SPECIAL KINDS OF DOMINATION PARAMETERS OF EDGE-DELETED GRAPHS

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Abstract. We study the behaviour of two domination parameters: the split domination number $\gamma_s(G)$ of a graph G and the maximal domination number $\gamma_m(G)$ of G after the deletion of an edge from G. The motivation of these problems comes from [2]. In [6] Vizing gave an upper bound for the size of a graph with a given domination number. Inspired by [5] we formulate Vizing type relation between $|E(G)|$, $|V(G)|$, $\Delta(G)$ and $\delta(G)$, where $\Delta(G)$ ($\delta(G)$) denotes the maximum (minimum) degree of G.

1. Introduction

By a graph G we mean a finite, not complete, undirected graph without loops and multiple edges, where $V(G)$ is the set of vertices and $E(G)$ is the set of edges of the graph G. A *path* joining vertices x_1 and x_n in G is the sequence of vertices $x_1, x_2, ..., x_n \in V(G)$ such that, $(x_i, x_{i+1}) \in E(G)$, for $i = 1, 2, ..., n-1$ and $n \geqslant 2$. We shall denote it by $P_G(x_1, x_n)$. The *open neighbourhood* of a vertex x in a graph G, denoted by $N_G(x)$, is the set of all vertices adjacent to x in G. The *closed neighbourhood* of x in a graph G is defined as $N_G[y] = N_G(x) \cup \{x\}$. Recall that the *degree of the vertex* x, denoted by $\delta_G(x)$, is the cardinality of the set $N_G(x)$. If $\delta_G(x) = 1$ ($\delta_G(x) = 0$), then x is said to be a *hanging vertex* (an *isolated* vertex) of G. An edge $(u, v) \in E(G)$ is a *hanging edge* of G if u or v is a hanging vertex of G. Denote by $\delta(G)$ ($\Delta(G)$) the minimum (the maximum) degree of G. By $G - e$ we mean a subgraph of G containing all the vertices of G with $E(G) - \{e\}$ as the edge set. If $X \subseteq V(G)$, then the notation $\langle X \rangle_G$ means the subgraph of G induced by a subset X. By P_n we define a graph on $n \geqslant 2$ vertices $x_1, x_2, ..., x_n$ and with the edge set $E(G) = \{(x_i, x_{i+1}) : i = 1, ..., n-1\}$.

A subset $D \subseteq V(G)$ is a *dominating set* of G if every vertex from $V(G) - D$ is adjacent to some vertex from D (for short: D is dominating in G). The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. In this paper we study two of many variations of the domination. A dominating set D of G is a *split dominating set* of G if the induced subgraph $\langle V(G) - D \rangle_G$ is disconnected (for short: D is split dominating in G). We note that the existence of such subset in a connected graph is assured only if it is different from a complete graph. The *split domination number* $\gamma_s(G)$ of G is the minimum cardinality of a split dominating set of G. A dominating set D of G is a *maximal dominating* set of G if $V(G) - D$ is not a dominating set of G. The *maximal domination number* $\gamma_m(G)$ of G is the minimum cardinality of a maximal dominating set of G. For a convenience, a subset which realizes number $\gamma(G)$, $\gamma_s(G)$, $\gamma_m(G)$ will be called a $\gamma(G)$ *– set*, a $\gamma_s(G)$ *– set* and a $\gamma_m(G)$ *– set*, respectively. For more information about split domination graphs and maximal domination graphs, the reader is referred to [4] and [3], respectively.

Any term not defined in this paper may be found in Berge [1].

2. Perliminaries

Note that the fact that the subset D is dominating in G is equivalent to the expression: for every $x \in V(G) - D$ $N_G(x) \cap D \neq \emptyset$. For a convenience, we shall sometimes write that D dominates x.

We begin with simple observations which will be useful in further investigations.
Let $D \subseteq V(G)$ be a split dominating set of G and $e = (x, y) \in E(G)$ be an arbitrary edge.

Proposition 2.1. *If $x, y \in D$, then D is a split dominating set of $G - e$.*

Proof. Suppose that D is a split dominating set of G. Since D also there must be dominating in G, for any $w \in V(G) - D$ holds $N_G(w) \cap D \neq \emptyset$. Moreover, $N_{G-e}(w) \cap D \neq \emptyset$ since $x, y \in D$, what means that D is dominating in $G - e$. There remains to show that the subgraph $\langle V(G) - D \rangle_{G-e}$ is disconnected. It is obvious from the definition of a split dominating set, that the subgraph $\langle V(G) - D \rangle_G$ is disconnected. This implies the existence of two vertices u, v belonging to $V(G) - D$ that every path $P_G(u, v)$ contains a vertex from D or there is no path joining u and v in G. Since the removal of the edge e from G does not change the above relationship between vertices u and v, the subgraph $\langle V(G) - D \rangle_{G-e}$ also is disconnected what completes the proof. ∎

Proposition 2.2. *If $x, y \in V(G) - D$, then D is a split dominating set of $G - e$.*

Proof. Let D be as in the assumption of the proposition. We can conclude from this that the vertex x is adjacent to some vertex from D in the graph G. Moreover, $N_{G-e}(x) = N_G(x) - \{y\}$. Combining the above facts we obtain $N_{G-e}(x) \cap D = N_G(x) \cap D \neq \emptyset$. Of course also $N_{G-e}(y) \cap D = N_G(y) \cap D \neq$

$\emptyset$. Reassuming, for any $w \in V(G) - D$, $N_{G-e}(w) \cap D \neq \emptyset$, what shows that the subset D is dominating in $G-e$. Since disconnectedness of the subgraph $\langle V(G) - D\rangle_G$ implies disconnectedness of the subgraph $\langle V(G) - D\rangle_{G-e}$, the proof is complete. ∎

Let $e = (x,y) \in E(G)$ and $D^- \subseteq V(G)$ be a split dominating set of $G - e$. It is easy to see

Remark 2.3. *If D^- is dominating in $G-e$, then D^- is dominating in G.*

Proposition 2.4. *If at least one of vertices x, y belongs to D^-, then D^- is a split dominating set of G.*

Proof. Let D^- be a split dominating set of $G-e$ and $x \in D^-$. By Remark 2.3 we conclude that D^- is dominating in G. Further, by the assumption it follows that the subgraph $H = \langle V(G) - D^-\rangle_{G-e}$ is disconnected. Therefore, there exist two vertices $u, v \in V(G) - D^-$ not joined by a path in H. Moreover, the adding the edge $e = (x,y)$ (for $y \in D^-$ or $y \in V(G) - D^-$) to the subgraph H leads to no path $P_{H+e}(u,v)$, what proves disconnectedness of $\langle V(G) - D^-\rangle_G$ and completes the proof. ∎

3. The split domination number of a graph with a removed edge

First we give the lower and upper bounds of the domination parameter $\gamma_s(G-e)$, for an edge different from a bridge of a graph G. Next it will be examined the case whenever the graph G has a bridge. Recall that by a $\gamma_s(G)$ - set we mean a split dominating set of G realizing the number $\gamma_s(G)$, and note a simple assertion

Remark 3.1. *If a graph H is disconnected, then for any $e \in E(H)$ the subgraph $H - e$ also is disconnected.*

Theorem 3.2. *Let $e \in E(G)$. If $G - e$ is connected, then*

$$\gamma_s(G-e) \leqslant \gamma_s(G) + 1. \tag{1}$$

Proof. Let D be $\gamma_s(G)$ - set, $e = (x,y)$ and $G - e$ is connected. First we clime that $x, y \in D$ or $x, y \in V(G) - D$. Then according to Proposition 2.1 and Proposition 2.2, respectively we obtain that D is a split dominating set of $G - e$. This means that $\gamma_s(G-e) \leqslant |D| = \gamma_s(G)$, as required. Further, let $x \in V(G) - D$ and $y \in D$. Assume additionally that $N_{G-e}(x) \cap D \neq \emptyset$. Thus D is split dominating in $G - e$. Moreover, disconnectedness of the subgraph $\langle V(G) - D\rangle_{G-e}$ follows immediately from the assumption about D and by Remark 3.1. Consequently, D is split dominating in $G - e$ and $\gamma_s(G-e) \leqslant |D| = \gamma_s(G)$.
Now, assume that $N_{G-e}(x) \cap D = \emptyset$. Then, it is easy to check that the

superset $D \cup \{x\}$ is dominating in $G-e$. Further, we state that the subgraph $\langle V(G)-(D \cup \{x\})\rangle_{G-e}$ is disconnected. Assuming the contrary, the vertex x should be an isolated vertex of $G-e$, but it is impossible by connectedness of $G-e$. Finally, $\gamma_s(G-e) \leqslant |D \cup \{x\}| = \gamma_s(G)+1$ and the proof is complete. ∎

Theorem 3.3. *Let $e \in E(G)$. If $G-e$ is connected, then*

$$\gamma_s(G)-1 \leqslant \gamma_s(G-e). \tag{2}$$

Proof. Let D^- be a $\gamma_s(G-e)$ - set and $e=(x,y) \in E(G)$ be an arbitrary edge of G. First note that if at least one of vertices x, y belongs to the subset D^-, then by Proposition 2.4, D^- is split dominating in G. As a consequence is the inequality $\gamma_s(G) \leqslant |D^-| = \gamma_s(G-e)$, as desired.
Now, let $x, y \in V(G)-D^-$.

Suppose that $|V(G)-D^-| = 2$ (i.e., $\langle V(G)-D^-\rangle_G \cong K_2$). Since a graph G is not complete, so there exist at least two nonadjacent vertices u and w in G. Obvious that $\{x,y\} \neq \{u,w\}$. Putting $D=V(G)-\{u,w\}$, we obtain that the subset D is split dominating in G, since $\langle V(G)-D\rangle_G$ is disconnected and by connectedness of G holds $N_G(u) \cap D \neq \emptyset$ and $N_G(w) \cap D \neq \emptyset$. Thus, $\gamma_s(G) \leqslant |D| = |D^-| = \gamma_s(G-e)$.

Suppose that $|V(G)-D^-| > 2$. If the subgraph $\langle V(G)-D^-\rangle_G$ is disconnected, then the subset D^- is split dominating in G, since D^- is a dominating set of G. As a consequence is $\gamma_s(G) \leqslant |D^-| = \gamma_s(G-e)$ and the inequality in (2) holds. Now suppose that $H = \langle V(G)-D^-\rangle_G$ is connected. It can observe that the superset $D^- \cup \{x\}$ is dominating in G. We shall prove that the subgraph $\langle V(G)-(D^- \cup \{x\})\rangle_G$ is disconnected. Noting that the subgraph $\langle V(G)-D^-\rangle_{G-e}$ is disconnected we can choose two vertices, say $x_1, x_2 \in V(G)-D^-$ such that every path $P_{G-e}(x_1,x_2)$ contains at least one vertex from D^-.
Consider the following cases:

(i) $x \neq x_1, x_2$.
Then every path $P_H(x_1,x_2)$ contains the edge (x,y). This means that after removal of the vertex x from the subgraph H, there exists no path joining the vertices x_1 and x_2 in the subgraph $\langle V(G)-(D^- \cup \{x\})\rangle_G$.
In consequence, the subgraph $\langle V(G)-(D^- \cup \{x\})\rangle_G$ is disconnected, what implies that the superset $D^- \cup \{x\}$ is split dominating in G. A simple calculation leads to the inequality in (2).

(ii) $x=x_1$and $y=x_2$.
First we show that there exists a vertex $z \in V(G)-D^-$ such that z is adjacent to one of vertices x or y in G—e. Indeed, since $|V(G)-D^-| > 2$, then there exists a vertex $u \in V(G)-D^-$, different from x and y. Next, by connectedness of $\langle V(G)-D^-\rangle_G$ we conclude that there

exists a path $P_H(x,u)$. It means that there must be a vertex $z \in V(G) - D^-$ different from x and y, adjacent to one of them in H. Thus z is belongs to the neighbourhood of x or y in $G - e$.
Without loss of generality, we may assume that z is adjacent to x in H, and arguing as in (i), we obtain that $D^- \cup \{x\}$ is split dominating in G. As the consequence is the inequality $\gamma_s(G) \leqslant |D^- \cup \{x\}| = \gamma_s(G-e) + 1$ and proof is complete. ∎

Theorem 3.2 and Theorem 3.3 lead immediately to the following conclusion.

Theorem 3.4. *Let $e \in E(G)$. If $G - e$ is connected, then*

$$\gamma_s(G) - 1 \leqslant \gamma_s(G-e) \leqslant \gamma_s(G) + 1.$$

Corollary 3.5. *Let H be a connected graph and $e \in E(\overline{H})$, where $\overline{H}$ is a complementary graph of H. Then putting $G = H + e$ and using inequality in (1) and in (2) we obtain*

$$\gamma_s(H) - 1 \leqslant \gamma_s(H+e) \leqslant \gamma_s(H) + 1,$$

where $H+e$ is a graph with $V(H+e) = V(H)$ and $E(H+e) = E(H) \cup \{e\}$.

We now study the case when an removed edge is a bridge of a graph. Recall that an edge of a connected graph G is called a *bridge* of G, if $G - e$ is disconnected.
Our aim is to prove the following

Theorem 3.6. *Let G be a connected graph without hanging vertices, then*

$$\gamma_s(G) - 1 \leqslant \gamma_s(G-e) \leqslant \gamma_s(G) + 1. \tag{3}$$

Proof. Let $e = (x,y)$ be an arbitrary edge of G. If $G - e$ is connected, then the result follows from Theorem 3.4. Assuming that $G - e$ is disconnected, denote by H_1 and H_2 the connected components of $G - e$. Note that $|V(H_1)|, |V(H_2)| \geqslant 2$, otherwise G would have a hanging vertex. First we prove

$$\gamma_s(G-e) \leqslant \gamma_s(G) + 1. \tag{4}$$

Let $D \subseteq V(G)$ be a $\gamma_s(G)$ - set and $x \in V(H_1)$, $y \in V(H_2)$. If $x,y \in D$ or $x,y \in V(G) - D$, then D is split dominating in $G - e$, by Proposition 2.1 and Proposition 2.2 respectively. In conclusion $\gamma_s(G-e) \leqslant |D| = \gamma_s(G)$ and the inequality in (4) holds. Suppose that $x \in D$, $y \in V(G) - D$ and put $D_1 = V(H_1) \cap D$, $D_2 = V(H_2) \cap D$ (of course $D_1 \neq \emptyset$, since $x \in D \cap V(H_1) = D_1$). Therefore, we observe that the subset D_1 is dominating in H_1. If $D_2 \neq \emptyset$ and D_2 dominates y in H_2, then $D_1 \cup D_2 = D$ is dominating in $G - e$. It is clear that the subgraph $\langle V(G) - D \rangle_{G-e}$ is disconnected since the subgraph $\langle V(G) - D \rangle_G$ is disconnected by the assumption about D. So, from the above investigations it follows that D is split dominating in $G-e$ and $\gamma_s(G-e) \leqslant |D| = \gamma_s(G)$. Thus, the inequality in (4) holds. If $D_2 = \emptyset$, then $V(H_2) = \{y\}$, what is impossible since $|V(H_2)| \geqslant 2$. Hence it remains to consider the case when D_2 does not dominate y in H_2. We states that the superset $D_2 \cup \{y\}$ is dominating in H_2. In a consequence, the set $D_1 \cup D_2 \cup \{y\} = D \cup \{y\}$ is dominating in $G-e$. Further, we shall prove disconnectedness of the subgraph $\langle V(G) - (D \cup \{y\})\rangle_{G-e}$. Note that, it suffices to show the existence of two vertices, say $w \in V(H_2) - (D_2 \cup \{y\})$ and $u \in V(H_1) - D_1$. Indeed, the existence of the vertex w follows from connectedness of H_2 and from the fact that D_2 does not dominate y. Assume that $V(H_1) = D_1$, then there exists a subset $D_0 \subset D_1$ such that $D_0 \cup D_2$ is a split dominating set of G which contradicts the assumption that D is a minimum split dominating set of G. Consequently $V(H_1) - D_1 \neq \emptyset$ and all this together proves disconnectedness of the subgraph $\langle V(G) - (D \cup \{y\})\rangle_{G-e}$. Finally, the superset $D \cup \{y\}$ is split dominating in $G-e$ and $\gamma_s(G-e) \leqslant |D \cup \{y\}| = \gamma_s(G)+1$, as required.

Now, we shall show that

$$(5) \qquad \gamma_s(G) - 1 \leqslant \gamma_s(G-e).$$

Let D^- be a $\gamma_s(G-e)$ - set and recall that $x \in V(H_1), y \in V(H_2)$. If x or $y \in D^-$, then D^- is split dominating in G, by Proposition 2.4. So, $\gamma_s(G) \leqslant |D^-| = \gamma_s(G-e)$ and the inequality in (5) holds. If $x, y \in V(G) - D^-$ and the subgraph $\langle V(G) - D^-\rangle_G$ is disconnected, then the set D^- is split dominating in G. Hence inequality in (5) holds in this case. It remains to consider the case while the subgraph $\langle V(G) - D^-\rangle_G$ is connected. If $|V(H_1)| = 2$, say $V(H_1) = \{x, u\}$, then $(x, u) \in E(G)$ since H_1 is connected. Moreover, $u \in D^-$ because $N_{G-e}(x) = \{u\}$ and D^- is dominating in $G-e$. Certainly, $D_0 = (D^- \cup \{x\}) - \{u\}$ is dominating in G. Now, we shall prove disconnectedness of the subgraph $H_0 = \langle V(G) - D_0\rangle_G$. Namely, it suffices to remark that $u, y \in V(H_0)$ and any path $P_G(u, y)$ contains the vertex x belonging to D_0. This means that the vertices u and y are not joined by a path in H_0. Hence D_0 is split dominating in G and $\gamma_s(G) \leqslant |D_0| = |D^-| = \gamma_s(G-e)$, thus inequality in (5) holds. Assume that $|V(H_1)| \geqslant 3$. Note that there exists a vertex $u \in V(H_1)$, such that u is different from x and $u \notin D^-$. Otherwise, it would exist a subset $D_0 \subset V(H_1)$ such that $D_0 \cup (D^- \cap V(H_2))$ is a split dominating set of $G-e$. But then we would have $|D_0 \cup (D^- \cap V(H_2))| < |D^-|$, contrary to the assumption that D^- is a $\gamma_s(G-e)$ - set. Consider the superset $D = D^- \cup \{x\}$. It is easy to observe that D is dominating in G. Since $u \in V(H_1) - D$, $y \in V(H_2) - D$ and any path $P_G(u, y)$ contains the vertex $x \in D$, then $\langle V(G) - D\rangle_G$ is disconnected, what implies that D is a split dominating set of G. Consequently, the set $D^- \cup \{x\} = D$ is split dominating in G and

$\gamma_s(G) \leqslant |D| = |D^- \cup \{x\}| = \gamma_s(G-e) + 1$, as required.
By inequality in (4) and in (5) the theorem is proved. ∎

Next, we discuss a graph G having hanging vertices taking into account the number $\gamma_s(G-e)$. We begin by stating the proposition.

Proposition 3.7. *Let D be a split dominating set of a connected graph G and $z \in D$, such that $N_G(z) \subseteq D$. Then $D - \{z\}$ is split dominating in G.*

Proof. First, we shall prove that $D - \{z\}$ is dominating in G. Let D be a split dominating set of G and $N_G(z) \subseteq D$. In order to do it we prove that $N_G(z) \cap (D - \{z\}) \neq \emptyset$. Since G is connected, so $N_G(z) \neq \emptyset$. From the fact, that $N_G(z) \subseteq D$ we obtain $N_G(z) \cap (D - \{z\}) \neq \emptyset$. Since D is a split dominating set of G, so $H = \langle V(G) - D \rangle_G$ is disconnected. This implies that there exist the vertices x_1 and x_2 such that there is no path $P_H(x_1, x_2)$. Since z is adjacent only to the vertices from D, we conclude that there is no path joining x_1, x_2 in $\langle V(G) - (D - \{z\}) \rangle_G$. This means that the subgraph $\langle V(G) - (D - \{z\}) \rangle_G$ is disconnected and the proof is complete. ∎

It has been proved in [4] the following statement

Theorem 3.8. [4]. *For any graph G with hanging vertices*

$$\gamma_s(G) = \gamma(G).$$

Furthermore, there exists a $\gamma_s(G)$ - set D containing all vertices adjacent to hanging vertices.

Remark 3.9. *It is not difficult to see that any hanging vertex of G belongs to $V(G) - D$, where D is mentioned in the second part of Theorem* 3.8. *Otherwise D would not be a minimal split dominating set of G, by Proposition* 3.7.

Now we prove the lemma, which will be useful in a proof of Theorem 3.12

Lemma 3.10. *Let G be a connected graph having at least two hanging vertices. If D is a $\gamma_s(G)$ - set, then the following conditions are equivalent:*

1. $|V(G) - D| = 2$;
2. $G \cong P_3$ *or* $G \cong P_4$.

Proof. Let D be a $\gamma_s(G)$ - set, containing all vertices adjacent to hanging vertices. If graph G has more than two hanging vertices, then $|V(G) - D| \geqslant 3$ so, we can assume that G has exactly two hanging vertices, say x_1, x_2.
Assume that $|V(G) - D| = 2$ i.e., $V(G) - D = \{x_1, x_2\}$.
If $|D| = 1$, then $G \cong P_3$.
Suppose that $|D| = 2$, say $D = \{u, w\}$. If $N_G(x_1) = N_G(x_2) = \{u\}$, then by connectedness of G it follows that $N_G(w) = \{u\}$. This means that w is

a hanging vertex of G, a contradiction. So it must be that $N_G(x_1) = \{u\}$ and $N_G(x_2) = \{w\}$. Moreover, $(u, v) \in E(G)$, since G is connected. All this together gives $G \cong P_4$, as required. Now, we shall show that the case when $|D| \geqslant 3$ can not occur.
Let $|D| \geqslant 3$, then there exists a vertex $z \in D$ which is not adjacent to x_1 and x_2, but z is adjacent to some vertex from D. Then, by Proposition 3.7, the subset $D - \{z\}$ is a split dominating set of G, but $|D| > |D - \{z\}|$ which is impossible since D is the minimum split dominating set of G. This contradiction proves that $|D| < 3$. If G is isomorphic to P_3 or to P_4, then the condition in 1. holds. Thus, the proof is complete. ∎

Note that by a simple calculation we obtain

Remark 3.11. *If $G \cong P_4$, then $\gamma_s(G) = 2$ and $\gamma_s(G - e) = 2$ for any $e \in E(G)$.*

Using the above assertions we prove the following theorem.

Theorem 3.12. *Let G be a connected graph with $|V(G)| \geqslant 4$. If G has at least two hanging vertices, then for any $e \in E(G)$*

(6) $$\gamma_s(G) \leqslant \gamma_s(G - e) \leqslant \gamma_s(G) + 1.$$

Proof. Let $e = (x, y) \in E(G)$. We shall prove the theorem by two steps. First we shall prove that

(7) $$\gamma_s(G - e) \leqslant \gamma_s(G) + 1.$$

Let x_1 and x_2 be two hanging vertices of G and let D be a $\gamma_s(G)$ - set having vertices adjacent to all hanging vertices of G (it is possible by Remark 3.9). Moreover, $x_1, x_2 \in V(G) - D$. Of course it can be $x_1 = x$ or $x_2 = x$). If $x, y \in D$ or $x, y \in V(G) - D$, then by Proposition 2.1 and 2.2 respectively, the subset D is split dominating in $G - e$. Hence $\gamma_s(G - e) \leqslant |D| = \gamma_s(G)$, as required in expression (7). It remains to consider the case when $x \in V(G) - D$ and $y \in D$.
If $|V(G)| = 4$, then $G \cong P_4$ and the inequality in (7) holds, by Remark 3.11. Now, assume that $|V(G)| > 4$, then by Lemma 3.10 $|V(G) - D| \geqslant 3$. Then, there exists a vertex $z \in V(G) - D$, different from x_1 and x_2, (of course it can be that $z = x$) such that all paths $P_G(x_1, z)$ and $P_G(x_2, z)$ contain a vertex from D (since G is connected and x_1, x_2 are hanging vertices). This means that the subgraph $\langle V(G) - D \rangle_{G-e}$ is disconnected. If additionally $N_{G-e}(x) \cap D \neq \emptyset$, then D is a split dominating set of $G - e$, hence (7) holds. If $N_{G-e}(x) \cap D = \emptyset$, then $D \cup \{x\}$ is split dominating in G. Thus $\gamma_s(G - e) \leqslant \gamma_s(G) + 1$.
Next we shall prove the second part of the inequality in (6), namely

(8) $$\gamma_s(G) \leqslant \gamma_s(G - e).$$

Let $x,y \in D^-$ or $x \in V(G) - D^-$ and $y \in D^-$. Then D^- also is split dominating in G, by Proposition 2.4. Hence $\gamma_s(G) \leqslant |D^-| = \gamma_s(G-e)$. Suppose that $x,y \in V(G) - D^-$. The number of hanging vertices can decrease when we remove a hanging edge so, we can observe that the graph $G-e$ has at least one hanging vertex, say x_1 (it can be that $x_1 = x$ or $x_1 = y$). Let D^- be a $\gamma_s(G-e)$ - set containing vertices adjacent to all hanging vertices in $G-e$. Therefore, according to Theorem 3.8, the hanging vertex x_1 of $G-e$ belongs to the set $V(G)-D^-$. Moreover, it turns out that x_1 is adjacent to no vertex from $V(G) - D^-$. From this, we conclude that the subgraph $\langle V(G) - D^- \rangle_G$ is disconnected. In addition, by Remark 2.3 we have that D^- is dominating in G. Then it follows easily from the above that the subset D^- is split dominating in G i.e. $\gamma_s(G) \leqslant |D^-| = \gamma_s(G-e)$, as desired in (8) and this completes the proof. ∎

Applying Theorem 3.12 we obtain

Corollary 3.13. *Let H be a connected graph and $e \in V(\overline{H})$, where $\overline{H}$ is a complementary graph of H. If $H+e$ has at least two hanging vertices, then*

$$\gamma_s(H) - 1 \leqslant \gamma_s(H+e) \leqslant \gamma_s(H). \tag{9}$$

Proof. Putting $G = H+e$ and using inequality in (7) and inequality in (8) lead to expression (9). ∎

It may be note that P_3 is the unique connected graph having at most three vertices and different from a complete graph. But in this case the inequalities in (6) are violated, since there exists no split dominating set of $P_3 - e$, for each $e \in E(P_3)$.

4. The maximal domination number of a graph with a removed edge

In this part of parer we study a similar subject with respect to the maximal domination number. Note that a simple observation leads to the following conclusion.

Remark 4.1. *Let D be a dominating set of G. Then D is a maximal dominating set of G, if and only if there exists a vertex $x \in D$, such that $N_G(x) \subseteq D$. Moreover, if x is an isolated vertex of G, then x belongs to every maximal dominating set of G.*

Using the above remark we prove

Theorem 4.2. *For any edge e of a graph G*

$$\gamma_m(G) - 1 \leqslant \gamma_m(G-e) \leqslant \gamma_m(G) + 1. \tag{10}$$

Proof. First we shall show an inequality

$$(11) \qquad \gamma_m(G-e) \leqslant \gamma_m(G)+1.$$

Let D be a $\gamma_m(G)$ - set. By the definition of a maximal dominating set of G and by Remark 4.1, we have that there exists a vertex $u \in D$ such that $N_G(u) \subseteq D$. Note that $N_{G-e}(u) \subseteq N_G(u) \subseteq D$, for any $e \in E(G)$. If D is dominating in $G-e$, then D is maximal dominating in $G-e$, since $N_{G-e}(u) \subseteq D$. Suppose on the contrary that D is not dominating in $G-e$. Assuming that $e=(x,y)$, this is possible if and only if $x \in V(G)-D, y \in D$ and $N_{G-e}(x) \cap D = \emptyset$. Consider the superset $D \cup \{x\}$. If x is not a hanging vertex of G, then $D \cup \{x\}$ is dominating in $G-e$ and $N_{G-e}(u) \subseteq D \cup \{x\}$. From the above facts it follows that $D \cup \{x\}$ is maximal dominating in $G-e$ and as a consequence is $\gamma_m(G-e) \leqslant |D \cup \{x\}| = \gamma_m(G)+1$. If x is a hanging vertex of G, then x is an isolated vertex of $G-e$ and the set $D_0 = (D-\{u\}) \cup \{x\}$ is maximal dominating in $G-e$, so $\gamma_m(G-e) \leqslant |D_0| = \gamma_m(G)$, as required.

Next, we shall prove the second part of the assertion

$$(12) \qquad \gamma_m(G)-1 \leqslant \gamma_m(G-e).$$

Let D^- be a $\gamma_m(G-e)$ - set and $e=(x,y) \in E(G)$. By Remark 4.1 it follows that there exists $u \in D^-$ such that $N_{G-e}(u) \subseteq D^-$. Consider the case while $x \in V(G)-D^-$ and $y=u \in D^-$. Then $D^- \cup \{x\}$ is dominating in G and $N_G(u) = N_{G-e}(u) \cup \{x\} \subseteq D^- \cup \{x\}$. Hence $D^+ \cup \{x\}$ is a maximal dominating set of G and we conclude that $\gamma_m(G) \leqslant |D^- \cup \{x\}| = \gamma_m(G-e)+1$, as required. To complete the proof we notice that in all other cases the set D^- also is maximal dominating in G, since D^- is dominating in G and $N_G[u] \subseteq D$. As a consequence is $\gamma_m(G) \leqslant |D| = \gamma_m(G-e)$. Finally, by (11) and (12) the result follows. ∎

Corollary 4.3. *Putting $G=H+e$ into* (11) *and* (12), *we obtain*

$$\gamma_m(H)-1 \leqslant \gamma_m(H+e) \leqslant \gamma_m(H)+1,$$

where $e \in E(\overline{H})$.

It may be to note that if G contains isolated vertices, then we are able to get the lower bound of $G-e$ which is better than in (10). First we prove the following assertion

Lemma 4.4. *For any edge $e \in E(G)$,*

$$\gamma(G) \leqslant \gamma(G-e) \leqslant \gamma(G)+1,$$

where $\gamma(G)$ is a domination number of G.

Proof. If $D^- \subseteq V(G)$ is a $\gamma(G-e)$ - set, then also D^- is dominating in G. As a consequence is $\gamma(G) \leqslant |D^-| = \gamma(G-e)$. If D is $\gamma(G)$ - set, then among of sets: D, $D \cup \{x\}$, $D \cup \{y\}$ always is a dominating set of $G - e$. This gives $\gamma(G - e) \leqslant \gamma(G) + 1$, completing the proof. ∎

In [3] it has been proved

Proposition 4.5. [3]. *For any graph G,*

$$\gamma(G) = \gamma_m(G),$$

if and only if G contains an isolated vertex.

There it follows easily from the above assertions that

Corollary 4.6. *For any graph G containing isolated vertices and for any $e \in E(G)$,*

$$\gamma_m(G) \leqslant \gamma_m(G - e) \leqslant \gamma_m(G) + 1.$$

Proof. The result follows immediately by Proposition 4.5 applied for $G - e$ and by Lemma 4.4. ∎

5. Further results

In [6] V.G. Vizing has obtained an upper bound for the number of edges in a graph with a given number of vertices and a given domination number. We give an upper bound for the split domination number of a connected graph with a given number of edges in terms of the maximum and the minimum degree. To do it we use the following result of Vizing type.

Theorem 5.1. [5]. *Let G a connected graph on n vertices and Δ be an integer, such that $\Delta \geqslant 3$. If $\Delta(G) \leqslant \Delta$, then*

$$|E(G)| \leqslant \Delta n + (\Delta + 1)\gamma(G), \tag{13}$$

where $\gamma(G)$ is the domination number of G.

Theorem 5.2. *Let G be a connected graph with $|V(G)| = n$, $|E(G)| = m$ and $\Delta \geqslant 3$ be an integer. If $\Delta(G) \leqslant \Delta$, then*

$$\gamma_s(G) \leqslant \frac{\Delta n - m - 2\Delta}{\Delta + 1} + \delta(G). \tag{14}$$

Proof. Let y be a vertex of G such that $\delta_G(y) = \delta(G)$ and S be a set of all isolated vertices of the subgraph $\langle V(G) - N_G[y]\rangle_G$. Put $G_1 = \langle N_G[y] \cup S\rangle_G$.

First we claim that $S \neq \emptyset$.

From connectedness of G, it follows that for any $w \in S$ holds $N_G(w) \cap N_G(y) \neq \emptyset$ and of course $N_G(y)$ dominates y in G_1. Thus, the subset $N_G(y)$ is dominating in G_1. Furthermore, any path joining y and some vertex $w \in S$ in G_1, contains a vertex from $N_G(y)$ (the existence of such path is assured by connectedness of G_1). All this together gives that $N_G(y)$ is a split dominating set of G_1. Now, two cases can occur.

If $V(G) - V(G_1) = \emptyset$, then $G = G_1$ and $N_G(y)$ is a split dominating set of G. Thus $\gamma_s(G) \leqslant |N_G(y)| = \delta(G)$. To prove the inequality in (14) it remains to show that $\frac{\Delta n - m - 2\Delta}{\Delta+1} \geqslant 0$ or equivalently $\Delta n - m - 2\Delta \geqslant 0$. Since $m \leqslant \delta(G)\Delta(G) \leqslant \delta(G)\Delta$ and $n = \delta(G) + 1 + |S|$, we obtain

$$\Delta n - m - 2\Delta \geqslant \Delta(|S| - 1)$$

and $\Delta(|S| - 1) \geqslant 0$, because of $|S| \geqslant 1$. Hence

$$\gamma_s(G) \leqslant \delta(G) \leqslant \frac{\Delta n - m - 2\Delta}{\Delta + 1} + \delta(G),$$

as required.
Assume that $V(G) - V(G_1) \neq \emptyset$. By G_2 we denote an induced subgraph by the set $V(G) - V(G_1)$ in G. Further, we put $m_1 = |E(G_1)| \geqslant 1$, $m_2 = |E(G_2)| \geqslant 0$, $m_{12} = |E(G) - (E(G_1) \cup E(G_2))|$. In other words m_{12} is a cardinality of the set of all edges $(u,v) \in E(G)$ such that $u \in V(G_1)$ and $v \in V(G_2)$. Note additionally, then it follows by connectedness of G that $m_{12} \geqslant 1$. Since $|N_G(y)| = \delta(G)$ and for any vertex $u \in N_G(y)$, $\delta_G(u) \geqslant \Delta(G) \geqslant \Delta$, then

$$m_1 + m_{12} \leqslant N_G(y)\Delta(G) \leqslant \delta(G)\Delta. \tag{15}$$

Assume that D_0 is a $\gamma(G_2)$ - set. Then using the fact that $\Delta(G_2) \leqslant \Delta(G) \leqslant \Delta$ and applying expression (13) we get

$$m_2 \leqslant n_2\Delta - (\Delta + 1)\gamma(G_2), \tag{16}$$

where $n_2 = |V(G_2)|$ and $\gamma(G_2) = |D_0|$. Recall that subset $N_G(y)$ is split dominating in G_1. Note that any vertex from S is adjacent to some vertex from $N_G(y)$, but not adjacent to y and not adjacent to any vertex from $V(G_2)$. Further, we can observe that the subset $D = N_G(y) \cup D_0$ is split dominating in G and $\gamma_s(G) \leqslant |D| = \delta(G) + \gamma(G_2)$. Thus

$$\gamma(G_2) \geqslant \gamma_s(G) + \delta(G). \tag{17}$$

Noting that $m = m_1 + m_{12} + m_2$, we have by (15) and (16) that

$$m \leqslant \delta(G)\Delta + \Delta n_2 - (\Delta + 1)\gamma(G_2). \tag{18}$$

Therefore, combining (17) and (18) we conclude that

$$m \leqslant \delta(G)\Delta + \Delta n_2 - (\Delta + 1)(\gamma_s(G) + \delta(G)). \tag{19}$$

Further putting $n_1 = |V(G_1)|$, we have $n_1 = \delta(G) + 1 + |S|$.
Since $n_2 = n - n_1 = n - \delta(G) - 1 - |S|$, then from (19) by a simple calculation we obtain

$$m \leqslant \Delta n - \Delta(|S| + 1) + (\Delta + 1)\delta(G) - (\Delta + 1)\gamma_s(G)$$

or equivalently $m \leqslant \Delta n - 2\Delta + (\Delta + 1)\delta(G) - (\Delta + 1)\gamma_s(G)$, because of $|S| \geqslant 1$. Thus, we deduce that the inequality in (14) holds.

Next we shall consider the case while $S=\emptyset$. Then $G_1=\langle N_G[y]\rangle_G$. If $V(G)-V(G_1)=\emptyset$, then $G=G_1$ and $n=\delta(G)+1$ or equivalently $\delta(G)=n-1$, which is impossible since G can not be a complete graph, as it was assumed. So it must be that $V(G)-V(G_1)\neq\emptyset$. Further, by connectedness of G, it follows that there exist vertices $u\in N_G(y)$ and $v\in V(G)-V(G_1)$ such that $(u,v)\in E(G)$. Since $S=\emptyset$, then v is not an isolated vertex in $\langle V(G)-N_G[y]\rangle_G=\langle V(G)-V(G_1)\rangle_G$. This implies that exists a vertex $z\in V(G)-V(G_1)$ adjacent to v in G. Hence $|V(G)-V(G_1)|\geqslant 2$. Let $G_{11}=\langle N_G[y]\cup\{v\}\rangle_G$ and $G_{22}=\langle V(G)-V(G_{11})\rangle_G$ and denote by n_1,n_2 and m_1,m_2 cardinalities of the sets $V(G_{11}),V(G_{22})$ and $E(G_{11}),E(G_{22})$, respectively. Observe also that $m_{12}=|E(G)-(E(G_{11})\cup E(G_{22}))|$. Supposing that D_2 is a $\gamma(G_{22})$ - set, we shall prove that the set $N_G(y)\cup D_2$ is split dominating in G. It is enough to see that $N_G(y)$ is dominating in G_{11} (or more precisely the set $N_G(y)$ dominates y and v in G_{11}) and the subgraph $\langle V(G_{11})-N_G[y]\rangle_{G_{11}}=\langle\{y,v\}\rangle_{G_{11}}$ is disconnected. For explanation, the adding the set D_2 to the set $N_G(y)$ guarantees a domination in G and disconnectedness of the subgraph $\langle V(G)-(N_G(y)\cup D_2)\rangle_G$. In a consequence, we obtain that $\gamma_s(G)\leqslant|N_G(y)\cup D_2|=\delta(y)+\gamma(G_{22})$, so

$$\gamma(G_{22})\geqslant\gamma_s(G)+\delta(y). \tag{20}$$

Since $m_1+m_{12}\leqslant\delta(G)\Delta(G)\leqslant\delta(G)\Delta$ and $m=m_1+m_{12}+m_2$, then $m\leqslant\delta(G)\Delta+m_2$. Further, applying the inequality in (13) for the subgraph G_{22} and using the inequality in (20) we get

$$m\leqslant\delta(G)\Delta+n_2\Delta-(\Delta+1)(\gamma_s(G)+\delta(y)).$$

Thus, setting $n_2=n-\delta(G)-2$, we can write the last inequality in form

$$m\leqslant\Delta n-2\Delta+(\Delta+1)\delta(G)-(\Delta+1)\gamma_s(G).$$

This is equivalent to the inequality in (14) and the theorem is proved. ∎

At the end we give the relationship between $\gamma_s(G)$ and $\gamma_m(G)$, for a graph with hanging vertices. We start with a simple assertion.

Proposition 5.3. *Let G be a connected graph. If D is a $\gamma_m(G)$ - set, then D is not the minimum dominating set of G.*

Proof. Let D be a $\gamma_m(G)$ - set. Then by Remark 4.1 and by connectedness of G it follows that there exists a vertex $w\in D$ such that $D\supseteq N_G(w)\neq\emptyset$. Thus we conclude that the subset $D-\{w\}\subset D$ is dominating in G. But this means that D can not be the minimum dominating set of G and proof is complete. ∎

Theorem 5.4. *Let G be a connected graph with hanging vertices. Then*

$$\gamma_m(G)=\gamma_s(G)+1.$$

Proof. Since, by Proposition 5.3, any maximal dominating set of G is not the minimum dominating set of G, then $\gamma(G) < \gamma_m(G)$ or equivalently $\gamma(G) + 1 \leqslant \gamma_m(G)$. Further, it follows by Remark 4.1 that $\gamma(G) = \gamma_s(G)$ and from the last inequality we obtain

$$\gamma_s(G) + 1 \leqslant \gamma_m(G). \tag{21}$$

On the other hand let D be a $\gamma_s(G)$ - set of G, containing the vertices adjacent to all hanging vertices of G. The existence of such a set is assured by Theorem 3.8. Let $w \in V(G) - D$ be a hanging vertex of G, by Remark 3.9. Then $D \cup \{w\}$ is a maximal dominating set of G, because $N_G[w] \subset D \cup \{w\}$. Thus $\gamma_m(G) \leqslant \gamma_s(G) + 1$ and by the inequality in (21) the result follows. ∎

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