Small 2-factors of Bipartite Graphs

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Abstract

A 2-factor F of a bipartite graph $G = (A, B; E), |A| = |B| = n$ is small if F comprises $\lfloor \frac{n}{2} \rfloor$ cycles. A set $\mathfrak{F}$ of small edge-disjoint 2-factors of G is maximal if $G - \mathfrak{F}$ does not contain a small 2-factor. We study the spectrum of maximal sets of small 2-factors.

1. Introduction

Let $G = (A, B; E)$ be a bipartite graph with $|A| = |B| = n$. Then a 2-factor F of G is called *small* if F consists of $\lfloor \frac{n}{2} \rfloor$ cycles, i.e., for n even, F consists of $n/2$ 4-cycles, for n odd, there is one 6-cycle in F and all the other cycles are of length 4. A set $\mathfrak{F}$ of small edge-disjoint 2-factors of the complete bipartite graph $K_{n,n}$ is said to be maximal if $K_{n,n} - \mathfrak{F}$, the graph obtained by deleting the edges of $\mathfrak{F}$ from $K_{n,n}$, contains no small 2-factor. The *spectrum* for maximal sets of small 2-factors of $K_{n,n}$ is the set *Spec*$(n) = \{k$: there exists a maximal set of k small (edge-disjoint) 2-factors of $K_{n,n}\}$.

Rees and Wallis [10] determined the spectrum of maximal sets of 1-factors of K_{2n}. Hoffman, Rodger and Rosa [4] determined the spectrum for maximal sets of 2-factors of the complete graph K_n and the spectrum of maximal sets of Hamiltonian 2-factors of K_n. Rees [11],[12] and Rees, Rosa, and Wallis

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[13] have, with a few exceptions, determined the spectrum for maximal sets of small 2-factors of K_{3k} (we call a factor of K_m small if it has $\lfloor \frac{m}{3} \rfloor$ cycles). The maximum value of *Spec*(n) for maximal sets of small 2-factors of K_n were determined, for some specific values of $n \neq 3k$ in [1] and in [2]. Bryant, El-Zanati, and Rodger [5] described the spectrum of maximal sets of Hamiltonian 2-factors of $K_{n,n}$. A survey of spectral problems for maximal sets of a wide variety of combinatorial objects can be found in [6]. In this paper we deal with *Spec*(n) for the maximal sets of small 2-factors of $K_{n,n}$. We determine *Spec*(n) for all n odd; for n even we provide some partial results.

2. n odd.

Throughout this section n will stand for an odd number.

Let $\delta(G)$ be a minimum degree of a graph G. The following theorem due to Wang enables us to determine the smallest value of *Spec*(n).

Theorem 1. *[9] Let t be a natural number, $G = (A, B; E)$ be a bipartite graph with $|A| = |B| = n \geq 2t+1$, and let $\delta(G) \geq \lceil \frac{n}{2} \rceil + 1$. Then G contains a 2-factor with exactly t components.*

As an immediate consequence we get

Corollary 2. *Let n be an odd number. Then* min Spec$(n) \geq \lceil \frac{n-1}{4} \rceil$.

Proof. From Theorem 2.1 it follows that if n is an odd number then a bipartite graph $G = (A, B; E)$ with $|A| = |B| = n$ and $\delta(G) \geq \lceil \frac{n}{2} \rceil + 1$ contains a small 2-factor. Hence, if $k \in$ *Spec*(n), then there is a maximal set $\mathfrak{F}$ of k small 2-factors so that the leave $L = K_{n,n} - \mathfrak{F}$, which is a regular graph of degree $n - 2k$, does not contain a small 2-factor, hence $n - 2k \leq \lceil \frac{n}{2} \rceil$; i.e. $k \geq \lceil \frac{n-1}{4} \rceil$. ■

As the maximum possible number of 2-factors of $K_{n,n}$ is $\frac{n-1}{2}$ the above Corollary implies that *Spec*$(n) \subseteq \{k; \lceil \frac{n-1}{4} \rceil \leq k \leq \frac{n-1}{2}\}$. We will show that, in fact, with an exception for $n = 5$ and a possible exception of $n = 9$, there is equality in the previous relation. For the proof of the statement we will need the following technical Lemma.

Lemma 3. *Let $t > 0, m$ be natural numbers, $0 \leq m \leq t/2, t-m \neq 2$. Then there exists a Latin $(t - m) \times t$ rectangle R on the symbol set $\{1, ..., t\}$ so that $R(i, i) = i$ for $i = 1, ..., t - m$, and, for $m > 0$, no element of the first m columns of R is from the set $\{t - m + 1, ..., t\}$.*

Proof. Let $t \geq 1, 0 \leq m \leq \frac{t}{2}, t-m \neq 2$. For $t-m \neq 6$, let A and B be a pair of orthogonal Latin squares of order $t-m$ on the symbol set $\{1, ..., t-m\}$, $A(i,i) = i$ and $B(i,i) = 1$ for $i = 1, ..., t-m$; it is notoriously known that a pair of orthogonal Latin squares of order n exists iff $n \neq 2, 6$, see e.g. [3], the properties $B(i,i) = 1, A(i,i) = i$ can be guaranteed by suitable permutation of rows and columns of A and B. For $t-m = 6$ we set

$$A = \begin{pmatrix} 1 & 3 & 6 & 2 & 4 & 5 \\ 6 & 2 & 4 & 5 & 3 & 1 \\ 5 & 6 & 3 & 1 & 2 & 4 \\ 3 & 1 & 5 & 4 & 6 & 2 \\ 2 & 4 & 1 & 6 & 5 & 3 \\ 4 & 5 & 2 & 3 & 1 & 6 \end{pmatrix}$$

If $m = 0$, then A has the required properties, i.e., we set $R = A$. Further, for t even and $m = t/2$, R is obtained by placing A into the first $t/2$ columns of R and any Latin square of order $t/2$ on the symbol set $\{t/2+1, ..., t\}$ into the last $t/2$ columns of R. Thus assume that $m > 0$ and $m \neq t/2$. We start with the case $t-m \neq 6$. Define a Latin square C of order $t-m$ as follows: for $m+1 \leq j \leq t-m$, if $B(i,j) = k$, where $t-2m+1 \leq k \leq t-m$ then $C(i,j) = B(i,j) + m$, otherwise $C(i,j) = A(i,j)$.

First of all, it is clear that no element occurs twice in a row or in a column of C. Further, C can be seen as obtained by changing some elements in the last $t-2m$ columns of A. Therefore, no element of the first m columns of C is from the set $\{t-m+1, ..., t\}$. Further, $C(i,i) = i$ for $i = 1, ..., t-m$ as $B(i,i) = 1 < t-2m+1$ $(m < t/2)$, thus $C(i,i) = A(i,i)$.

To cover the case $t-m = 6$, that means $(t,m) = (i, i-6), i = 7, ..., 11$, we construct the Latin square C from A (defined for $t-m = 6$) by changing some of its elements. For $(t,m) = (7,1)$, we set $C(i,i+1) = 7, i = 1, ..., 4$, and $C(5,1) = 7$, otherwise $C(i,j) = A(i,j)$; for $(t,m) = (8,2)$, $C(i,i+1) = 7, i = 1, ..., 4$, $C(i,i+2) = 8, i = 1,2,3$, and $C(4,1) = 8$; for $(t,m) = (9,3)$, $C(i,i+1) = 7, i = 1,2,3$, $C(i+3,i) = 8, i = 1,2,3$, and $C(i,i+2) = 9, i = 1,2,3$; for $(t,m) = (10,4)$, $C(i,i+1) = 7, i = 1,2$, $C(i,i+1) = 8, i = 3,4$, $C(i,i+2) = 9, i = 1,2$, and $C(i,i+3) = 10, i = 1,2$; and for $(t,m) = (11,5)$, we set $C(6,i) = i+6, i = 1, ..., 5$.

To finish the proof we show, by applying Ryser's Theorem, see [7], that C can be completed to a Latin square D of order t, as then R with the required properties might be obtained by taking the first $t-m$ rows of D (in fact we need only to extend C into Latin $(t-m) \times t$ rectangle). This means we need to show that each element of the symbol set $\{1, ..., t\}$ occurs in C at least $(t-m) + (t-m) - t = t-2m$ times. For $t-m \neq 6$, each element of the set $\{t-2m+1, ..., t-m\}$ occurs in the last $t-2m$ columns of B exactly $t-2m$ times, thus each element of $\{t-m+1,...,t\}$ occurs in C exactly $t-2m$ times. Moreover, as A and B are orthogonal, if we take $t-2m$ occurrences of a fixed element k in the last $t-2m$ columns of B, then their mates in A (elements occupying in A the same cells as k in B) are distinct elements from $\{1,...,t-m\}$. Hence, when replacing elements of A to construct C, we substitute each element of $\{1,...,t-m\}$ at most m times, so each element of $\{1,...,t-m\}$ occurs in C at least $(t-m)-m=t-2m$ times. For $t-m=6$, when constructing the matrix C, any fixed value from $\{t-m+1,...,t\}$ has replaced in A distinct elements. Thus also in this case C satisfies the conditions of Ryser's theorem. ■

With this in hand we are ready to prove:

Theorem 4. *Let $n \geq 3$ be an odd number. Then, for $n \notin \{5,9\}$,* $\mathrm{Spec}(n) = \{k; \lceil \frac{n-1}{4} \rceil \leq k \leq \frac{n-1}{2}\}$, $\mathrm{Spec}(5)=\{1\}$, $\mathrm{Spec}(9) \subset \{3,4\}$.

Remark 1. *Thus, the only open case is whether* $2 \in \mathrm{Spec}(9)$.

Proof. Trivially, $Spec(3)=1$. Let F be a small 2-factor of $K_{5,5}$. It is easy to check that $K_{5,5}-F$ does not contain a small 2-factor. This implies that $1 \in Spec(5)$ but $2 \notin Spec(5)$. Thus, $Spec(5)=\{1\}$.

Now let $n>5$ and $\lceil \frac{n-1}{4} \rceil \leq k \leq \frac{n-1}{2}, (n,k) \notin \{(7,2),(9,2)\}$. We will construct a maximal set of k small 2-factors $F_1,...,F_k$ of $K_{n,n}$. Let A, B form the bipartition of the vertex set of $K_{n,n}$, $A=\{a_1,...,a_n\}, B=\{b_1,...,b_n\}$. The 2-factors F_i's will be constructed so that the subgraph of the leave $L(n,k)=K_{n,n}-\bigcup_{i=1}^{k} F_i$ induced by the set $C=\{a_1,...,a_s\} \cup \{b_{n-s+2},...,b_n\}$, $s=n-2k$, is a complete bipartite graph $K_{s,s-1}$. Then clearly $L(n,k)$ does not contain a small 2-factor as $s-1$ is an even number.

Set $t=\frac{n-1}{2}, m=\frac{s-1}{2}$. Then $k=t-m \neq 2$ as we excluded pairs $(n,k)=(7,2),(9,2)$, and $m \leq \frac{t}{2}$ is implied by $\lceil \frac{n-1}{4} \rceil \leq k$. Let R be a Latin $(t-m) \times t$ rectangle with properties guarantied by the previous Lemma. Now we are ready to define the sought k small edge-disjoint 2-factors $F_1,...,F_k$.

If, for $i \neq j$, $R(i,j)=l$ then the 4-cycle $a_{2j}b_{2l}a_{2j+1}b_{2l+1}a_{2j}$ belongs to F_i. The (only) 6-cycle of F_i is the cycle $a_1b_{2i}a_{2i+1}b_1a_{2i}b_{2i+1}a_1$. As each row of R comprises distinct elements $F_i, i=1,...,k$ is a small 2-factor. Further, since R is a Latin rectangle with the poroperty $R(i,i)=i$, the 2-factors F_i are pairwise edge-disjoint. Finally, no element in the first m columns of R is from the set $\{t-m+1,...,t\}=\{k+1,..,t\}$. This means, no vertex from $\{a_i, i=1,...,s\ (=2m+1)\}$ is in $\bigcup_i F_i$ adjacent to any vertex in $\{b_j, j=2k+2,...,n\}=\{b_j, j=n-s+2,...,n\}$. Thus, the subgraph of $L(n,k)$ induced by C is isomorphic to $K_{s,s-1}$.

At the very end we show that $2 \in Spec(7)$. In this case we cannot use the above construction because there does not exist a Latin 2×3 rectangle R with properties as in Lemma 2.3. The question whether $2 \in Spec(9)$ is open as we do not have such 2×4 rectangle. We set $F_1 = \{a_1b_1a_2b_2a_1, a_3b_3a_4b_4a_3, a_5b_5a_6b_6a_7b_7a_5\}$, $F_2 = \{a_1b_3a_5b_4a_1, a_6b_1a_7b_2a_6, a_2b_5a_3b_7a_4b_6a_2\}$. Then the leave $K_{7,7} - (F_1 \cup F_2)$ contains a complete bipartite graph $K_{3,2}$ on the vertex set $\{a_3, a_4, a_5\} \cup \{b_1, b_2\}$, thus the leave does not possess a small 2-factor. ■

3. n even

Throughout this section n will stand for an even number. This case is much more complicated than the case of odd n. We will be able to determine $Spec(n)$ only for $n \leq 8$, otherwise we provide some partial results.

3.1. $Spec(n)$ for $n \leq 8$

We will make use of the following Lemmas.

Lemma 1. *For all n, $n/2 \in$ Spec(n).*

Proof. To prove the statement we need to show that $K_{n,n} = G(A, B; E), A = \{a_1, ..., a_n\}, B = \{b_1, ..., b_n\}$ has a factorization into small 2-factors. Consider a 1-factorization $\{H_1, ..., H_{n/2}\}$ of $K_{n/2,n/2} = G'(A', B'; E')$, $A' = B' = \{1, ..., n/2\}$. Then we can define $n/2$ small edge-disjoint 2-factors $F_1, ..., F_{n/2}$ of G as follows: If an edge $ij, i \in A', j \in B'$, belongs to the 1-factor H_t then the 4-cycle $a_{2i-1}b_{2j-1}a_{2i}b_{2j}a_{2i-1}$ is in F_t. ■

Lemma 2. *For $n > 2, 1 \notin$ Spec(n).*

Proof. Let F be a small 2-factor of $G = K_{n,n}$. WLOG we assume that $F = \{a_{2i-1}b_{2i-1}a_{2i}b_{2i}a_{2i-1} : 1 \leq i \leq n/2\}$. Then $\{a_{2i-1}b_{2i+1}a_{2i}b_{2i+2}a_{2i-1} : 1 \leq i \leq n/2\}$, where the indices are taken $\bmod n$ from the set $\{1, ..., n\}$, is a small 2-factor of $G - F$, hence $1 \notin Spec(n)$. ■.

Lemma 3. *If $n = 6, 8$, then $2 \notin$ Spec(n).*

Proof. Let $K_{n,n} = G(A, B; E)$ and F be a small 2-factor of G. Then F determines a partition $F(A)$ of A into $n/2$ 2-subsets as well as a partition $F(B)$ of B into $n/2$ 2-subsets, where two vertices of A (of B) belong to the same part of $F(A)$ ($F(B)$) if they are on the same cycle of F.

For $n = 6$, if H is a small 2-factor of $G - F$ then either $F(A) = H(A)$ or $F(B) = H(B)$ implying that $G - (F \cup H)$ is a small 2-factor T, where $T(A) = F(A) = H(A)$ or $T(B) = F(B) = H(B)$, hence $2 \notin Spec(6)$.

For $n = 8$ the situation is a bit more involved. Let H be a small 2-factor of $G-F$. We show how to construct a small 2-factor T of $G-(F\cup H)$. Suppose first that $F(B)\cap H(B) = \emptyset$. Then there is a small 2-factor T of $G-(F\cup H)$ so that $T(A) = H(A)$ and if a cycle $a_i b_i a_j b_j a_i \in H$ then $a_i x a_j y a_i \in T$, where $\{x, b_i\}, \{y, b_j\} \in F(B)$. So, by symmetry, we are left with the case $F(A) \cap H(A) \neq \emptyset$ and $F(B) \cap H(B) \neq \emptyset$. In a similar way as above we define T in the case if there is no cycle C in H so that $V(C) \cap F(A) \neq \emptyset$, and $V(C) \cap F(B) \neq \emptyset$. An inspection of all remaining possibilities shows that also in these cases there is a small 2-factor in $G-(F\cup H\}$. ■

Theorem 4. Spec(2) = {1}, Spec(4) = {2}, Spec(6) = {3}, Spec(8) = {3, 4}.

Proof. *Spec*(2) = 1 is equivalent to stating that $K_{2,2}$ has a small 2-factor while Lemmas 3.1 and 3.2 imply *Spec*(4) = {2}. Lemmas 3.1-3.3 show that *Spec*(6) = {3}. The following 8×3 matrix $C = [c_{ij}]$ defines three small edge-disjoint 2-factors F_1, F_2, F_3 of $K_{8,8} = G(A, B; E)$, $A = \{a_1, ..., a_8\}, B = \{b_1, ..., b_8\}$ by: if $c_{ij} = \{x, y\}$ then a_i is in F_j on a 4-cycle with b_x and b_y. The three factors form a maximal set as $G-(F_1\cup F_2\cup F_3)$ comprises two cycles of length 8, the first one on vertices $\{a_1, ..., a_4\}\cup\{b_1, ..., b_4\}$ the second on $\{a_5, ..., a_8\}\cup\{b_5, ..., b_8\}$.

$$C = \begin{pmatrix} 5,6 & 4,7 & 3,8 \\ 5,6 & 1,8 & 4,7 \\ 7,8 & 2,5 & 1,6 \\ 7,8 & 3,6 & 2,5 \\ 1,2 & 3,6 & 4,7 \\ 1,2 & 4,7 & 3,8 \\ 3,4 & 1,8 & 2,5 \\ 3,4 & 2,5 & 1,6 \end{pmatrix}$$

Together with Lemmas 3.1-3.3 this implies that *Spec*(8) = {3, 4}. ■

3.2. A sufficient condition for the existence of a small 2-factor

As for the other values of n, it is difficult to describe *Spec*(n) because the answer to the following question is not known yet.

Given a bipartite graph $G = (A, B; E), |A| = |B| = 2t$. What is the smallest number d so that the condition $\delta(G) \geq d$ guarantees that G contains a small 2-factor, that is, a 2-factor comprising t cycles of length 4?

Thus, we do not have a lower bound on min *Spec*(n). It is conjectured that, see[8], $\delta(G) \geq t+1$ guarantees a small 2-factor in G. If true, it would be the best possible. In the same paper it is shown, to support the conjecture, that $\delta(G) \geq t+1$ guarantees a small 2-factor without one edge, i.e., G contains $t-1$ cycles of length 4 and a path of length 4 so that all of them are vertex-disjoint.

3.3. *Spec*(n) for $n \geq 10$.

With respect to the result of Hong Wang stated above we believe that the following is true:

Conjecture 5. *For* $n \geq n_0$, $\mathrm{Spec}(n) = \{\lfloor \frac{n+4}{4} \rfloor, ..., \frac{n}{2}\}$.

In what follows we provide partial results supporting this Conjecture. We show that the expected minimum value $\lfloor \frac{n+4}{4} \rfloor$, up to two possible exceptions, belongs to *Spec*(n) and that the "second half" of the conjectured interval is in *Spec*(n) as well. Namely,

Theorem 6. *If* $n \geq 10, n \notin \{12, 18\}$, *then* $\lfloor \frac{n}{4} \rfloor + 1 \in \mathrm{Spec}(n)$.

and

Theorem 7. *If* $n \geq 16$, *then* $\{i, \lceil \frac{3n+4}{8} \rceil + d \leq i \leq \frac{n}{2}\} \subset \mathrm{Spec}(n)$, *where* $d = 1$ *for* $n \in \{36, 38\}$, *otherwise* $d = 0$.

3.4. Minimum value of *Spec*(n)

In this subsection we prove Theorem 3.6. The construction proving the statement is rather complicated and is based on the following technical lemma.

Let $N_k = \{1, 2, ..., k\}$, and $N_k^* = N_k \cup \{\infty\}$. We set, for $i \in N_k$, $\infty \pm i = \infty$. Consider a sequence $S = (a_1, a_2, ...a_k)$, $a_i \in N_k^*, 1 \leq i \leq k$. In what follows a finite difference $a_i - i$ will always be taken modulo k from the set N_k. The sequence S is a k_1- *sequence* if the following conditions are satisfied:

(i) $a_1 = 1$,

(ii) $a_{k-1} = k$;

(iii) Each element of N_k^* occurs in S at most once (i.e., exactly one element of N_k^* does not occur in S);

(iv) $\{a_i - i; 1 \leq i \leq k\} = N_k^* - \{k-1\}$ (each element from $N_k^* - \{k-1\}$ occurs exactly once as a difference $a_i - i$).

The sequence S is a k_2- *sequence* if S satisfies (ii)-(iii), and

(i') $a_1 = 2$;

(iv') $\{a_i - i; 1 \leq i \leq k\} = N_k^* - \{k-1, k\}$, ((i') and (ii) imply that the number 1 occurs twice as a difference $a_i - i$).

Lemma 8. *A* k_i-*sequence,* $i = 1, 2$, *exists for any* $k \in N - \{1, 2, 4\}$.

Proof. For both, k_1-sequences and k_2−sequences, we prove the existence by recursive constructions. We start with k_1−sequences. It is easy to check that $S_3 = (1, 3, \infty)$, $S_5 = (1, 4, \infty, 5, 3)$, $S_6 = (1, \infty, 5, 2, 6, 3)$, $S_7 = (1, 6, 5, 2, \infty, 7, 3)$, and $S_8 = (1, \infty, 7, 6, 2, 4, 8, 3)$ are 3_1-, 5_1-, 6_1-, 7_1-, and 8_1-sequence, respectively. Now we show how to construct, for $k \geq 5$, a $(k+4)_1-$ sequence S' from a k_1-sequence $S = (a_1, a_2, ..., a_k)$. Set $S' = (b_1, b_2, ..., b_{k+4}) = (1, \infty, k+3, k+2, a_3+2, a_4+2, ..., a_{k-2}+2, 2, 5, k+4, 3)$ for k even, and $S' = (b_1, b_2, ..., b_{k+4}) = (1, k+3, k+2, k+1, a_3+2, a_4+2, ..., a_{k-2}+2, 2, 5, k+4, 3)$ for k odd (that is, $b_i = a_{i-2}+2, 5 \leq i \leq k$).

Thus S' satisfies (i) and (ii). Further, $b_i \in N^*_{k+4}$ (for $5 \leq i \leq k, b_i = a_{i-2}+2 \leq k+2$ or b_i equals ∞ for k odd) hence $b_i \in N^*_k$. As $a_3, ..., a_{k-2}$ are distinct numbers, $2 \leq a_i \leq k-1, a_i \neq 3$ for $3 \leq i \leq k-2$ or a_i equals ∞ for k odd, also $b_5, ..., b_k$ are distinct, and $4 \leq b_i \leq k+1, b_i \neq 5, 5 \leq i \leq k$, or b_i may equal ∞ for k odd, which, together with the definition of b_i for $1 \leq i \leq 4$ and $k+1 \leq i \leq k+4$, implies (iii).

To see that S' fulfills (iv) we prove that each k_1- sequence $C = (c_1, c_2, ..., c_k)$ obtained by the above recursive construction satisfies,

for k even:

(a) $c_i - i = 2(k/2 - i + 1)$ for $3 \leq i \leq k/2$, that is, $\{c_i - i; 1 \leq i \leq k/2\} = \{2, 4, 6, ..., k-4, k\} \cup \{\infty\}$; and

(b) For $k/2+1 \leq i \leq k-2$, it is $i > c_i$, and $\{c_i - i; k/2+1 \leq i \leq k\} = \{2j+1; 0 \leq j \leq k/2-2\} \cup \{k-2\}$;

for k odd:

(c) $c_i - i = 2(\frac{k+1}{2} - i)$ for $2 \leq i \leq \frac{k-1}{2}$, that is, $\{c_i - i; 1 \leq i \leq \frac{k-1}{2}\} = \{2, 4, 6, ..., k-3, k\}$;

(d) For $\frac{k+1}{2} \leq i \leq k-2$, if $c_i \neq \infty$ then $i > c_i$, and $\{c_i - i; \frac{k+1}{2} \leq i \leq k\} = \{2j+1; 0 \leq j \leq \frac{k-3}{2}\} \cup \{\infty\}$.

First we deal with k even. Clearly S_6 and S_8 satisfy (a) and (b). To finish the proof we show that if S has the properties then also S' does. By the definition of $S', b_3 - 3 = k, b_4 - 4 = k-2$, and $b_i - i = a_{i-2} + 2 - i = a_{i-2} - (i-2) = 2(k/2 - (i-2) + 1) = 2(\frac{k+4}{2} - i + 1)$ for $5 \leq i \leq \frac{k+4}{2}$, thus (a) follows. Further, for $\frac{k+4}{2} + 1 \leq i \leq k, b_i = a_{i-2} + 2 < (i-2) + 2 = i, b_{k+1} = 2 < k+1, b_{k+2} = 5 < k+3$, hence $b_i < i$ for $\frac{k+4}{2} + 1 \leq i \leq k+2$. It is, for $\frac{k+4}{2} + 1 \leq i \leq k$, $b_i - i = a_{i-2} + 2 - i = a_{i-2} - (i-2) < 0$. Therefore, if $a_{i-2} - (i-2) \equiv d (\text{mod}\, k), d \in N_k$ then $b_i - i \equiv d + 4 (\text{mod}\, k+4)$. As $a_{k-1} - (k-1) = 1$ and $a_k - k = 3$, $\{a_i - i; k/2+1 \leq i \leq k-2\} = \{2j+1; 2 \leq j \leq k/2-2\} \cup \{k-2\}$. Hence $\{b_i - i; \frac{k+4}{2} + 1 \leq i \leq k\} = \{2j+1; 4 \leq i \leq k/2\} \cup \{k+2\}$. Since $\{b_i - i; k+1 \leq i \leq k+4\} = \{1, 3, 5, 7\}$, (b) follows. The proof of (c) and (d) for k odd is omitted as it can be obtained from the even case by slight modifications.

As to constructing k_2- sequences, the starters are $S_3 = (2,3,\infty)$, $S_5 = (2,4,\infty,5,3)$, $S_6 = (2,\infty,5,1,6,4)$, $S_7 = (2,5,\infty,6,3,7,4)$, $S_8 = (2,6,5,\infty,3,1,8,5)$, $S_9 = (2,8,7,6,3,\infty,1,9,5)$, $S_{10} = (2,8,7,6,3,\infty,4,1,10,5)$, $S_{11} = (2,10,9,8,7,4,3,\infty,1,11,5)$, $S_{12} = (2,10,9,8,7,4,\infty,3,6,1,12,5)$, and $S_{13} = (2,12,11,10,9,8,3,\infty,7,4,1,13,5)$.

Let $S = (a_1, a_2, ..., a_k)$, $k \geq 8$, be a k_2- sequence. Then $S' = (b_1, b_2, ..., b_{k+6})$ $= (2, k+4, k+3, k+2, a_2+3, a_3+3, ..., a_{k-2}+3, 3, 8, 1, k+6, 5)$ for k even, and $S' = (2, k+5, k+4, k+3, a_2+3, a_3+3, ..., a_{k-2}+3, 3, 8, 1, k+6, 5)$ for k odd, is a $(k+6)_2-$sequence as well.

The properties (i') and (ii) follow from the definition of S'. Moreover, the same type of argument as before implies (iii). To see (iv') the reader can verify the following using the same reasoning as for k_1-sequences. For any sequence $(c_1, c_2, ..., c_k)$ constructed in the above manner we have $\{c_i - i; 1 \leq i \leq \lfloor\frac{k-1}{2}\rfloor\} = \{2j; 1 \leq j \leq \lfloor\frac{k-3}{2}\rfloor\} \cup \{1\}$, and $\{c_i - i; \lfloor\frac{k+1}{2}\rfloor \leq i \leq k\} = \{2j+1, 0 \leq j \leq \lfloor\frac{k-3}{2}\rfloor\} \cup \{\infty\} \cup A$, where $A = \emptyset$ for k odd, and $A = \{k-2\}$ for k even. ■

Proof of Theorem 3.6. For $n = 10, 16, 20$, $\lfloor\frac{n+4}{4}\rfloor = k$ small edge-disjoint 2-factors $F_1, ..., F_k$ of $K_{n,n} = G(A, B; E)$, $A = \{a_1, ..., a_n\}, B = \{b_1, ..., b_n\}$ are given by the matrix C_n, where $c_{ij} = \{x, y\}$ means that $a_i \in A$ is in F_j on a 4$-$ cycle with b_x and b_y.

$$C_{10} = \begin{pmatrix} 5,8 & 6,9 & 7,10 \\ 7,10 & 6,9 & 4,8 \\ 4,9 & 5,8 & 7,10 \\ 5,8 & 4,10 & 6,9 \\ 7,10 & 1,2 & 3,5 \\ 4,9 & 3,7 & 1,2 \\ 1,2 & 4,10 & 3,5 \\ 1,2 & 3,7 & 6,9 \\ 3,6 & 5,8 & 1,2 \\ 3,6 & 1,2 & 4,8 \end{pmatrix}$$

$$C_{16} = \begin{pmatrix} 2,3 & 6,12 & 15,16 & 4,5 & 13,14 \\ 13,14 & 4,5 & 2,12 & 1,6 & 15,16 \\ 4,12 & 15,16 & 1,6 & 13,14 & 2,3 \\ 1,5 & 6,12 & 15,16 & 2,3 & 13,14 \\ 13,14 & 3,1 & 2,12 & 4,5 & 15,16 \\ 4,12 & 15,16 & 3,5 & 13,14 & 1,6 \\ 2,3 & 13,14 & 10,11 & 8,9 & 7,12 \\ 1,5 & 13,14 & 10,11 & 8,9 & 7,12 \\ 6,7 & 4,5 & 13,14 & 10,11 & 8,9 \\ 6,7 & 3,1 & 13,14 & 10,11 & 8,9 \\ 8,9 & 2,7 & 1,6 & 15,16 & 10,11 \\ 8,9 & 2,7 & 3,5 & 15,16 & 10,11 \\ 10,11 & 8,9 & 4,7 & 1,6 & 2,3 \\ 10,11 & 8,9 & 4,7 & 2,3 & 1,6 \\ 15,16 & 10,11 & 8,9 & 7,12 & 4,5 \\ 15,16 & 10,11 & 8,9 & 7,12 & 4,5 \end{pmatrix}$$

$$C_{20} = \begin{pmatrix} 2,3 & 17,18 & 8,16 & 19,20 & 4,5 & 6,7 \\ 19,20 & 4,5 & 17,18 & 2,16 & 6,7 & 1,8 \\ 4,16 & 19,20 & 6,7 & 17,18 & 1,8 & 2,3 \\ 17,18 & 6,16 & 19,20 & 1,8 & 2,3 & 4,5 \\ 1,5 & 17,18 & 8,16 & 19,20 & 2,3 & 6,7 \\ 19,20 & 3,7 & 17,18 & 2,16 & 4,5 & 1,8 \\ 4,16 & 19,20 & 1,5 & 17,18 & 6,7 & 2,3 \\ 17,18 & 6,16 & 19,20 & 3,7 & 1,8 & 4,5 \\ 8,9 & 14,15 & 2,3 & 12,13 & 17,18 & 10,11 \\ 8,9 & 14,15 & 6,7 & 12,13 & 17,18 & 10,11 \\ 10,11 & 2,9 & 14,15 & 1,8 & 12,13 & 19,20 \\ 10,11 & 2,9 & 14,15 & 4,5 & 12,13 & 19,20 \\ 1,5 & 10,11 & 4,9 & 14,15 & 19,20 & 12,13 \\ 2,3 & 10,11 & 4,9 & 14,15 & 19,20 & 12,13 \\ 12,13 & 3,7 & 10,11 & 6,9 & 14,15 & 17,18 \\ 12,13 & 4,5 & 10,11 & 6,9 & 14,15 & 17,18 \\ 6,7 & 12,13 & 1,5 & 10,11 & 16,9 & 14,15 \\ 6,7 & 12,13 & 2,3 & 10,11 & 16,9 & 14,15 \\ 14,15 & 1,8 & 12,13 & 4,5 & 10,11 & 16,9 \\ 14,15 & 1,8 & 12,13 & 3,7 & 10,11 & 16,9 \end{pmatrix}$$

$L_n = K_{n,n} - \bigcup_{i=1}^{k} F_i$ contains, for $n = 10$, a complete bipartite graph $H = K_{3,4}$ on the set $\{a_1, ..., a_4\} \cup \{b_1, ..., b_3\}$ and a matching edge-disjoint with H and covering $\{a_1, ..., a_4\}$, thus L_{10} does not contain a small 2-factor, which yields 3 $\in Spec(10)$. For $n = 16, 20$, L_n contains a complete bipartite graph $H = K_{n-2k,n-2k-1}$ on the set $\{a_1, ..., a_{n-2k}\} \cup \{b_{n-2k+1}, ..., b_{2n-4k-1}\}$ and a matching covering $\{a_1, ..., a_{n-2k}\}$ which is edge disjoint with $\dot{H}$. Hence $5 \in Spec(16), 6 \in Spec(20)$.

First we consider the case $n \equiv 2(\mathrm{mod}\, 4)$, $n \geq 14, n \neq 18$. Let $K_{n,n} = G(A, B; E)$, $A = \{a_1, a_2, ..., a_n\}$, $B = \{1, 2, ..., 2k\} \cup \{1', 2', ..., (2k-1)'\} \cup \{\infty_1, \infty_2, \infty_3\}$, where $k = \frac{n-2}{4}$. We will construct $k+1 = \lfloor \frac{n+4}{4} \rfloor$ small edge-disjoint 2-factors $F_1, F_2, ..., F_{k+1}$ of G so that the leave $L_n = K_{n,n} - \bigcup_{i=1}^{k+1} F_i$ does not possess a small 2-factor. The 2-factors $F_i s$ will be constructed so that the leave L_n will contain a complete bipartite graph $H = K_{2k,2k-1}$ on the vertex set $\{a_1, ..., a_{2k}\} \cup \{1', ..., (2k-1)'\}$ and a matching edge disjoint with H covering $A' = \{a_1, ..., a_{2k}\}$, thus L does not contain a small 2-factor. To simplify the description of the construction of $F_i's$ we will construct a matrix $C_n = [c_{ij}], 1 \leq i \leq n, 1 \leq j \leq k+1$ so that the element c_{ij} equals the pair of vertices of B which are neighbours of the vertex a_i in the small 2-factor F_j. Thus, we need to construct a matrix C_n so that each element of C_n is a pair of vertices of B, where

(i) for $1 \leq i \leq n$, the $i-th$ row of C_n comprises $2(k+1)$ distinct vertices of B;

(ii) for $1 \leq j \leq k+1$, each vertex of B occurs exactly twice in the $j-th$ column of C_n, and if $c_{ij} = \{x, y\}$ then there is $l, l \neq i$, so that $c_{lj} = \{x, y\}$ (the vertices a_i, a_l, x, y form a 4-cycle in F_j).

We start with the first row of C_n. Let $S = (s_1, s_2, ..., s_k)$ be a k_1-sequence. As $k \notin \{1, 2, 4\}$, the existence of k_1- sequence is guaranteed by Lemma 3.6. Then, if $s_j = \infty$, we set $c_{1j} = \{\infty_2, \infty_3\}$, and $c_{1,k-1} = \{2s_{k-1}, \infty_1\} = \{2k, \infty_1\}$, otherwise, for $1 \leq j \leq k, j \neq k+1$ and $s_j \neq \infty$, $c_{1j} = \{2s_j, 2s_j + 1\}$. Let t be the only element of N_k^* which does not occur in S (see (iii) in the definition of k_1-sequence), then we set $c_{1,k+1} = \{2t, 2t+1\}$. Thus,

$$\bigcup_{j=1}^{k+1} c_{1j} = \{2, 3, ..., 2k\} \cup \{\infty_1, \infty_2, \infty_3\}, \tag{3.1}$$

this is, no vertex of B occurs is the first row more than once, and

Claim 1. The vertex a_1 is in the leave L_n adjacent to vertices of $\{1', 2', ..., (2k-1)'\} \cup \{1\}$.

The next $k-1$ rows of C_n will be constructed in a recursive manner. All values in c_{ij} are taken modulo $2k$ from the set N_{2k}, $\infty_i + 2 = \infty_i$ for $i = 1, 2, 3$. For $2 \leq i \leq k$, we set

(a) If $c_{i-1,k+1} = \{x, y\}$ then $c_{i,k+1} = \{x+2, y+2\}$;
(b) If $c_{i-1,k} = \{x, y\}$ then $c_{i,1} = \{x+2, y+2\}$
(c) If $c_{i-1,j} = \{x, y\}$ then $c_{i,j+1} = \{x+2, y+2\}$ for $1 \leq j \leq k-1$.

Hence, from (3.1) and from (a)-(c), for $2 \le i \le k$, $\bigcup_{j=1}^{k+1} c_{ij} = (N_{2k} - \{2i - 1\}) \cup \{\infty_1, \infty_2, \infty_3\}$, that is, no vertex of B occurs in the $i-th$ row more than once, and

Claim 2. The vertex $a_i, i = 2, ..., k$, is adjacent in L_n to vertices $\{1', 2', ..., (2k-1)'\} \cup \{2i - 1\}$.

From (b) and (c) we get

(d) $c_{31} = \{4, \infty_1\}$, and for $2 \le i \le k$, $i \ne 3$, if $s_{k+2-i} \ne \infty$, then $c_{i1} = \{2s_{k+2-i} + 2(i-1), 2s_{k+2-i} + 2(i-1) + 1\}$, otherwise $c_{ij} = \{\infty_2, \infty_3\}$;

(e) if $c_{k,j-1} = \{x, y\}$ then $c_{1j} = \{x+2, y+2\}$.

We show now that the vertices of B placed so far in the first column of C_n (i.e., in the first k rows of the first column) are all distinct. Assume to the contrary that the same vertex appears in c_{1i} and $c_{1j}, i \ne j$. From (d) it follows that then either $2s_{k+2-i} + 2(i-1) = 2s_{k+2-j} + 2(j-1)$ or $2s_{k+2-i} + 2(i-1) + 1 = 2s_{k+2-j} + 2(j-1) + 1$. In either case we get $s_{k+2-i} + i = 2s_{k+2-j} + j$ which implies $s_{k+2-i} - (k+2-i) = s_{k+2-j} - (k+2-j)$. However this contradicts the property (iv) in the definition of k_1-sequence. Further, vertices 1 and $2k$ do not occur yet in the first column. If they did, there would be i so that $a_{i1} = \{2s_{k+2-i} + 2(i-1), 2s_{k+2-i} + 2(i-1) + 1\} = \{2k, 1\}$, i.e., $2s_{k+2-i} + 2(i-1) = 2k$ which in turn implies $s_{k+2-i} - (k+2-i) = -1 \equiv k - 1 (\bmod k)$ contradicting again the property (iv) in the definition of k_1- sequence. Summarizing the above discussion we get that

$$\{c_{i1}; i = 1, ..., k\} = \{\{2,3\}, \{4, \infty_1\}, \{6,7\}, ..., \{2k-2, 2k-1\}, \{\infty_2, \infty_3\}\}. \tag{3.2}$$

Let $S = (s_1, ..., s_k)$ be now a k_2-sequence. The $(k+1)-th$ row, except of the first term, is defined in the same way as the first row. Formally, $c_{k+1,1} = \{1, 2s_1 + 1\} = \{1, 5\}$, $c_{k+1,k-1} = \{2k, \infty_1\}$, for $1 \le j \le k, j \ne k-1, s_j \ne \infty, c_{k+1,j} = \{2s_j, 2s_j + 1\}$, if $s_j = \infty$ then $c_{k+1,j} = \{\infty_2, \infty_3\}$, and $c_{k+1,k+1} = \{2t, 2t+1\}$, where t is the only element of N_k^* which does not occur in S. For $k+2 \le i \le 2k$ and $1 \le j \le k+1$, c_{ij} is defined by (a),(b), and (c). By the same token as above we get:

$$\{c_{i1}; k+1 \le i \le 2k\} = \{\{1,5\}, \{4, \infty_1\}, \{6,7\}, ..., \{2k-2, 2k-1\}, \{\infty_2, \infty_3\}\} \tag{3.3}$$

and consequently,

$$\bigcup_{j=1}^{k} c_{ij} = \{\infty_1, \infty_2, \infty_3\} \cup N_{2k} - \{2(i-k+1)\}, \; i = k+1, ..., j, \tag{3.4}$$

where the number $2(i-k+1)$ is taken $\mathrm{mod}(2k)$ from N_{2k}.

That is, no vertex occurs twice in the first column and rows $k+1$ up to $2k$. Combining (3.2) and (3.3), each pair $\{2j, 2j+1\}, 3 \le j \le k-1$ of vertices occurs twice in the first $2k$ rows of the first column as well as pairs $\{\infty_2, \infty_3\}$ and $\{4, \infty_1\}$; pairs $\{1,5\}$ and $\{2,3\}$ occur there once. Further, (3.4) implies

Claim 3. The vertex $a_i, i = k+1, ..., 2k$, is adjacent in L_n to vertices $\{1', 2', ..., (2k-1)'\} \cup \{2(i-k+1)\}$.

Now we illustrate the first part of the construction of C_n for $n = 26$. Then $k = 6$, $S = (1, \infty, 5, 2, 6, 3)$ and $S' = (2, \infty, 5, 1, 6, 4)$ are 6_1- and 6_2-sequence, respectively. The first 12 rows of C_{26} are

$$\begin{pmatrix}
2,3 & \infty_2,\infty_3 & 10,11 & 4,5 & 12,\infty_1 & 6,7 & 8,9 \\
8,9 & 4,5 & \infty_2,\infty_3 & 12,1 & 6,7 & 2,\infty_1 & 10,11 \\
4,\infty_1 & 10,11 & 6,7 & \infty_2,\infty_3 & 2,3 & 8,9 & 12,1 \\
10,11 & 6,\infty_1 & 12,1 & 8,9 & \infty_2,\infty_3 & 4,5 & 2,3 \\
6,7 & 12,1 & 8,\infty_1 & 2,3 & 10,11 & \infty_2,\infty_3 & 4,5 \\
\infty_2,\infty_3 & 8,9 & 2,3 & 10,\infty_1 & 4,5 & 12,1 & 6,7 \\
1,5 & \infty_2,\infty_3 & 10,11 & 2,3 & 12,\infty_1 & 8,9 & 6,7 \\
10,11 & 3,7 & \infty_2,\infty_3 & 12,1 & 4,5 & 2,\infty_1 & 8,9 \\
4,\infty_1 & 12,1 & 5,9 & \infty_2,\infty_3 & 2,3 & 6,7 & 10,11 \\
8,9 & 6,\infty_1 & 2,3 & 7,11 & \infty_2,\infty_3 & 4,5 & 12,1 \\
6,7 & 10,11 & 8,\infty_1 & 4,5 & 9,1 & \infty_2,\infty_3 & 2,3 \\
\infty_2,\infty_3 & 8,9 & 12,1 & 10,\infty_1 & 6,7 & 11,3 & 4,5
\end{pmatrix}$$

The last $n-2k$ rows of the matrix C_n will be formed by a $(n-2k) \times (k+1)$ matrix $C'_n = [c'_{ij}]$ defined as follows:

(f) $c'_{11} = \{2,3\}, c'_{21} = \{1,5\}, c'_{31} = c'_{41} = \{2k, 1'\}, c'_{2i-1,1} = c'_{2i,1} = \{(2i-4)', (2i-3)'\}, i = 3, ..., n/2-k$,

(g) for $j = 2, ..., k, c'_{1,j} = c'_{2,j} = c_{n-2k,j-1}, c'_{1,k+1} = c'_{2,k+1} = \{1', \infty_1\}$.

(h) $c'_{n-2k-1,k+1} = c'_{n-2k,k+1} = \{\infty_2, \infty_3\}$, for all other $i = 3, ..., n-2k-2$, $j = 2, ..., k$, we set if $c'_{ij} = \{x_1, x_2\}$ then $c'_{i+2,j+1} = \{\overline{x}_1, \overline{x}_2\}$, where $\overline{x}_i = x_i$ if $x_i \in \{1', ..., (2k-1)'\}$, and $\overline{x}_i = x_i + 2$ if $x_i \in \{1, ..., 2k\}$ (the sum $x_i + 2$ is taken $\mathrm{mod}\, 2k$ from the set N_{2k}).

From f), (3.2), and (3.3) we see that the first column of C_n contains each element twice and if a pair occurs in the first column, it occurs there exactly twice. This is true for all columns of C_n, as C_n is defined in a cyclic way. Further, (g) and (h) imply that no element occurs more than once in each row of C'_n, as the only conflict (=having an element twice in a row) could be with the elements of $\{1, 2, ..., 2k\}$, where we add 2 to the element in the next column. However, there is no conflict since there is no conflict in the third and fourth rows of C'_n as $2k \neq 4$ for $k \geq 3$.

Thus, the matrix C_n defines $k+1$ small edge-disjoint 2-factors of $K_{n,n}$. From Claim 1,2, and 3 one can see that the leave L_n contains a complete bipartite graph $H = K_{2k,2k-1}$ on the vertex set $\{a_1, ..., a_{2k}\} \cup \{1', ..., (2k-1)'\}$ and a matching edge disjoint with H covering $A' = \{a_1, ..., a_{2k}\}$, thus L_n does not contain a small 2-factor.

Again we illustrate the construction of C' for $n = 26$.

$$\begin{pmatrix}
2,3 & 10',11' & 8',9' & 6',7' & 4',5' & 2',3' & 1',\infty_1 \\
1,5 & 10',11' & 8',9' & 6',7' & 4',5' & 2',3' & 1',\infty_1 \\
12,1' & 4,5 & 10',11' & 8',9' & 6',7' & 4',5' & 2',3' \\
12,1' & 3,7 & 10,11' & 8',9' & 6',7' & 4',5' & 2',3' \\
2',3' & 2,1' & 6,7 & 10',11' & 8',9' & 6',7' & 4',5' \\
2',3' & 2,1' & 5,9 & 10',11' & 8',9' & 6',7' & 4',5' \\
4',5' & 2',3' & 4,1' & 8,9 & 10',11' & 8',9' & 6',7' \\
4',5' & 2'3' & 4,1' & 7,11 & 10',11' & 8',9' & 6',7' \\
6',7' & 4',5' & 2',3' & 6,1' & 10,11 & 10',11' & 8',9' \\
6',7' & 4',5' & 2',3' & 6,1' & 1,9 & 10',11' & 8',9' \\
8',9' & 6',7' & 4',5' & 2',3' & 8,1' & 1,12 & 10',11' \\
8',9' & 6',7' & 4',5' & 2',3' & 8,1' & 3,11 & 10',11' \\
10',11' & 8',9' & 6',7' & 4',5' & 2',3' & 10,1' & \infty_2,\infty_3 \\
10',11' & 8',9' & 6',7' & 4',5' & 2',3' & 10,1' & \infty_2,\infty_3
\end{pmatrix}$$

To construct $\frac{n}{4}+1$ small edge-disjoint 2-factors of $K_{n,n}$ for $n \equiv 0 (\text{mod}\, 4), n \geq 24$, we define an $n \times (\frac{n}{4} + 1)$ matrix $D_n = [d_{ij}]$ with the same properties (i) and (ii) as matrix C_n for the case $n \equiv 2 (\text{mod}\, 4)$.

Let $k = \frac{n-4}{4}$. In this case the bipartition of $K_{n,n}$ is given by $A = \{a_1, ..., a_n\}$, $B = \{1, ..., 2k\} \cup \{1', ..., (2k-1)'\} \cup \{\infty_1, ..., \infty_5\}$. The first $2k$ rows of D_n are obtained by a slight modification of the first $2k$ rows of the matrix $C_{n-2} = [c_{ij}]$. For $i = 2, ..., k$, set $d_{i,i-1} = d_{1,k} = \{\infty_4, \infty_5\}$; for $i = 1, ..., k$, set $d_{i,k+2} = c_{i,i-1} = \{2i+4, 2i+5\}$, where the numbers $2i + j, j = 4, 5$ are taken mod $2k$ from the set N_{2k}. Thus the elements in the i-th row and the last column of D_n are formed by numbers which where substituted in the i-th row by the pair $\{\infty_4, \infty_5\}$. For all other i, j set $d_{ij} = c_{ij}$. As to the rows $k+1, ..., 2k$ we proceed as follows. For $k \geq 5$, the last number of k_1- sequence equals 3, and thus $c_{2,1} = \{8, 9\}$. Choose $i \in \{k+1, ..., 2k\}$ so that $c_{i,1} = \{8, 9\}$, cf. (3.3) to see that such i exists. For $j = 0, ..., k-1$, set $d_{i+j,1+j} = \{\infty_4, \infty_5\}$ and $d_{i+j,k+2} = c_{i+j,1+j} = \{8+2j, 9+2j\}$, where the indices $i+j, 1+j$ are taken mod k from the set N_k. Thus, also for the rows $k+1, ..., 2k$, the elements in the last column are elements which were substituted in the given row by the pair $\{\infty_4, \infty_5\}$, and we can conclude that the first $2k$ rows of D_n satisfy conditions (i) and (ii). Now we illustrate our construction for $n = 28$.

$$\begin{pmatrix}
2,3 & \infty_2,\infty_3 & 10,11 & 4,5 & 12,\infty_1 & \infty_4,\infty_5 & 8,9 & 6,7 \\
\infty_4,\infty_5 & 4,5 & \infty_2,\infty_3 & 1,12 & 6,7 & 2,\infty_1 & 10,11 & 8,9 \\
4,\infty_1 & \infty_4,\infty_5 & 6,7 & \infty_2,\infty_3 & 2,3 & 8,9 & 1,12 & 10,11 \\
10,11 & 6,\infty_1 & \infty_4,\infty_5 & 8,9 & \infty_2,\infty_3 & 4,5 & 2,3 & 1,12 \\
6,7 & 1,12 & 8,\infty_1 & \infty_4,\infty_5 & 10,11 & \infty_2,\infty_3 & 4,5 & 2,3 \\
\infty_2,\infty_3 & 8,9 & 2,3 & 10,\infty_1 & \infty_4,\infty_5 & 1,12 & 6,7 & 4,5 \\
1,5 & \infty_2,\infty_3 & 10,11 & \infty_4,\infty_5 & 12,\infty_1 & 8,9 & 6,7 & 2,3 \\
10,11 & 3,7 & \infty_2,\infty_3 & 1,12 & \infty_4,\infty_5 & 2,\infty_1 & 8,9 & 4,5 \\
4,\infty_1 & 1,12 & 5,9 & \infty_2,\infty_3 & 2,3 & \infty_4,\infty_5 & 10,11 & 6,7 \\
\infty_4,\infty_5 & 6,\infty_1 & 2,3 & 7,11 & \infty_2,\infty_3 & 4,5 & 1,12 & 8,9 \\
6,7 & \infty_4,\infty_5 & 8,\infty_1 & 4,5 & 9,1 & \infty_2,\infty_3 & 2,3 & 10,11 \\
\infty_2,\infty_3 & 8,9 & \infty_4,\infty_5 & 10,\infty_1 & 6,7 & 11,3 & 4,5 & 1,12
\end{pmatrix}$$

The last $n - 2k$ rows of D_n are formed by an $(n - 2k) \times (k + 2)$ matrix $D'_n = [d'_{ij}]$ defined in a very similar way as the matrix C'_n for the case $n \equiv 2 (\mathrm{mod}\, 4)$. We obtain the first column of D'_n by adding twice the pair $\{8, 9\}$ to the first column of C'_{n-2}, add to the second column twice the pair $\{10, 11\}$, etc. Formally,

(a) $d'_{11} = \{2,3\}, d'_{21} = \{1,5\}, d'_{31} = d'_{41} = \{8,9\}$, $d'_{51} = d'_{61} = \{8',9'\}$, $d'_{2n-k-3,1} = d'_{2n-k-2,1} = \{2k,1'\}, d'_{2i-1,1} = d'_{2i,1} = \{(2i-6)',(2i-5)'\}$ for $i \in \{4, ..., n/2 - k - 2, n/2 - k\}\}$,

(b) for $j = 2, ..., k + 1$, if $d'_{n-2k,j-1} = \{x_1, x_2\}$ then $d'_{1,j} = d'_{2,j} = \{\overline{x}_1, \overline{x}_2\}$, where $\overline{x}_i = x_i$ if $x_i \in \{1', ..., (2k-1)', \infty_4, \infty_5\}$, and $\overline{x}_i = x_i + 2$ if $x_i \in \{1, ..., 2k\}$;

(c) for $i = 3, ..., n-2k-2$, $j = 2, ..., k+1$, $d'_{n-2k-3,k+1} = d'_{n-2k-2,k+1} = d'_{n-2k-1,k+2} = d'_{n-2k,k+2} = \{\infty_2, \infty_3\}$, $d'_{n-2k-1,k+1} = d'_{n-2k,k+1} = \{\infty_4, \infty_5\}$, $d'_{n-2k-7,k+1}, = d'_{n-2k-6,k+1} = d'_{n-2k-5,k+2} = d'_{n-2k-4,k+2} = \{1', \infty_1\}$, otherwise, if $d'_{ij} = \{x_1, x_2\}$ then $d'_{i+2,j+1} = \{\overline{x}_1, \overline{x}_2\}$, where $\overline{x}_i = x_i$ if $x_i \in \{1', ..., (2k-1)'\}$, and $\overline{x}_i = x_i + 2$ if $x_i \in \{1, ..., 2k\}$ (the sum $x_i + 2$ is taken $\mathrm{mod}\, 2k$ from the set N_{2k}).

As D'_n is defined in a cyclic way the only conflict (=having an element twice in a row) could be with the elements of $\{1, 2, ..., 2k\}$, where we add 2 to the element in the next column. However, as there is no conflict in the third and fourth rows of D'_n

$$\begin{pmatrix}
8,9 & 4,5 & (2k-2)',(2k-1)' & 6,1' & \ldots \\
8,9 & 3,7 & (2k-2)',(2k-1)' & 6,1' & \ldots
\end{pmatrix}$$

there is no conflict in D'_n which is defined in a cyclic manner. Thus the matrix D_n satisfies conditions (i) and (ii), that is, D_n defines $k + 2$ small edge-disjoint 2-factors F_j of $K_{n,n}$. From the construction of D_n it also follows that, for $i = 1, \ldots, 2k$, $\bigcup_{j=1}^{k+2} d_{ij} = \bigcup_{i=1}^{k+1} c_{ij} \cup \{\infty_4, \infty_5\}$, ($c_{ij}$ being elements of the matrix C_{n-2}). Thus Claim 1,2, and 3 is valid also in this case, hence F_j's form a maximal set of small 2-factors. Bellow we illustrate the construction of D_n' for $n = 28$.

$$\begin{pmatrix}
2,3 & 10',11' & 4,1' & 6',7' & 4',5' & 2',3' & 8',9' & \infty_4,\infty_5 \\
1,5 & 10',11' & 4,1' & 6',7' & 4',5' & 2',3' & 8',9' & \infty_4,\infty_5 \\
8,9 & 4,5 & 10',11' & 6,1' & 6',7' & 4',5' & 2',3' & 8',9' \\
8,9 & 3,7 & 10',11' & 6,1' & 6',7' & 4',5' & 2',3' & 8',9' \\
8',9' & 10,11 & 6,7 & 10',11' & 8,1' & 6',7' & 4',5' & 2',3' \\
8',9' & 10,11 & 5,9 & 10',11' & 8,1' & 6',7' & 4',5' & 2',3' \\
2',3' & 8',9' & 1,12 & 8,9 & 10',11' & 10,1' & 6',7' & 4',5' \\
2',3' & 8',9' & 1,12 & 7,11 & 10',11' & 10,1' & 6',7' & 4',5' \\
4',5' & 2',3' & 8',9' & 2,3 & 10,11 & 10',11' & \infty_1,1' & 6',7' \\
4',5' & 2',3' & 8',9' & 2,3 & 9,1 & 10',11' & \infty_1,1' & 6',7' \\
6',7' & 4',5' & 2',3' & 8',9' & 4,5 & 1,12 & 10',11' & \infty_1,1' \\
6',7' & 4',5' & 2',3' & 8',9' & 4,5 & 11,3 & 10',11' & \infty_1,1' \\
12,1' & 6',7' & 4',5' & 2',3' & 8',9' & 6,7 & \infty_2,\infty_3 & 10',11' \\
12,1' & 6',7' & 4',5' & 2',3' & 8',9' & 6,7 & \infty_2,\infty_3 & 10',11' \\
10',11' & 2,1' & 6',7' & 4',5' & 2',3' & 8',9' & \infty_4,\infty_5 & \infty_2,\infty_3 \\
10',11' & 2,1' & 6',7' & 4',5' & 2',3' & 8',9' & \infty_4,\infty_5 & \infty_2,\infty_3
\end{pmatrix}$$

■

3.5. Proof of Theorem 3.7

The following lemma will be the key igredience of the proof.

Lemma 9. *If* $k \in$ Spec(t) *then* $\frac{n}{2} - (\frac{t}{2} - k)$ $\in$Spec(n) *for all even* $n \geq 2t$.

Proof. If $k = \frac{t}{2}$ then the statement claims that $\frac{n}{2} \in$ *Spec*(n) which follows from Lemma 3.1. For $k < t/2$, let $C = [c_{ij}]$ be a Latin square of order $n/2$ on the symbol set $\{1, \ldots, n/2\}$ with a Latin subsquare D of order $t/2$ on the symbol set $\{1, \ldots, t/2\}$ in the upper right-hand corner of C. Such a square is well known to exist, see [3], for any $n \geq 2t$. We will construct a set of $\frac{n-t}{2} + k$ small 2-factors F_j of $K_{n,n} = (A, B; E)$, $A = \{a_1, \ldots, a_n\}, B = \{b_1, \ldots, b_n\}$. For $j = 1, \ldots, n/2$, the $j-th$ column of C defines a small 2-factor H_j comprising 4-cycles $a_{2i-1}b_{2m-1}a_{2i}b_{2m}a_{2i-1}$, where $m = c_{ij}$ and $i = 1, \ldots, n/2$. As C is a Latin square the 2-factors H_j are pairwise edge-disjoint. Further, the upper right-hand corner of C is formed by the Latin square D, thus, for $j = \frac{n-t}{2} + 1, \ldots, \frac{n}{2}$, the vertices of $A' = \{a_1, \ldots, a_t\}$

are in H_j on 4-cycles only with vertices of $B' = \{b_1, ..., b_t\}$. Let $F'_1, ..., F'_k$ be a maximal set of small edge-disjoint 2-factors of $K_{t,t} = (A', B'; E)$, $A' = \{a_1, ..., a_t\}, B' = \{b_1, ..., b_t\}$, $L' = K_{t,t} - \bigcup_{i=1}^{k} F'_j$. To construct a set of $\frac{n-t}{2} + k$ small 2-factors F_j of $K_{n,n}$ we set, for $j = 1, ..., \frac{n-t}{2}$, $F_j = H_j$; for $j = \frac{n-t}{2} + 1, ..., \frac{n-t}{2} + k$, we obtain F_j by replacing in H_j 4-cycles on vertices of $A' \cup B'$ by 4-cycles of $F'_{j-\frac{n-t}{2}}$. Formally, $F_j = F'_{j-\frac{n-t}{2}} \cup \{ a_{2i-1}b_{2m-1}a_{2i}b_{2m}a_{2i-1}$, where $m = c_{ij}$, $\frac{t}{2} + 1 \le i \le \frac{n}{2}\}$. As H'_js and F'_js are pairwise edge-disjoint the factors F_j have the property as well.To see that they form a maximal set of 2-factors it is sufficient to note that in the leave $L = K_{n,n} - \bigcup_{j=1}^{\frac{n-t}{2}+k} F_j$ vertices of A' and B' are not adjacent to any vertex of $A - A'$ and $B - B'$ and the subgraph of L induced by $A' \cup B'$ is isomorphic to L'. Hence, $\frac{n-t}{2} + k = \frac{n}{2} - (\frac{t}{2} - k) \in$ *Spec*(n). ■

With this in hand we are ready to prove Theorem 3.7.

Proof of Theorem 3.7. From Theorem 3.6 it follows that, for $t \equiv 2 (\mathrm{mod}\, 4)$, $t \notin \{6, 18\}$, $\frac{t+2}{4} \in$ *Spec*(t), $6 \in$ *Spec*(20), and from Theorem 3.4, $3 \in$ *Spec*(8). Combining with Lemma 3.9 we get:

(i) if $t \le \frac{n}{2}, t \equiv 2(\mathrm{mod}\, 4), t \notin \{6, 18\}$, then $\frac{n}{2} - (\frac{t}{2} - \frac{t+2}{4}) = \frac{n}{2} - \frac{t-2}{4} \in$ *Spec*(n);

(ii) for $20 \le \frac{n}{2}, \frac{n}{2} - 4 \in$ *Spec*(n);

(iii) for $8 \le \frac{n}{2}$, $\frac{n}{2} - 1 \in$ *Spec*(n).

Let t_n be the largest number from the set $T = \{8, 20\} \cup \{4j+2, j \ge 2, j \ne 4\}$ so that $t_n \le \frac{n}{2}$. Then, for $n \notin A = \{16, 18, 36, 38, 40, 42\}$, $t_n = 4\lfloor\frac{n-4}{8}\rfloor + 2$, $t_{16} = t_{18} = 8, t_{36} = t_{38} = 14, t_{40} = t_{42} = 20$. Take a fixed n. Then, applying the numbers $t \in T, t \le t_n$, to (i),(ii), and (iii) we get that, if $n \notin A$, then *Spec*$(n) \supset \{l, \frac{n}{2} - \frac{t_n-2}{4} \le l \le \frac{n}{2}\} = \{l, \frac{n}{2} - \lfloor\frac{n-4}{8}\rfloor \le l \le \frac{n}{2}\} = \{l, \lceil\frac{3n+4}{8}\rceil \le l \le \frac{n}{2}\}$; $\{\frac{n}{2} - 1, \frac{n}{2}\} = \{l, \lceil\frac{3n+4}{8}\rceil \le l \le \frac{n}{2}\} \subset$ *Spec*(n) for $n = 16, 18$; $\{\frac{n}{2} - 3, ..., \frac{n}{2}\} = \{l, \lceil\frac{3n+4}{8}\rceil + 1 \le l \le \frac{n}{2}\} \subset$ *Spec*(n) for $n = 36, 38$, and $\{\frac{n}{2} - 4, \frac{n}{2}\} = \{l, \lceil\frac{3n+4}{8}\rceil \le l \le \frac{n}{2}\} \subset$ *Spec*(n) for $n = 40, 42$.
■

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