Linear Vertex Arboricity, Independence Number and Clique Cover Number

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ABSTRACT. The linear vertex-arboricity of a graph G is defined to the minimum number of subsets into which the vertex-set G can be partitioned so that every subset induces a linear forest. In this paper, we give the upper and lower bounds for sum and product of linear vertex-arboricity with independence number and with clique cover number respectively. All of these bounds are sharp.

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1 Introduction

Throughout this paper, all graphs are simple and finite. Let $G = (V, E)$ be a graph. A subset of V is called an LV-set if it induces a linear forest in G. A partition of V is called an LV-partition if every subset in the partition is an LV-set. *Linear-vertex arboricity* of G, denoted by $\rho'(G)$, is the smallest number of subsets into which the vertex set V can be partitioned so that the partition is an LV-partition.

A clique is a subset of V such that its induced subgraph is a complete graph. A clique is *maximum* if no other clique of G is of larger order. *Clique number* of G, denoted by $\omega(G)$, is the number of vertices in a maximum clique of G. A *clique cover* of size k is a partition of the vertex set V into $V_1, V_2, \ldots, V_k$, such that each V_i is a clique, $1 \le i \le k$. The *clique cover number* of G, denoted by $\vartheta(G)$, is the number of cliques in a smallest clique cover of G. An *independent set* is a subset of vertices, no two of which are adjacent. $\alpha(G)$ is the number of vertices in an independent set of maximum order. It is also called the *independence number* of G.

In [1], Chartrand et al introduced the concept of vertex-arboricity as a generalization of vertex coloring. Linear vertex-arboricity of graphs was first mentioned by Harary [2]. This is followed by a series of papers by other researchers. For examples, see [3] to [6]. In this paper, we consider the sum and product of linear vertex-arboricity with given independence number and given clique cover number respectively. We shall obtain various lower and upper bomlds, all of which are sharp. For terms and notations not defined in the paper, refer to [7].

2 Lemmas

In this section, we shall present some Lemmas. The first Lemma can be found in [6].

Lemma 2.1. *Suppose M is a maximum clique of a graph $G = (V, E)$, and N is a clique in $\langle V \backslash M \rangle$. Then there exists a one-to-one mapping $f\colon N \to M$ such that for any $n \in N$, $nf(n) \notin E$.*

The following Lemma is a direct consequence of Lemma 2.1.

Lemma 2.2. *For any graph $G = (V, E)$, we can construct a sequence of cliques $Q_1, Q_2, \ldots, Q_h$ with the following properties:*

1. $|Q_i| \ge |Q_j| \ge 1$ *if* $j > i$,
2. $Q_i \cap Q_j = \emptyset$ *if* $i \ne j$,
3. Q_i *is a maximum clique in* $G - \bigcup_{j=1}^{i-1} Q_j$,
4. $\bigcup_{j=1}^{h} Q_j = V$, *and*

5. *For each $1 \le i \le h$, we may label elements of Q_i by v_i^k, where $1 \le k \le |Q_i|$, so that $v_j^k v_{j+1}^k \notin E$ for $1 \le j \le h-1$ and $1 \le k \le |Q_{j+1}|$.*

A sequence of cliques of $G = (V, E)$ is called a *standard sequence* of cliques, or simply a *standard sequence* if it satisfies the above conditions. The following Lemma can also be found in [6].

Lemma 2.3. *Suppose a standard sequence is embedded into $\mathbb{Z}^+ \times \mathbb{Z}^+$ so that the j-th clique lies in the j-column, and for each clique, the k-th vertex lies in the k-th row. Then any four vertices in four adjacent columns and in the same row is an LV-set.*

Lemma 2.4. *If G is a graph of order n, then*

$$\rho'(G) + \vartheta(G) \le n + 1, \tag{1}$$

$$\rho'(G) \cdot \vartheta(G) \le \lfloor (n+3)^2/8 \rfloor. \tag{2}$$

Proof: We shall use mathematical induction to obtain the upper bounds for (1) and (2). We can verify that $\rho'(G) + \vartheta(G) \le n + 1$ for $n = 1$ or 2. Suppose that $\rho'(G) + \vartheta(G) \le n+1$ holds for $n = 1, 2, \ldots, k-1$, where $k \ge 3$. Consider any graph G of order k. If G is edgeless, then $\rho'(G) + \vartheta(G) = 1 + k$. Suppose G has at least one edge uv. Let $G = \langle V \backslash \{u, v\} \rangle$ and $G'' = \langle \{u, v\} \rangle$. Then

$$\rho'(G) + \vartheta(G) \le \rho'(G') + \vartheta(G') + \rho'(G'') + \vartheta(G'') \le k + 1,$$

which establishes the upper bound of (1).

To establish the upper bound of (2), we first verify that $\rho'(G) \cdot \vartheta(G) \le \lfloor (n+3)^2/8 \rfloor$ for $1 \le n \le 16$. Suppose that $\rho'(G) \cdot \vartheta(G) \le \lfloor (n+3)^2/8 \rfloor$ for $1 \le n \le k-1$, where $k \ge 17$. Consider any graph G of order k. Let $Q_1, Q_2, \ldots, Q_h$ be a standard sequence of cliques. We shall denote the number of s-cliques in this sequence by $f(s)$.

Suppose $|Q_4| \ge 4$. Let $Q_i = \{u_i^1, u_i^2, u_i^3, u_i^4, \ldots\}$ for $i = 1, 2, 3, 4$. Because the sequence is standard, $A_j = \{u_1^j, u_2^j, u_3^j, u_4^j\}$ are LV-sets for $j = 1, 2, 3, 4$. Let $G^* = \left\langle V - (\bigcup_{j=1}^4 A_j) \right\rangle$. Then

$$\begin{aligned}
\rho'(G) \cdot \vartheta(G) &\le \{\rho'(G^*) + 4\}\{\vartheta(G^*) + 4\} \\
&\le \rho'(G^*) \cdot \vartheta(G^*) + 4\{\rho'(G^*) + \vartheta(G^*)\} + 16 \\
&\le \frac{(k - 16 + 3)}{8} + 4\{k - 16 + 1\} + 16 \\
&\le \frac{(k+3)^2}{8}.
\end{aligned}$$

So we may assume that $\sum_{r\geq 4} f(r) \leq 3$. Suppose $\sum_{r\geq 4} f(r) = 1$ and $w = |Q_1|$. Also suppose the numbers of 3-, 2- and 1-cliques are :x, y and z respectively. Then

$$k = w + 3x + 2y + z, \tag{3}$$

$$\rho'(G) \leq (w+1)/2 + \epsilon_1 + (3x + 2y + \epsilon_2)/4, \tag{4}$$

$$\vartheta(G) \leq (1 - \epsilon_1) + x + y + z, \tag{5}$$

where $\epsilon_1 = 0$ if there exists one vertex in Q_1 which is not adjacent to vertices in any 1-clique of the standard sequence, and $\epsilon_1 = 1$ otherwise. If $\epsilon_1 = 1$ then $\vartheta(G) \leq x + y + z$. To obtain (4), we partition Q_1 into $\frac{w}{2}$ 2-subsets if w is even, and $\frac{w-1}{2}$ 2-subsets plus one 1-subset if w is odd. By Lemma 2.3, we may partition vertices in 3- and 2-cliques into sets of four vertices along the lines $y = 1$, 2 and 3, each inducing one linear forest in G. In the right side of (4), ϵ_2 is a correctional term to allow for subsets having less than four vertices. Substituting (3) into (4) and (5), we get

$$\rho'(G) \cdot \vartheta(G) \leq \frac{(k + w - z + 2 + \epsilon_2 + 4\epsilon_1)(k - w - x + z + 2 - 2\epsilon_1)}{8}$$
$$\leq \frac{\{k + 2 + (\epsilon_2 - x + 2\epsilon_1)/2\}^2}{8}$$

The Lemma follows if we manage to chose ϵ_2 to satisfy $\epsilon_2 \leq 2 + x - 2\epsilon_1$. In the following cases/sub-cases, let $x + y \equiv t \pmod 4$, where $0 \leq t \leq 3$.

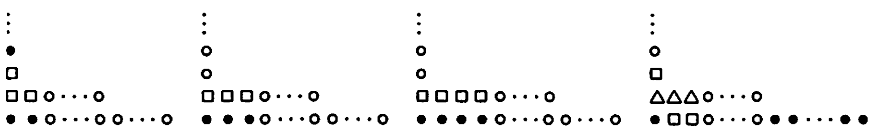
Case 1. $x = 0$.

Sub-case 1.1 $\epsilon_1 = 1$

If $t = 0$, then we may choose $\epsilon_2 = 0$ because the number of 2-cliques is a multiple of 4. If $t = 1$, then four vertices of Q_1 and one 2-clique may form two LV-sets (Figure 1a). Therefore

$$\rho'(G) \leq (w - 4 + 1)/2 + \epsilon_1 + 2 + (3x + 2y - 2)/4$$
$$\leq (w+1)/2 + \epsilon_1 + (3x + 2y)/4,$$

and we may choose $\epsilon_2 = 0$.



(a) $t = 1,\ \epsilon_1 = 1$ (b) $t = 2,\ \epsilon_1 = 1$ (c) $t = 3,\ \epsilon_1 = 1$ (d) $t = 2,\ \epsilon_1 = 0$

Figure 1 $x = 0$

If $t = 2$ or 3, then two vertices of Q_1 and two or three 2-cliques respectively, may be partitioned into two LV-sets, (Figure lb and 1c). So

$$\rho'(G) \leq (w - 2 + 1)/2 + \epsilon_1 + 2 + (3x + 2y - 4)/4$$
$$= (w + 1)/2 + \epsilon_1 + (3x + 2y)/4,$$

and we may also choose $\epsilon_2 = 0$.

Sub-case 1.2 $\epsilon_1 = 0$

If $t = 0$ then $\epsilon_2 = 0$ because the number of 2-cliques is a multiple of 4. If $t = 1$ or 3 then $\epsilon_2 = 2$ because the total number of vertices in 2-cliques plus 2 is a multiple of 4. If $t = 2$, then three vertices of Q_1 together two 2-cliques and all the 1-cliques may be partitioned into three LV-sets (Figure 1d). So

$$\rho'(G) \leq (w - 3 + 1)/2 + 3 + (3x + 2y - 4)/4$$
$$= (w + 1)/2 + \epsilon_1 + (3x + 2y + 2)/4,$$

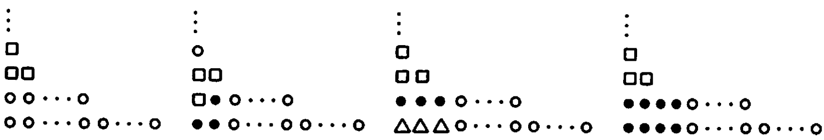
and we may also choose $\epsilon_2 = 2$.

Case 2. $x = 1$.

Sub-case 2.1 $\epsilon_1 = 1$

If $t = 0$, then two vertices of Q_1 together with one vertex in the 3-clique forms one LV-set (Figure 2a), and

$$\rho'(G) \leq (w - 2 + 1)/2 + \epsilon_1 + 1 + (3x + 2y - 1)/4$$
$$= (w + 1)/2 + \epsilon_1 + (3x + 2y - 1)/4$$



(a) $t = 0$ (b) $t = 1$ (c) $t = 2$ (d) $t = 3$

Figure 2. $x = 1$, $\epsilon_1 = 1$

If $t = 1$, then three vertices of Q_1 together with vertices in the 3-clique two form two LV-sets (Figure 2b), and

$$\rho'(G) \leq (w - 3 + 1)/2 + \epsilon_1 + 2 + (3x + 2y - 3)/4$$
$$= (w + 1)/2 + \epsilon_1 + (3x + 2y - 1)/4.$$

If $t = 2$, then four vertices of Q_1 together with vertices in the 3-clique and one 2-clique to form three LV-sets (Figure 2c), and

$$\rho'(G) \leq (w - 4 + 1)/2 + \epsilon_1 + 3 + (3x + 2y - 5)/4$$
$$= (w + 1)/2 + \epsilon_1 + (3x + 2y - 1)/4.$$

If $t = 3$, then four vertices of Q_1 together with vertices in the 3-clique and two 2-clique to form three LV-sets (Figure 2d), and

$$\begin{aligned}\rho'(G) &\leq (w-4+1)/2+\epsilon_1+3+(3x+2y-7)/4 \\ &= (w+1)/2+\epsilon_1+(3x+2y-3)/4.\end{aligned}$$

So for any t, we may also choose $\epsilon_2 = 0$.

Sub-case 2.2 $\epsilon_1 = 0$

For any value of t, argument of sub-case 2.1 is still valid if we replace ϵ_1 by 1, and we get

$$\begin{aligned}\rho'(G) &\leq (w+1)/2+1+(3x+2y-1)/4 \\ &= (w+1)/2+(3x+2y+3)/4.\end{aligned}$$

Therefore for any t, we may choose $\epsilon_2 = 3$.

Cases where $x \geq 2$, and where $\sum_{r\geq 4} f(r) = 0$, 2 and 3 may be dealt with in the similar manner. □

3 Main results

Theorem 3.1. *If $G = (V, E)$ is a graph of order n, then*

$$\lceil\sqrt{2n}\rceil \leq \rho'(G)+\alpha(G) \leq n+1, \quad (6)$$

$$\lceil n/2\rceil \leq \rho'(G)\cdot\alpha(G) \leq \lfloor(n+3)2/8\rfloor. \quad (7)$$

Proof: Let $\rho'(G) = k$. It follows that $\alpha(G) \geq n/(2k)$ and $\rho'(G)\cdot\alpha(G) \geq n/2$. Therefore

$$\rho'(G)+\alpha(G) \geq k+n/(2k) = \left[\sqrt{k}-\sqrt{n/(2k)}\right]^2+\sqrt{2n},$$

from which $\rho'(G)+\alpha(G) \geq \lceil\sqrt{2n}\rceil$ follows. The upper bounds follows from Lemma 2.4 because the clique cover number is in general not less than the independence number. □

The bounds in Theorem 3.1 are sharp. For the upper bound of (6), equality holds for a null graph G on n vertices, and for the lower bound of (7), equality holds for a complete graph on n vertices. Stronger results on sharpness of other bounds are obtained in the following two theorems.

Theorem 3.2. *For any positive integer n, there exists a graph G of order n such that*

$$\rho'(G)+\alpha(G) = \lceil\sqrt{2n}\rceil.$$

Proof: Let m be the positive integer such that $2(m-1)^2 < n \leq 2m^2$ and let $l = n-2(m-1)^2$, then $0 < l \leq 2m^2-2(m-1)^2 = 4m-2$.

If $l \leq 2m-2$, then $2m-2 < \sqrt{2n} < 2m-1$. We construct $G = A_1 \vee A_2 \vee \cdots \vee A_{m-1}$, where A_i are paths of order $2m$ for $1 \leq i \leq m-2$ and of order $l+2$ for $i = m-1$. We can see that $|G| = n$, $\rho'(G) = m-1$, $\alpha(G) = m$, $\rho'(G) + \alpha(G) = 2m-1 = \lceil\sqrt{2n}\rceil$.

If $2m-1 \leq l$, then $2m-1 < \sqrt{2n} \leq 2m$. We construct $G = A_1 \vee A_2 \vee \cdots \vee A_{m-1}$, where A_i are paths of order $2m$ for $1 \leq i \leq m-1$ and of order $l+2-2m$ for $i = m$. We can see that $|G| = n$, $\rho'(G) = m$, $\alpha(G) = m$, $\rho'(G) + \alpha(G) = 2m = \lceil\sqrt{2n}\rceil$. □

Theorem 3.3. *For any positive integer $n \geq 4$, there exists a graph G of order n such that*

$$\rho'(G) \cdot \alpha(G) = \left\lfloor \frac{(n+3)^2}{8} \right\rfloor. \tag{8}$$

Proof: Let $G = N_t \vee K_{n-t}$, where N_t is a null graph of order t. Then $\alpha(G) = t$ and $\rho'(G) = \lfloor (n+3-t)/2 \rfloor$. Set $t = (n+2+s)/2$, where $s \equiv n \mod (4)$, then $\alpha(G) = (n+2+s)/2$, $\rho'(G) = (n+4-s)/4$, and

$$\rho'(G) \cdot \alpha(G) = \frac{(n+3)^2 - (s-1)^2}{8} \tag{9}$$

Because $\rho'(G) \cdot \alpha(G)$ is an integer and $(s-1)^2/8 \leq 1/2$, (8) follows from (9). □

For any graph G, we have $\omega(\bar{G}) = \alpha(G)$. The following corollary is a direct consequence of Theorem 3.1.

Corollary 3.4. *For any graph G of order n,*

$$\lceil\sqrt{2n}\rceil \leq \rho'(G) + \omega(\bar{G}) \leq n+1,$$
$$\lceil n/2 \rceil \leq \rho'(G) \cdot \omega(\bar{G}) \leq \lfloor (n+3)^2/8 \rfloor.$$

and all of the above bounds are sharp

Now we can show that the upper bounds and lower bounds for sum and product of linear vertex-arboricity and clique cover number is the same as those in Theorem 3.1.

Theorem 3.5. *If G is a graph of order n, then*

$$\lceil\sqrt{2n}\rceil \leq \rho'(G) + \vartheta(G) \leq n+1, \tag{10}$$
$$\lceil n/2 \rceil \leq \rho'(G) \cdot \vartheta(G) \leq \lfloor (n+3)^2/8 \rfloor. \tag{11}$$

and all of the bounds are sharp.

Proof: The upper bounds of both (10) and (11) is given in Lemma 2.4. The lower bounds of both (10) and (11) follows from Theorem 3.1 and the fact that $\vartheta(G) \geq \alpha(G)$ for any graph G. □

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