Bounding the Roots of Independence Polynomials

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Abstract

The *independence polynomial* of graph G is the function $i(G,x) = \sum i_k x^k$ where i_k is the number of independent sets of cardinality k in G. We ask the following question: for fixed independence number β, how large can the modulus of a root of $i(G,x)$ be, as a function of the n, the number of vertices? We show that the answer is $(n/\beta)^{\beta-1} + O(n^{\beta-2})$.

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1 Introduction

Let G be a graph (finite and simple) with independence number β and let i_k denote the number of independent sets of size k. The *independence polynomial* of G is defined by

$$i(G,x) = \sum_{k=0}^{\beta} i_k x^k.$$

As is the case with other graphs polynomials, such as chromatic polynomials (c.f. [10], matching polynomials [6, 7], and others, it is natural to consider the nature and location of the roots. Some results on independence polynomials can be found in [2, 8]. It is known ([4]; see also [2]) that independence polynomials always have a real root; in fact, a root of smallest modulus is necessarily real. A natural question is: how large can the modulus of a root of an independence polynomial be? In [2] it was shown that for a well covered graph with independence number β (that is, a graph all of whose maximal independent sets have size β), all of the roots of its independence polynomial lie in $|z| \leq \beta$. We shall

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show here that no such bound exists for general graphs in terms of the independence number. In fact, it may be surprising how accurately the largest modulus of a root of a independence polynomial of a graph of order n (and independence number β) can be determined. We omit the case of $\beta = 1$ in the statement of Theorem 1 as such graphs are complete graphs, and $i(K_n, x) = 1 + nx$.

Theorem 1 *Let $\beta \geq 2$ be fixed. Let $r_\beta(n)$ denote the maximum modulus of a root of the independence polynomial of a graph of order n with independence number β. Then*

$$r_\beta(n) = \left(\frac{n}{\beta - 1}\right)^{\beta-1} + O(n^{\beta-2}).$$

We begin our proof of Theorem 1 by showing the upper bound. To do so, we shall need the following two lemmas.

Lemma 2 *Fix $\beta \geq 2$. Let $\sum_{i=0}^{\beta} i_k x^k$ be the independence polynomial of a graph G of independence number β, and suppose that $i_\beta \leq K$, where K is a constant depending only on β. Then*

$$i_{\beta-1} \leq \left(\frac{n}{\beta - 1}\right)^{\beta-1} + O(n^{\beta-2}).$$

<u>Proof</u> As there are at most K independent sets of size β, we can pick vertices $v_1, \ldots, v_L$ $(L \leq K)$ such that if $S = \{v_1, \ldots, v_L\}$, then $\beta(G - S) = \beta - 1$ (we simply recursively remove a vertex from each independent set of size β until this is no longer possible). Now in [5] it was shown that certain bounds exists for the number of cliques of each size in a graph. Applying these to the complement of G, we derive that

$$\frac{n}{\beta} = \left(\frac{i_1}{\binom{\beta}{1}}\right)^{1/1} \geq \left(\frac{i_2}{\binom{\beta}{2}}\right)^{1/2} \geq \cdots \geq \left(\frac{i_\beta}{\binom{\beta}{\beta}}\right)^{1/\beta}$$

and so

$$\frac{i_k}{\binom{\beta}{k}} \leq \left(\frac{n}{\beta}\right)^k \qquad \text{for } k = 1, \ldots, \beta - 1. \tag{1}$$

Using this applied to $G - S$ and by considering how independent sets of size $\beta - 1$ intersect the set S, we derive that

$$\begin{aligned} i_{\beta-1} &\leq \sum_{i=0}^{L} \binom{L}{i} i_{\beta-1-i}(G - S) \\ &\leq i_{\beta-1}(G - S) + \sum_{i=1}^{L} \binom{L}{i} \left(\frac{n - L}{\beta - 1}\right)^{\beta-1-i} \end{aligned}$$

$$\leq \left(\frac{n}{\beta-1}\right)^{\beta-1} + O(n^{\beta-2}).\square$$

Lemma 3 *For any graph G of order n and independence number β,*

$$\frac{i_{k-1}}{i_k} \leq \left(\frac{n}{\beta-1}\right)^{\beta-1} + O(n^{\beta-2})$$

for all $k = 1, \ldots, \beta$.

Proof From (1) it follows that for $k = 1, \ldots, \beta - 1$,

$$\frac{i_{k-1}}{i_k} \leq i_{k-1} \leq \binom{\beta}{k-1}\left(\frac{n}{\beta}\right)^{k-1} = O(n^{\beta-2}).$$

Thus it suffices to show

$$\frac{i_{\beta-1}}{i_\beta} \leq \left(\frac{n}{\beta-1}\right)^{\beta-1} + O(n^{\beta-2}).$$

Now again from (1) we have

$$i_{\beta-1} \leq \frac{n^{\beta-1}}{\beta^{\beta-2}} \tag{2}$$

and so it is easy to verify that if

$$\frac{i_{\beta-1}}{i_\beta} > \left(\frac{n}{\beta-1}\right)^{\beta-1}$$

then

$$i_\beta < \frac{(\beta-1)^{\beta-1}}{\beta^{\beta-2}}.$$

The key observation is that the right hand side is a constant (depending on β).

We now set $K = 2(\beta-1)^{\beta-1}/\beta^{\beta-2}$. Then if $i_{\beta-1} \leq K$, we have from the previous lemma that

$$\frac{i_{\beta-1}}{i_\beta} \leq i_{\beta-1} \leq \left(\frac{n}{\beta-1}\right)^{\beta-1} + O(n^{\beta-2}).$$

On the other hand, if $i_\beta > K$, then from (2),

$$\begin{aligned}\frac{i_{\beta-1}}{i_\beta} &< \frac{n^{\beta-1}/\beta^{\beta-2}}{2(\beta-1)^{\beta-1}/\beta^{\beta-2}} \\ &= \frac{n^{\beta-1}}{2(\beta-1)^{\beta-1}} \\ &< \left(\frac{n}{\beta-1}\right)^{\beta-1} + O(n^{\beta-2}),\end{aligned}$$

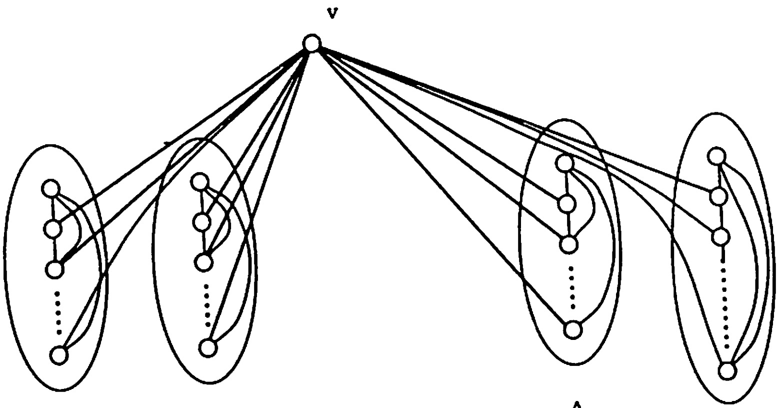

Figure 1: $G_{l,\beta}$

and we are done. □

We complete now the proof of the upper bound for Theorem 1. The *Eneström-Kakeya Theorem* (c.f. [1]) states that all of the roots of $i(G,x)$ lie in

$$|z| \leq \max\left\{\frac{i_{k-1}}{i_k} : i = 1, \ldots, k\right\}.$$

From Lemma 3 we see that the right hand side is at most

$$\left(\frac{n}{\beta - 1}\right)^{\beta-1} + O(n^{\beta-2}).$$

Turning now to the lower bound, we require the existence of a certain family of graphs. We write n as $l(\beta-1)+r$ where $r \in \{1, \ldots, \beta-1\}$ (as we can assume that n is sufficiently large, we take $n \geq 2\beta$, so that $l \geq 2$). We form a graph $G_{l,\beta}$ on disjoint sets $\{v\}, A_1, \ldots, A_{\beta-2}, A_{\beta-1}$, where

(i) A_i induces a complete graph of order l, for $i = 1, \ldots, \beta - 2$,

(ii) $A_{\beta-1}$ induces a complete graph of order $l + r - 1$, and

(iii) for each $i = 1, \ldots, \beta - 1$, v is joined to all but one vertex of A_i.

(see figure 1).

It is not hard to see that $G_{l,\beta}$ has order n and independence number β. If F is any graph, then the independence polynomial of F is simply the product of the independence polynomials of its components, and for any vertex u of F,

$$i(F,x) = x \cdot i(F - N[u], x) + i(F - u, x), \tag{3}$$

where $N[u] = \{u\} \cup \{y \in V(F) : uy \in E(F)\}$ denotes the closed neighbourhood of u in F (this formula follows by partitioning the independent sets of F into those that do and those that do not include vertex u). We apply this formula with $F = G$ and $u = v$ we derive that

$$\begin{aligned} i(G_{l,\beta}, x) &= x \cdot i(G_{l,\beta} - N[v], x) + i(G_{l,\beta} - v, x) \\ &= x \cdot i(\overline{K_{\beta-1}}, x) + i((\beta - 2)K_l \cup K_{l+r-1}, x) \\ &= x(1+x)^{\beta-1} + (1+lx)^{\beta-2}(1 + (l+r-1)x). \end{aligned}$$

It follows that

$$\begin{aligned} i(G_{l,\beta}, -l^{\beta-1}) &= -l^{\beta-1}(1-l^{\beta-1})^{\beta-1} + (1-l^{\beta})^{\beta-2}(1-(l+r-1)l^{\beta-1}) \\ &= (-1)^{\beta-1}\left[(l^{\beta}-1)^{\beta-2}((l+r-1)l^{\beta-1}-1)) - (l^{\beta}-l)^{\beta-1}\right] \end{aligned}$$

Now the term inside the square parentheses in the last line is at least

$$(l^{\beta}-1)^{\beta-1} - (l^{\beta}-l)^{\beta-1} > 0$$

as $l \geq 2$. We conclude that $i(G_{l,\beta}, -l^{\beta-1})$ has sign $(-1)^{\beta-1}$. However, $i(G_{l,\beta}, x)$ is a polynomial of degree β with positive leading coefficient, and hence has sign $(-1)^{\beta}$ as $x \to -\infty$. Thus $i(G_{l,\beta}, x)$ has a real root to the left of

$$-l^{\beta-1} = -\left(\frac{n-r}{\beta-1}\right)^{\beta-1} \leq -\left(\frac{n-\beta+1}{\beta-1}\right)^{\beta-1},$$

and hence to the left of $-\left(n/(\beta-1)\right)^{\beta-1} + O(n^{\beta-2})$. This ends the proof of the main result. □

While Theorem 1 provides a complete answer to the problem of determining the maximum modulus of roots of independence polynomials (at least for fixed independence number), the question becomes increasingly interesting for various families of graphs.

One might think that the real roots of the independence polynomials of a simple class such as trees might be better behaved, but such is not the case. Of course, as the independence number of a tree of order n is at least $n/2$ (as the graph is bipartite), it does not make sense to fix the independence number for trees. Nevertheless, there are indeed real roots of large absolute values for trees. Consider the k-star $K_{1,k}$ with vertices $\{w, v_1, \ldots, v_k\}$ and edges $wu_1, \ldots, wu_k$. We form a new tree T_n by attaching a new leaf u_i to each vertex v_i ($i = 1, \ldots, k$). Indeed T_n has order $n = 2k+1$, and by the formula (3) for calculating independence polynomials, we derive that

$$\begin{aligned} i(T_n, x) &= i(kK_2, x) + x \cdot i(\overline{K_k}, x) \\ &= (1+2x)^k + x(1+x)^k \end{aligned}$$

Now from this formula, we see that

$$i(T_n, -2^{k-1}) = (1-2^k)^k - 2^{k-1}(1-2^{k-1})^k$$

so that

$$\begin{aligned} (-1)^k i(T_n, -2^{k-1}) &= (2^k - 1)^k \left(1 - \frac{1}{2}\left(\frac{2^k-2}{2^k-1}\right)^k\right) \\ &> 0. \end{aligned}$$

Hence $i(T_n, -2^{k-1})$ has sign $(-1)^k$. On the other hand, $i(T_n, x)$ is a monic polynomial of degree $\beta(T_n) = k+1$, so it follows that $i(T_n, x)$ has a root in $(-\infty, -2^{k-1}) = (-\infty, -2^{(n-3)/2})$.

On the other hand, line graphs of trees have real roots of much smaller absolute value. In this case, the independence number (being the matching number of the original graph) can be quite small compared to the order (for example, the independence number of the line graph of a star is trivially 1). Here we see that indeed one can bound the absolute value of the real roots of the independence polynomial in terms of only the independence number.

Theorem 4 *Let G be the line graph of a tree T, and let the independence number of G be β. Then all roots of $i(G,x)$ have modulus at most $\binom{\beta}{2}$.*

Proof A result of Newton's (c.f. [3]) states that if $p(x) = \sum_{i=0}^{d} a_i x^i$ is a real polynomial with all real roots, then

$$a_i^2 \geq a_{i-1}a_{i+1}$$

for $i = 1, \ldots, d-1$, that is the sequence $a_0, \ldots, a_d$ is *log concave*. Recall from the proof of Theorem 1 the Eneström–Kakeya Theorem, which states that the roots of the real polynomial $p(x) = \sum_{i=0}^{d} a_i x^i$ are bounded by

$$\max\left\{\frac{a_{k-1}}{a_k} : k = 0, \ldots, d-1\right\},$$

and simple inequalities show that for any log concave sequence $a_0, \ldots, a_d$ of positive terms, the maximum of this set occurs for $k = d$. Thus any real polynomial $p(x) = \sum_{i=0}^{d} a_i x^i$ that has all real roots has all of its roots bounded in absolute value by a_{d-1}/a_d.

Now the *matching polynomial* [6] of a graph H of order n is given by $m(H,x) = \sum m_i(-1)^i x^{n-2i}$, where m_i is the number of matchings of size i in H. A well known result [9] states that matching polynomials always have all real roots. It follows that as $i(G,x) = \sum m_i x^i = x^{-n} m(T, -1/\sqrt{x})$, all the roots of $i(G,x)$ are real as well. Hence as noted above, to prove that the modulus of all the roots of $i(G,x)$ are bounded by $\binom{\beta}{2}$, all we need to do is to show that

$$\frac{m_{\beta-1}}{m_\beta} \leq \binom{\beta}{2},$$

where β is the matching number (i.e. the size of a maximum matching) of T.

Suppose that M' is a matching of T of cardinality $a-1$. Then we claim that there is a matching M of G of cardinality β such that the symmetric difference $M \triangle M'$ of M and M' is an odd path. Let's choose a maximum matching M of G that has the most edges in common with M'. Then as $M \triangle M'$ is a subgraph of maximum degree 2, it consists of even cycles and paths. There can be no even cycles or paths as otherwise we could 'flip' edges to get another maximum matching with more edges in common with M'. It follows that $M \triangle M'$ is the disjoint union of odd paths, and it is not hard to see from the facts that M is a maximum matching and M' has size one smaller than that of M that $M \triangle M'$ must be exactly an odd path whose end edges are in M.

Thus for every matching M' of cardinality $\beta - 1$ there is a matching M of cardinality a and two edges e_1, e_2 such that $M \triangle M'$ is the unique (odd) path in T from e_1 to e_2. Any choice of M, e_1 and e_2 determine at most one such M' (as the uniqueness of the path from e_1 to e_2 in T allows one to recover the matching M'). It follows that the number of such matchings M' is at most the number of ordered pairs $(M, \{e, f\})$, where M is a matching of size a and e and f are edges of T. It follows that

$$m_{\beta-1} \leq m_\beta \binom{\beta}{2},$$

that is,

$$\frac{m_{\beta-1}}{m_\beta} \leq \binom{\beta}{2},$$

and as noted before, we are done. □

We point out that the argument can be carried through almost in its entirety for line graphs in general. The one sticking point is bounding $m_{\beta-1}/m_\beta$ for graphs, rather than just for trees. By counting the number of alternating paths that can be formed when the 'odd' edges are from a fixed matching of size

β, one can show by a straightforward argument that $m_{\beta-1}/m_\beta \leq f(\beta)$ for some function f. This will show that for line graphs in general, the roots of their independence polynomials are bounded by a function of the independence number alone, although the bound derived from the argument is exponential in the independence number and likely far from the truth.

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