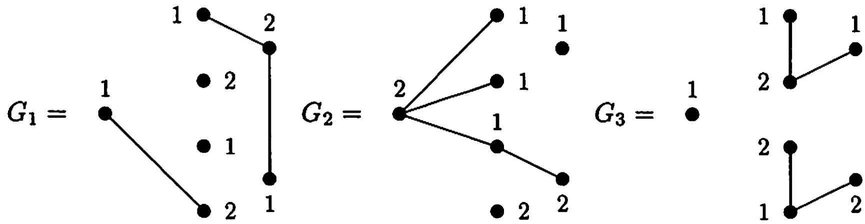


Figure 4 – Spanning subgraphs. This is based on the CCE-orientation in Figure 3. In G_1, vertex v_i is labeled 1 if (x_i, y_i) is an arc, and 2 otherwise. Its edges are v_iv_j if x_ix_j and y_iy_j have the same orientation. In G_2, vertex v_i is labeled 1 if (x_i, z_i) is an arc, and 2 otherwise. Its edges are v_iv_j if x_ix_j and z_iz_j have the same orientation. In G_3, vertex v_i is labeled 1 if (y_i, z_i) is an arc, and 2 otherwise. Its edges are v_iv_j if y_iy_j and z_iz_j have the same orientation. The proof of Theorem 3 shows each of these labelings is a 2-coloring, and that every edge of G is in at least one of the three subgraphs.

$\{1, \ldots, 9\}$ and edges $x_{i1}x_{j1}, \ldots, x_{i9}y_{j9}, y_{i1}y_{j1}, \ldots, y_{i4}y_{j4}$ for any distinct i, j in $\{1, 2, 3\}$. In this proof, we identify x_{0k} (resp. y_{0k}) as x_{3k} (resp. y_{3k}), x_{4k} (resp. y_{4k}) as x_{1k} (resp. y_{1k}), x_{i0} as x_{i9}, x_{i10} as x_{i1}, y_{i0} as y_{i4}, and y_{i5} as y_{i1}. Now construct a digraph D as follows:

$$\begin{aligned}
V(D) &= V(G); \\
A(D) &= \{(y_{i1}, x_{ik}) \mid 1 \le i \le 3 \text{ and } 1 \le k \le 9\} \\
&\cup \{(x_{ik}, y_{i2}) \mid 1 \le i \le 3 \text{ and } 1 \le k \le 9\} \\
&\cup \{(x_{ik}, x_{i(k-1)}) \mid 1 \le i \le 3 \text{ and } 1 \le k \le 9\} \\
&\cup \{(x_{ik}, x_{(i+1)(k-1)}) \mid 1 \le i \le 3 \text{ and } 1 \le k \le 9\} \\
&\cup \{(y_{ik}, y_{i(k-1)}) \mid 1 \le i \le 3 \text{ and } 1 \le k \le 4\} \\
&\cup \{(y_{ik}, y_{(i+1)(k-1)}) \mid 1 \le i \le 3 \text{ and } 1 \le k \le 4\}.
\end{aligned}$$

We list the in-neighbors and the out-neighbors of each vertex in D in Table 1. In the table, $i = 1$, 2, or 3. Let H' be the CCE-graph of D. From Table 1, we can see that $x_{i1}, \ldots, x_{i9}$ form a clique in H' since y_{i1} is a common in-neighbor and y_{i2} is a common out-neighbor in D for $i = 1$, 2, 3. In addition, x_{ik} and $x_{(i+1)k}$ are adjacent in H' since $x_{i(k+1)}$ is a common in-neighbor and $x_{(i+1)(k-1)}$ is a common out-neighbor for $i = 1$, 2, 3 and $k = 1, \ldots, 9$. Similarly y_{ik} and $y_{(i+1)k}$ are adjacent in H' for $i = 1$, 2, 3 and $k = 1, \ldots, 4$. Other than those pairs of vertices, the only pairs whose vertices share a common out-neighbor are x_{ik} and y_{i3}, x_{ik} and $y_{(i-1)3}$, x_{ik} and y_{i1}, and x_{ik} and $y_{(i+1)1}$ for $i = 1$, 2, 3 and $k = 1, \ldots 9$. However, it is easy to check that vertices of none of those pairs share a common in-neighbor. We have just shown that $H = H'$ and the theorem follows. □

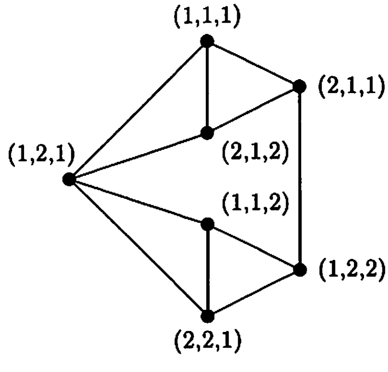


Figure 5 – A coloring. The labels are a concatenation of the labels in Figure 4. The proof of Theorem 3 shows this is a coloring.

vertex v	the in-neighbors of v	the out-neighbors of v
x_{ik} $(1 \le k \le 9)$	y_{i1}, $x_{i(k+1)}$, $x_{(i-1)(k+1)}$	y_{i2}, $x_{i(k-1)}$, $x_{(i+1)(k-1)}$
y_{i1}	y_{i2}, $y_{(i-1)2}$	x_{i1}, ..., x_{i9}, y_{i4}, $y_{(i+1)4}$
y_{i2}	x_{i1}, ..., x_{i9}, y_{i3}, $y_{(i-1)3}$	y_{i1}, $y_{(i+1)1}$
y_{ik} $(k = 3, 4)$	$y_{i(k+1)}$, $y_{(i-1)(k+1)}$	$y_{i(k-1)}$, $y_{(i+1)(k-1)}$

Table 1 – The in-neighbors and the out-neighbors of each vertex in D.

3 Another family of graphs that are CCE-orientable

In this section, we present a new family of graphs that are CCE-orientable. Let $G = (V, E)$ be a graph. Given an orientation F of G, we define a matrix A_F to be the matrix whose (i, j)-entry is equal to -1 (resp. 1) if $(i, j) \in F$ (resp. $(j, i) \in F$), and zero otherwise. An orientation F is said to be *Pfaffian* if the determinant of A_F is equal to the square of the number of perfect matchings in G. If there exists a Pfaffian orientation of G, then G is said to be *Pfaffian orientable.* The Pfaffian orientation has been studied in relation to the number of perfect matchings in a given graph. Computing the number of perfect matchings in a graph is an NP-hard problem, but it becomes an easy problem if G is Pfaffian orientable and we have a Pfaffian orientation of G. Some equivalent conditions for a graph G being Pfaffian orientable are given in [5] and [12]. One simple sufficient condition is that G is planar, which is obtained by Kasteleyn [7, 8]. Little [11] has obtained a more general condition, which is that G contains no subdivision of $K_{3,3}$.

Let C be an even cycle in G and F be an orientation of G. The cycle C can be traversed in clockwise or counterclockwise direction. Given a direction W for traversing C, let $N_C(W)$ be the number of edges on C whose orientations in F agree with W. Since C is an even cycle, $N_C(W)$ has the same parity whether W is clockwise or counterclockwise. An even cycle C in G is said to be *oddly oriented relative to* F if $N_C(W)$ is odd for some direction W for traversing C. A cycle C of length 4 is called a *completable square* if the subgraph induced by $V(C)$ is not the complete graph K_4. We present the following theorem without proof (for the proof, refer to [12]).

Theorem 6 *[12, p.321] Let G be any graph with an even number of vertices and F be an orientation of G. Then the following two properties are equivalent:*

(i) F is a Pfaffian orientation of G.

(ii) An even cycle C in G is oddly oriented relative to F if $G - V(C)$ contains a perfect matching.

Theorem 7 *Let G be a Pfaffian orientable graph. Assume that for each completable square C, $G - V(C)$ contains a perfect matching. Then G is CCE-orientable.*

Proof. If there is no completable square in G, then any orientation of G is a CCE-orientation. Now suppose that there is at least one completable square C in G. Since $G - V(C)$ has a perfect matching, the number of the vertices in $G - V(C)$ is even and so is the number of those in G. Let F be a Pfaffian orientation of G. By Theorem 6, every completable square is oddly oriented relative to F. This means that there cannot exist four vertices u, v, w, x such that (w,u), (w,v), (u,x), $(v,x) \in F$. Thus F is a CCE-orientation of G. □

A graph G is said to be *2-extendable* if for every pair of nonadjacent edges e_1 and e_2 of G, there exists a perfect matching containing both e_1 and e_2. For the characterization of 2-extendable graphs, refer to [12]. The next corollary follows from Theorem 7.

Corollary 8 *Any 2-extendable planar graph G is CCE-orientable.*

Proof. Since G is planar, G is Pfaffian orientable. Let C be a completable square, and e_1 and e_2 be a pair of nonadjacent edges on C. Since G is 2-extendable, there exists a perfect matching containing both e_1 and e_2. Thus $G - V(C)$ contains a perfect matching and the corollary follows from Theorem 7. □

4 Closing remark

In this paper, we answered two questions proposed by Bak *et al.* [1] by finding two families of graphs that are not CCE-orientable and finding a CCE-graph that is not CCE-orientable. However, it is still open to find a graph that is not a CCE-graph. In fact, we do not know whether or not all members of those two families of graphs given in Theorems 2 and 3 are CCE-graphs.

In Section 3, we have shown that planar graphs satisfying a certain property are CCE-orientable. We propose the problem of proving or disproving that any planar graph is CCE-orientable. Finally, we wish to characterize an CCE-orientable graph.

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