A note on choosability with separation for planar graphs

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Abstract

A graph G is (p, q, r)-choosable, if for every list assignment L with $|L(v)| \geq p$ for each $v \in V(G)$ and $|L(u) \cap L(v)| \leq p - r$ whenever u, v are adjacent vertices, G is q-tuple L-colorable. We give an alternative proof of $(4t, t, 3t)$-choosability for the planar graphs and construct a triangle-free planar graph on 119 vertices which is not $(3, 1, 1)$-choosable (and so neither 3-choosable). We also propose some problems.

1 Introduction

The concept of the list colorings and choosability was introduced by Vizing [6] and independently by Erdős, Rubin, and Taylor[1]. A ***list assignment*** of G is a function L that assigns to each vertex $v \in V(G)$ a list $L(v)$ of colors. An *L-coloring* is a function $\lambda : V(G) \to \bigcup_v L(v)$ such that $\lambda(v) \in L(v)$ for each $v \in V(G)$ and $\lambda(u) \neq \lambda(v)$ whenever u, v are adjacent vertices of G. If G admits an L-coloring, it is called *L-colorable*. Graph G is *m-choosable* if for every list assignment L with $|L(v)| \geq m$ for each $v \in V(G)$, there exists an L-coloring of G. The *list chromatic number* (or *choice number*) $\chi_l(G)$ of G is the smallest number m for which G is m-choosable.

A list assignment L with $|L(v)| \geq p$ for each $v \in V(G)$ and $|L(u) \cap L(v)| \leq p - r$ for every pair of adjacent vertices $u, v \in V(G)$ is called *(p, r) list assignment*. A graph G is *(p, q, r)-choosable* if for every (p, r) list

*Supported in part by the Ministry of Science and Technology of Slovenia, Research Project J1-0502-0101-98.

assignment L, we can choose for every vertex $v \in V(G)$ a subset $\Lambda(v) \subseteq L(v)$ with $|\Lambda(v)| = q$ so that $\Lambda(u) \cap \Lambda(v) = \emptyset$ for every pair of adjacent vertices $u, v \in V(G)$. This concept was recently introduced by Kratochvíl, Tuza, and Voigt [3], and it is called *choosability with separation* (see also Tuza [5]). They proved that every planar graph is $(4t, t, 3t)$-choosable and every triangle-free planar graph is $(3t, t, 2t)$-choosable. In fact, this is also true for graphs embeddable in the torus, projective plane or the Klein bottle. One can prove this using Theorem 4.1 from [3] and Euler's formula for graphs on surfaces. Using Wagner's Theorem [9] one can similarly show that K_5-minor-free graphs are also $(4t, t, 3t)$-choosable.

Voigt [7] constructs a triangle-free planar graph on 166 vertices which is not $(3, 1, 1)$-choosable. Gutner [2] constructs such graph on 164 vertices.

In this paper, we give an alternative proof of $(4t, t, 3t)$-choosability of the planar graphs. Combining the above mentioned constructions, we obtain a triangle-free planar graph on 119 vertices which is not $(3, 1, 1)$-choosable (and so neither 3-choosable). Finally, we propose several problems.

2 The $(4t, t, 3t)$-choosability of planar graphs

The proof in [3] of $(4t, t, 3t)$-choosability of the planar graphs relies on Euler's formula and Hall's theorem. Here we give straightforward proof of this assertion. The $(4, 1, 3)$-choosability of planar graphs is an easy consequence of the following lemma. The proof of this lemma is similar to the proof of planar 5-choosability [4].

Lemma 1 *Let G be a connected planar graph with outerwalk $C = x_1 \cdots x_n x_1$. Let L be a list assignment of G such that $|L(x)| \geq 3$ for every $x \in V(C)$, $|L(x)| \geq 4$ for every $x \in V(G) \setminus V(C)$, and $|L(x) \cap L(y)| \leq 1$ whenever x, y are adjacent vertices of G. Let $\lambda(x_1) \in L(x_1)$ and $\lambda(x_n) \in L(x_n) \setminus \{\lambda(x_1)\}$. Then, λ can be extended to an L-coloring of G.*

Proof. Suppose that the lemma is false and G is a counterexample with $|V(G)|$ as small as possible.

(1) *C is 2-connected.* Suppose that C is not 2-connected. Now, by the minimality, extend λ to the block which contains x_n and x_1. After that repeat the following procedure until G is colored: choose block B with precisely one colored vertex, say u. Let $v \in V(B)$ be a neighbor of u on the outerwalk of B. Set $\lambda(v) \in L(v) \setminus \{\lambda(u)\}$ and by the minimality, extend λ of $\{u, v\}$ to B.

(2) *C is a chordless cycle.* Suppose that edge uv is a chord of C. Let C_1 and C_2 be the two chordless cycles of the graph $C \cup \{uv\}$. Let $G_1 = \mathrm{Int}(C_1)$ and $G_2 = \mathrm{Int}(C_2)$. We may assume that $x_n x_1 \in E(G_1)$. Now, by the

minimality, extend λ of $\{x_n, x_1\}$ to G_1 and after that extend λ of $\{u, v\}$ to G_2, a contradiction.

(3) *C has a chord.* Suppose that C is chordless. Note that, by the assumptions of L, we can choose $\lambda(x_2) \in L(x_2) \setminus \{\lambda(x_1)\}$ so that the following is satisfied: if $n = 3$ then $\lambda(x_2) \neq \lambda(x_3)$ and if $n \neq 3$ then $\lambda(x_2) \notin L(x_3)$. Now, let $G' = G \setminus \{x_2\}$ and L' be the list assignment of G' such that $L'(x) = L(x) \setminus \{\lambda(x_2)\}$ if $x \in N(x_2) \setminus \{x_n, x_1\}$ and $L'(x) = L(x)$ otherwise. By (1) and (2), G' and L' satisfy the assumptions of the lemma. So, by the minimality, we can extend λ to G', a contradiction.

Obviously, by (2) and (3), we obtain a contradiction which completes the proof. □

Note that this lemma and its proof can be easily generalized to the q-tuple colorings and then we obtain $(4t, t, 3t)$-choosability for the planar graphs.

3 A not $(3, 1, 1)$-choosable triangle-free planar graph

We leave the reader to verify the following lemma.

Lemma 2 *Graph G' from Figure 1 is not L'-colorable.*

Now we will construct graph G and a list assignment L of G as follows. Take nine copies $G_i, i = 0, \ldots, 8$ of G'. For $i = 0, \ldots, 8$, let L_i be the the list assignment of G'_i obtained from L' by replacing the colors (z_1, x, y, z_2) with that from the tuple t_i:

$$\begin{array}{lll} t_0 = (7,5,8,9) & t_1 = (8,5,9,10) & t_2 = (9,5,10,6) \\ t_3 = (5,6,10,7) & t_4 = (6,7,10,9) & t_5 = (10,7,9,6) \\ t_6 = (7,6,9,8) & t_7 = (9,6,8,7) & t_8 = (6,7,8,5). \end{array}$$

Identify all nine u vertices of the graphs G_i, $i = 0, \ldots, 8$ and afterward identify all nine v vertices of these graphs; finally identify w_2 of G_i with w_1 of G_{i+1} for $i = 0, \ldots, 8$ (index modulo 9). Let $L(u) = \{5, 6, 7\}$ and $L(v) = \{8, 9, 10\}$ and for every other vertex $x \in V(G)$ let $L(x) = L_i(x)$ whenever $x \in G_i$. Note that it is well defined since $L_i(w_2) = L_{i+1}(w_1)$ for $i = 0, \ldots, 8$ (index modulo 9).

It is not hard to observe that G is a triangle-free planar graph and L is a $(3, 1)$ list assignment of G. Graph G is of order $12 \cdot 9 + 9 + 2 = 119$. In every L-coloring of G, there always exists $i \in \{0, \ldots, 8\}$ such that colors of the vertices u and v are equal to the second and the third color of the tuple

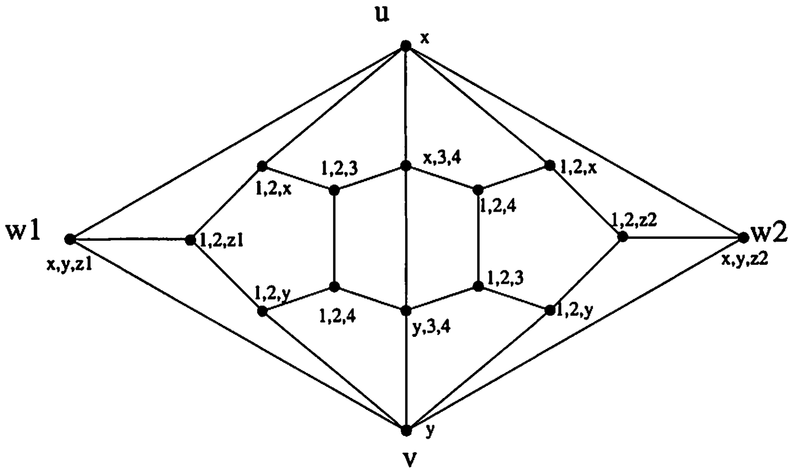

Figure 1: Graph G' with list assignment L'

t_i, respectively. Then from Lemma 2, it follows that the coloring of u and v cannot be extended to G_i and so to G. This contradicts the existence of an L-coloring of G. Thus, we obtain the following result.

Proposition 1 *G is a triangle-free planar graph on 119 vertices which is not $(3,1,1)$-choosable (and so neither 3-choosable).*

We conclude this paper with the following problems. In [3], it was asked if planar graphs are (4, 1, 2)-choosable. But the answer of the following problem is not known also.

Problem 1 *Is every planar graph $(3,1,2)$-choosable?*

Let $f : N \to N$ be a function, G be a graph, and let $\chi = \chi(G)$. Denote by $\mathcal{L}(G, f)$ the set of lists assignments L of G such that L assigns at least one color to every vertex of G and $|L(u) \cap L(v)| \geq f(\chi)$ whenever u, v are adjacent vertices of G. Note that the condition on the list assignments from $\mathcal{L}(G, f)$ is in some way opposite to that in the concept of choosability with separation. Instead of forbidding, it requires that adjacent vertices have many colors in common.

Problem 2 *Is there a function f such that every graph G is L-colorable for every $L \in \mathcal{L}(G, f)$?*

By Voigt and Wirth [8], if such a function f exists, then $f(x) \geq x+1$. Similarly for the planar graphs one can ask the following two problems. If these problems are true, then they will be beautiful generalizations of The Grötzsch Theorem and The Four Color Theorem (in the concept of list colorings).

Problem 3 *Let G be a connected nontrivial triangle-free planar graph and let L be an arbitrary list assignment of G such that $|L(u) \cap L(v)| \geq 3$ whenever u, v are adjacent vertices of G. Is G L-colorable?*

Problem 4 *Let G be a connected nontrivial planar graph and let L be an arbitrary list assignment of G such that $|L(u) \cap L(v)| \geq 4$ whenever u, v are adjacent vertices of G. Is G L-colorable?*

Remark. Problem 2 was recently solved by Graham Brightwell from London School of Economics. He proved that such a function does not exist. Moreover, in the realm of bipartite graphs, exists such a function. Bellow is his proof.

Solution of Problem 2 (Brightwell). We show the following.

(i) If G is bipartite, and L is a list assignment such that $|L(u)| \geq 1$ for all vertices u and $|L(u) \cap L(v)| \geq 2$ whenever u and v are adjacent, then (G, L) admits a list coloring.

(ii) For any $k \in N$, there is a 3-colorable graph G and a list assignment L such that $|L(u) \cap L(v)| \geq k$ for every adjacent pair (u, v), and yet (G, L) does not admit a list coloring.

In the language of the problem, this means that $f(2)$ can be taken to be 2, while no value of $f(3)$ suffices.

(i) Suppose that G is bipartite, and let the two bipartite classes be U and V. Given a list assignment L such that $|L(u)| \geq 1$ and $|L(u) \cap L(v)| \geq 2$ whenever u and v are adjacent, we define a list-coloring of G as follows.

First we take a total order $\prec$ of the set of colors used. Then, for each vertex u of U, we assign the $\prec$-least color in $L(u)$, while for each vertex v of V, we assign the $\prec$-greatest color in $L(v)$. The condition on the lists now ensures that different colors are assigned to u and v whenever u and v are adjacent.

(ii) Given any positive integer k, define the graph G as a complete 3-partite graph, with each vertex class of size $\binom{3k}{2k}$. For each subset A of $\{1, \ldots, 3k\}$ of size $2k$, there is one vertex in each part with A as its list. Note that any two vertices of G have lists with at least k colors in common, so the lists satisfy the given condition. On the other hand, at least $k+1$

colors must be used for each part, in any list coloring, since if only k colors are used on some part then there is some vertex that is assigned a color not in its list. Since the graph is complete 3-partite, the sets of colors used for each part must be disjoint, so at least $3k+3$ colors are used in all. But this is a contradiction, since only a total of $3k$ colors appear in the lists. □

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