PATH SPECTRA AND FORBIDDEN FAMILIES

ALLEN G. FULLER AND RONALD J. GOULD

ABSTRACT. The *path spectrum*, $sp(G)$, of a graph G is the set of all lengths of maximal paths in G. The path spectrum is *continuous* if $sp(G) = \{\ell, \ell+1, \ldots, m\}$ for some $\ell \leq m$. A graph whose path spectrum consists of a single element is called *scenic* and is by definition continuous. In this paper, we determine when a $\{K_{1,3}, S\}$-free graph has a continuous path spectrum where S is one of $C_3, P_4, P_5, P_6, Z_1, Z_2, Z_3, N, B$, or W.

1. INTRODUCTION

All graphs considered in this paper are simple graphs, no loops or multiple edges are allowed. For terms not defined here, see [4]. A graph G is *hamiltonian* if G contains a cycle spanning the vertex set of G. A path P in G is *maximal* if it cannot be extended to a longer path by adding an edge and a vertex to one of the end vertices of P. A graph G is $\{H_1, H_2, \ldots, H_k\}$-*free* ($k \geq 1$) if G contains no induced subgraph isomorphic to an H_i, $1 \leq i \leq k$

The *path spectrum* of a connected graph G, $sp(G)$, is the set of lengths of all maximal paths in G. The path spectra of graphs have been studied in [5] and [2]. In [5] and [2], the focus of the work is on determining whether a given set of integers is in the path spectrum of some graph. Also, in [5], Jacobson *et al.* asked about the complexity of computing the path spectrum of a given graph G. They considered the related question of whether there is a maximal path of length k. This question is NP-hard since if k is one less than the order of G, the problem asks whether the graph has a hamiltonian path. Hence, the path spectrum question for an arbitrary graph was determined to be NP-complete.

However, Bedrossian in [1] proved the following (see Figure 1 for drawings of some of the graphs).

Theorem 1. Let R and S be connected graphs with $R, S \neq P_3$, and let G be a 2-connected graph that is not a cycle. Then G being $\{R, S\}$-free implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, P_6, C_3, Z_1, Z_2, B, N$, or W.

Research partially supported by O.N.R. Grant N00014-97-1-0499.

Faudree and Gould in [3] improved on the work of Bedrossian to get the following theorem.

Theorem 2. Let R and S be connected graphs with $R, S \neq P_3$, and let G be a 2-connected graph of order $n \geq 10$. Then G being $\{R, S\}$-free implies G is hamiltonian if and only if $R = K_{1,3}$ and $S = P_4, P_5, P_6, C_3, Z_1, Z_2, Z_3, B$, N, or W.

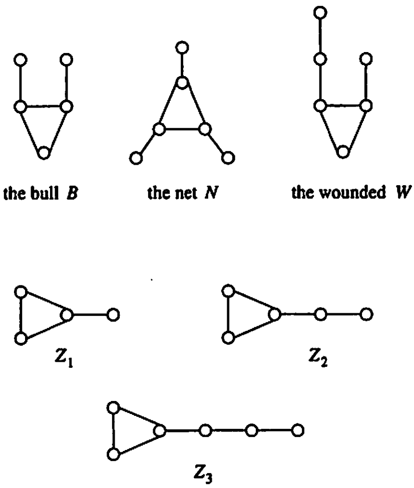

FIGURE 1. The graphs B, N, W, Z_1, Z_2, Z_3.

Since we know that these 2-connected, $\{R, S\}$-free graphs are hamiltonian (and hence have a hamiltonian path), we ask what can be said about the path spectrum of such graphs. In particular, are they continuous? By a *continuous path spectrum*, we mean that $sp(G) = \{\ell, \ell+1, \ldots, m\}$ where ℓ is the length of the shortest maximal path in G and m is the length of the longest maximal path in G. Note a path spectrum consisting of only one element is continuous. A graph with such a path spectrum is called *scenic*. Thomassen characterized when a traceable graph is scenic in [7]. Jacobson, Kézdy, and Lehel also studied scenic graphs in [6].

We need some notation to state Thomassen's result. A matching of t edges will be denoted by tK_2. A graph that is a complete graph minus a matching with $1 \leq t \leq n/2$ will be denoted by $K_n - tK_2$. A complete bipartite graph plus (resp. minus) an edge is denoted by $K_{p,p} + K_2$ (resp. $K_{p,p} - K_2$). The graph obtained by adding an edge to each partite set of $K_{p,p}$ is denoted by $K_{p,p} + 2K_2$. If $H \in \{K_3, 2K_2, K_{1,q}\}$, the graph $K_{p,p+1} + H$ denotes the graph formed by adding all the edges of H to the largest partite set of $K_{p,p+1}$. The *cube* is the graph $K_{4,4} - 4K_2$ and the *prism* is the graph formed from K_6 by removing the edges of a six-cycle. The following result is Thomassen's characterization of traceable, scenic graphs.

Theorem 3. [7] A traceable graph is scenic if and only if it belongs to one of the following families:

$$\begin{aligned}
\Phi[K_n] &= \{K_n, K_n - tK_2\ (1 \le t \le n/2)\}, \\
\Phi[K_{p,p}] &= \{K_{p,p}, K_{p,p} - K_2, K_{p,p} + K_2, K_{p,p} + 2K_2\}, \\
\Phi[K_{p,p+1}] &= \{K_{p,p+1}, K_{p,p+1} + K_3, K_{p,p+1} + 2K_2, \\
&\quad K_{p,p+1} + K_{1,q}\ (1 \le q \le p)\}, \\
\Psi &= \{P_n, C_n, prism, cube\}.
\end{aligned}$$

We answer the question concerning the 2-connected, $\{R, S\}$-free graphs in Theorem 2 that have continuous path spectra in the following result.

Theorem 4. Let G be a 2-connected, $\{K_{1,3}, S\}$-free graph of order $n \ge 10$ where S is one of C_3, P_4, P_5, P_6, Z_1, Z_2, Z_3, N, B, or W. Then G has a continuous path spectrum if and only if S is one of the graphs C_3, P_4, Z_1 or Z_2. Furthermore, G is scenic if and only if G is one of $K_n, K_n - tK_2$ or C_n.

The proof of this theorem is in Section 3. Two preparatory propositions are in the next section.

2. Two Results

Proposition 2.1. Let G be a 2-connected, $\{K_{1,3}, Z_2\}$-free graph of order n. Let P be a maximal u–v path of order $m < n$. Then P can be extended to a maximal u–v path of order $m + 1$.

Proof. Let P be $u = x_1, x_2, \ldots, x_m = v$. Since G is connected, $m < n$, and P is maximal, there is a vertex w in $V(G) - V(P)$ such that w is adjacent to a vertex x_j, $1 < j < n$, on P. Also, since G is 2-connected, there is at least one other path from w to P. Consider the collection C of these paths that have the shortest length. Let Q be the path from C that hits closest to x_j. Suppose that Q hits P at x_k and with out loss of generality that $j < k < n$. Let Q be $x_k = z_1, z_2, \ldots, z_\ell = w$.

CASE 1: Suppose that $k \ge j+3$. We first note that $\langle\{x_{j-1}, x_j, x_{j+1}, w\}\rangle$ is a claw centered at x_j and that $\langle\{x_{k-1}, x_k, x_{k+1}, z_2\}\rangle$ forms a claw centered at x_k. Observe that w cannot be adjacent to x_{j-1} or x_{j+1} otherwise Q would not be the closest path to x_j from w to P. Also, z_2 is not adjacent to x_{k-1} or else Q would not hit closest to x_j. Now if z_2 is adjacent to x_{k+1}, then P can be easily extended as follows:

$$u = x_1, x_2, \ldots, x_k, z_2, x_{k+1}, \ldots, x_m = v.$$

Thus, suppose that z_2 is not adjacent to x_{k-1} or x_{k+1}. Consequently, $x_{j-1}x_{j+1}$ and $x_{k-1}x_{k+1}$ must be edges of G.

Now, if Q has 3 or more vertices, then $\langle\{x_{k-1}, x_k, x_{k+1}, z_2, z_3\}\rangle$ forms a Z_2. Thus, either $x_{k-1}z_2$, $x_{k+1}z_2$, $x_{k-1}z_3$, $x_{k+1}z_3$, or x_kz_3 is an edge of G. Since Q is the shortest path from w to P, $x_{k-1}z_3$, $x_{k+1}z_3$, and x_kz_3 cannot be edges in G. Since Q hits closest to x_j, $x_{k-1}z_2$ cannot be an edge in G. If x_{k+1} is adjacent to z_2, then P can be extended as above. Therefore, we assume that Q has only two vertices; that is, w is adjacent to x_k.

Next, we note that $\{x_{j-1}, x_j, x_{j+1}, w, x_k\}$ induces a Z_2. Hence at least one of the following edges is in G: $x_{j-1}x_k$, $x_{j+1}x_k$, or x_jx_k. (The pairs wx_{j-1} and wx_{j+1} were eliminated since Q hits closest to x_j.) If $x_{j-1}x_k$ (or similarly $x_{j+1}x_k$) is an edge, then P can be extended as follows:

$$u = x_1, x_2, \ldots, x_{j-1}, x_k, w, x_j, x_{j+1}, \ldots, x_{k-1}, x_{k+1}, \ldots, x_m = v.$$

Therefore, assume that x_jx_k is an edge in G.

Note that by a symmetric argument on x_k, the edges wx_{k+1}, $x_{k-1}x_j$, and $x_{k+1}x_j$ can shown to extend the path P.

Before proceeding, we make the following notational convention and two observations. We will denote the subpath $\{x_a, x_{a+1}, \ldots, x_b\}$ of P as $[x_a, x_b]$. Now we observe that if x_{j-1} and x_{k+1} are adjacent to adjacent vertices of $[x_{j+1}, x_{k-1}]$, then P can be extended. To see why, suppose that x_{j-1} is adjacent to x_i and that x_{k+1} is adjacent to x_{i+1}. Then a path of order $m+1$ can be formed as follows:

$$u = x_1, x_2, \ldots, x_{j-1}, x_i, x_{i-1}, \ldots, x_j, w, x_k, x_{k-1} \ldots, \\ x_{i+1}, x_{k+1}, \ldots, x_m = v.$$

Secondly, we note that if x_j and x_k are adjacent to adjacent vertices of $[x_{j+1}, x_{k-1}]$, then P can be extended to a path of order $m+1$. To see this, suppose without loss of generality that x_j is adjacent to x_i and x_k is adjacent to x_{i+1}. Then a path of order $m+1$ is formed as follows:

$$u = x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_i, x_j, w, x_k, x_{i+1}, \ldots, \\ x_{k-1}, x_{k+1}, \ldots, x_m = v.$$

Now, notice that $\langle\{w, x_j, x_k, x_{k-1}, x_{k-2}\}\rangle$ and $\langle\{w, x_j, x_k, x_{j+1}, x_{j+2}\}\rangle$ each forms a Z_2. We will only consider the Z_2 induced by $\{w, x_j, x_k, x_{k-1}, x_{k-2}\}$ in detail since the Z_2 $\langle\{w, x_j, x_k, x_{j+1}, x_{j+2}\}\rangle$ is symmetric. We see that at least one of the following pairs is an edge of G: x_jx_{k-2}, x_jx_{k-1}, or $x_{k-2}x_k$. If either of the edges x_jx_{k-2} or x_jx_{k-1} is an edge of G, then x_{j-1} and x_{k+1} are adjacent to adjacent vertices in $[x_{j+1}, x_{k-1}]$, and thus P can be extended. Hence, we assume that x_jx_{k-2} and x_jx_{k-1} are not edges in G but that $x_{k-2}x_k$ is an edge in G. By symmetry, we assume that x_kx_{j+2} and x_kx_{j+1} are not edges in G but that x_jx_{j+2} is an edge in G. If $j+2 = k-2$, then x_j and x_k are adjacent to adjacent vertices in $[x_{j+1}, x_{k-1}]$ and P can be extended. If $j+2 \neq k-2$, we apply arguments similar to the preceding arguments to the following Z_2's: $\langle\{w, x_j, x_k, x_{k-2}, x_{k-3}\}\rangle$ and $\langle\{w, x_j, x_k, x_{j+2}, x_{j+3}\}\rangle$. We see that the only edges that do not immediately lead to a path of length $m+1$ are x_jx_{j+3} and x_k, x_{k-3}. We continue the process until the path extends or we reach a point where x_j and x_k are adjacent to adjacent vertices in $[x_{j+1}, x_{k-1}]$ which also implies P can be extended.

Thus, we see that if $k \geq j+3$, P can be extended to a path of length $m+1$.

CASE 2: Suppose that $k = j+2$. Then, by the arguments of Case 1, we may assume that the edges $x_{j-1}x_{j+1}$ and $x_{k-1}x_{k+1}$ and that w is adjacent to x_k. Thus P can be extended as follows:

$$u = x_1, \ldots, x_{j-1}, x_{j+1}, x_j, w, x_k, x_{k+1}, \ldots, x_m = v.$$

CASE 3: Suppose that $k = j+1$. Note that if w is adjacent to both x_j and x_k, P can easily be extended by exactly one vertex. Thus, we assume that w is not adjacent to x_k; that is, Q has at least three vertices. Thus, $\langle\{x_j, x_k, x_{k+1}, z_2\}\rangle$ forms a $K_{1,3}$ centered at x_k. If either x_jz_2 or $x_{k+1}z_2$ is an edge in G, P is easily seen to be extendable. Hence, we suppose that x_jx_{k+1} is an edge of G. Since $\langle\{x_j, x_k, x_{j-1}, w\}\rangle$ forms a claw centered at x_j, we assume by symmetry that $x_{j-1}x_k \in E(G)$.

Now, we note that if Q has more than three vertices, a Z_2 is formed by $\langle\{x_j, x_k, x_{k+1}, z_2, z_3\}\rangle$. Observe that $x_{k+1}z_3$ cannot be an edge of G or Q would not be the shortest path from w to P. If z_2 is adjacent to x_{k+1}, P is easily seen to be extendable to a path of length $m+1$. Thus, we suppose that x_jz_3 is an edge of G. Observe that this is really the case when Q has exactly three vertices as z_3 assumes the role of w.

Thus, suppose that Q has three vertices, say w, z, and x_k. Note that $\langle\{w, x_{j-1}, x_j, x_{k+1}\}\rangle$ forms a claw centered at x_j. Since Q is the shortest path from w to P (except for wx_j),$x_{j-1}x_{k-1}$ must be an edge in G. However, we see that $\{x_{j-1}, x_{k+1}, x_k, z, w\}$ induces a Z_2. Since Q is the shortest path, the only possible edges that can exist are $x_{j-1}z$ and $x_{k+1}z$. Clearly if $x_{k+1}z$ is an edge of G, P can be extended. Now, if $x_{j-1}z \in E(G)$, P can be extended as follows:

$$u = x_1, \ldots, x_{j-1}, z, x_k, x_j, x_{k+1}, \ldots, x_m = v.$$

Thus, when $k = j+1$, P can be extended to a path of length $m+1$. □

Proposition 2.2. Let G be a 2-connected, $\{K_{1,3}, P_4\}$-free graph of order n. Let P be a maximal u–v path of order $m < n$. Then P can be extended to a maximal u–v path of order $m+1$.

Proof. Let P be $u = x_1, x_2, \dots, x_m = v$. Since G is connected and $m < n$, there is a vertex w in $V(G) - V(P)$ such that w is adjacent to a vertex x_j on P. Also, since G is 2-connected, there is at least one other path from w to P. Consider the collection C of these paths that have the shortest length. Among this collection let Q be the path that hits closest to x_j. Suppose that Q hits P at x_k and with out loss of generality that $j < k < n$. Let Q be $x_k = z_1, z_2, \dots, z_\ell = w$. Observe that since G is P_4-free, $\ell \leq 3$.

CASE 1: Suppose that $k \geq j + 2$. First, we note that since G is claw-free, $x_{j-1}x_{j+1}$ and $x_{k-1}x_{k+1}$ are edges in G. Next, we observe that $\ell = 2$. To see why, suppose $\ell = 3$. Then $wz_2x_kx_{k-1}$ forms a P_4. Note that the addition of any edge to this P_4 contradicts the choice of Q. Hence, $\ell = 2$; that is, w is adjacent to x_k.

Now, we see that if $k = j + 2$, P can be extended as follows:

$$u = x_1, x_2, \dots, x_j, w, x_k, x_{k-1}, x_{k+1}, \dots, x_m = v.$$

Thus, we assume that $k > j + 2$ and observe that $\langle\{x_{j-1}, x_j, w, x_k\}\rangle$ forms a P_4. The vertex w cannot be adjacent to x_{j-1} (contradicts the choice of Q). If x_{j-1} is adjacent to x_k, then P can be extended as follows:

$$u = x_1, x_2, \dots, x_{j-1}, x_k, w, x_j, x_{j+1}, \dots, x_{k-1}, x_{k+1}, \dots, x_m = v.$$

Thus, we suppose that x_j is adjacent to x_k.

Now, we see that $\langle\{x_{j+1}, x_j, x_k, x_{k+1}\}\rangle$ forms a P_4. If x_{j+1} is adjacent to x_{k+1}, then P can be extended as follows:

$$u = x_1, x_2, \dots, x_j, w, x_k, x_{k-1}, x_{k-2}, \dots, x_{j+1}, x_{k+1}, \dots, x_m = v.$$

If x_{j+1} is adjacent to x_k, then P can be extended as follows:

$$u = x_1, x_2, \dots, x_j, w, x_k, x_{j+1}, x_{j+2}, \dots x_{k-1}, x_{k+1} \dots, x_m = v.$$

Finally, if x_j is adjacent to x_{k+1}, then P can be extended as follows:

$$u = x_1, x_2, \dots, x_{j-1}, x_{j+1}, x_{j+2}, \dots, x_k, w, x_j, x_{k+1}, \dots, x_m = v.$$

CASE 2: Suppose that $k = j + 1$. If $\ell = 2$, then P is easily extendable. Hence, we assume that $\ell = 3$. Then $\langle\{w, z_2, x_k, x_{k+1}\}\rangle$ forms a P_4. The edges wx_k and wx_{k+1} cannot be in G by the choice of Q. Thus, $x_{k+1}z_2$ must be an edge in G. Consequently, P is easily seen to be extendable. □

3. Proof of Theorem 4

Proof. First we note that if G is a 2-connected, $\{K_{1,3}, C_3\}$-free graph, then G is a cycle, $C_n, n \geq 10$. Also note that a 2-connected, $\{K_{1,3}, Z_1\}$-free graph is either a cycle or a complete graph minus a matching. By Theorem 3, the only 2-connected, $\{K_{1,3}, S\}$-free scenic graphs of order $n \geq 10$ are $K_n, K_n - tK_2$, and C_n. Thus, these graphs have continuous path spectra.

Now suppose G is a nonscenic, 2-connected, $\{K_{1,3}, S\}$-free graph of order $n \geq 10$ where S is P_4 (or Z_2). Then by choosing the shortest maximal path in G and repeatedly applying Proposition 2.2 (or Proposition 2.1), we see that the path spectrum of G is continuous.

Finally, suppose that G is a nonscenic, 2-connected $\{K_{1,3}, S\}$-free graph of order $n \geq 10$ where S is one of B, N, W, P_5, P_6, or Z_3. We consider the graph H in Figure 2. The path spectrum of H is easily seen to be

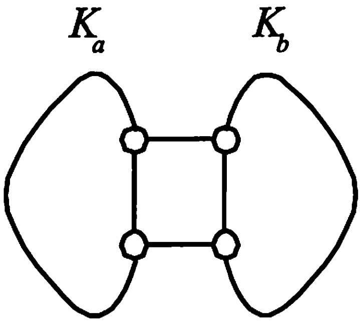

FIGURE 2. The graph H with $b > a+1$, $a \geq 4$.

$sp(H) = \{a-1, a+1, a+2, \ldots, a+b-1\}$. The graph H is also free of claws, B's, N's, W's, P_5's, P_6's, and Z_3's. □

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DIVISION OF NATURAL SCIENCES AND NURSING, GORDON COLLEGE, BARNESVILLE, GA 30204

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, EMORY UNIVERSITY, ATLANTA, GA 30322